Algebra - I
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Lecture - 70
Finitely generated modules and the Noetherian condition

In this lecture, I am going to discuss Finitely generated modules and the relationship with the ascending chain condition for modules, which is also known as the Noetherian condition. So, let us start with a finitely generated modules.

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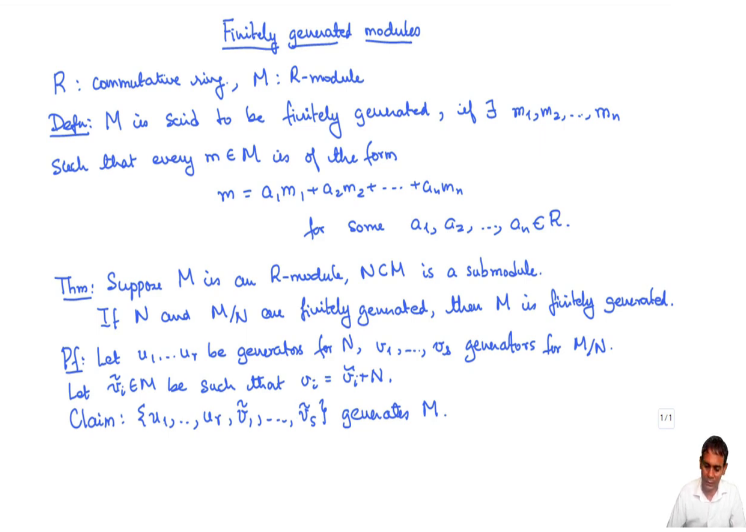

So, we will take R to be a commutative ring and we define an R-module M to be finitely generated. So, M is an R module, if you can find a finite number of elements of M that generated. In the sense, if there exist elements m 1, m 2, m n for some finite n such that every element m in M is of the form m is equal to a 1 m 1 plus a 2 m 2 plus a n m n for some elements a 1, a 2, a n in R.

For example, if R is a field, then R modules are just vector spaces over the field and a finitely generated R-module is just a finite dimensional vector space over the field. In frontend example of an R-module that is not finitely generated, you could again just take R to be a field and take an infinite dimensional vector space over the field.

Now, a very help useful property of R modules that we will use later on is the following. So, let us put it down as a theorem. So, suppose a M is an R module; N, a sub module of M and suppose, we know that N and M mod N, the quotient module are finitely generated, then we can conclude that M is also finitely generated.

The proof of this is not very difficult. You just write down generators of N, generators of M mod N and somehow using them, you construct a finite set of generators for M itself. So, firstly, you write down generators. Let u 1 up to u r be generators for N, v 1 dot dot dot v s generators for M mod N and now, we will put these together to construct a generating set for M.

Now, the problem is that these v 1, v 2, v s, they are in the quotient space M mod N, they are not in M. So, what we will do is we will just lift them up to M. So, let v i tilde in M be such that v i is v i tilde plus N. So, it is the image; v i is the image of v i tilde in M mod N.

Then, I claim that this set you take u 1 up to u r and then, you take v 1 tilde up to v s tilde; this generates M. So, what do I need to show? I need to show that given any element m of M, I can write it as combination of these things multiplied by scalars; so, a linear combination of these elements.

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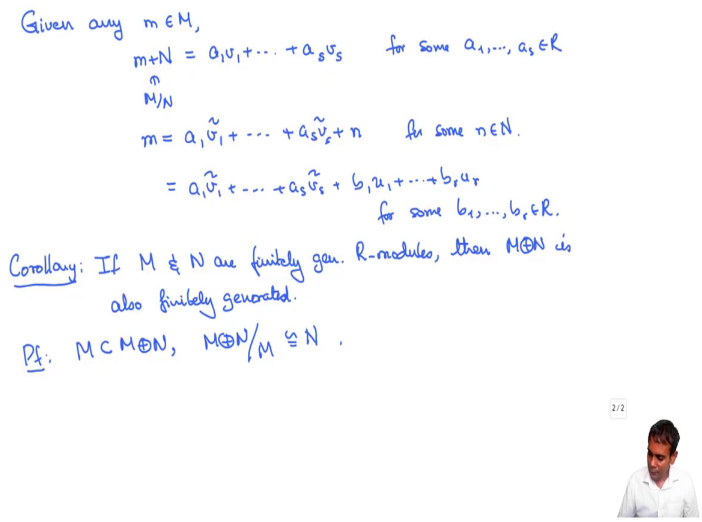

So, take any given any m in M, let. So, its image in M mod N is m plus N ok. So, this is an element of M mod N and this because this v 1, v 2, v s generate M mod N is of the form m plus N is of the form a 1, v 1 plus a s, v s and so that means, that m is of the form a 1, v 1 tilde plus a s, v s tilde plus some element n of N.

But then, this element n is in N. So, it can be written as. So, this is for some a 1, a s in R and this N can be written as a 1, a s v s tilde plus b 1 u 1 plus b r u r for some b 1 b r in R. So, at the end of the day, we have written M as a linear combination of the vectors, when the module elements v 1 tilde up to v s tilde and u 1 up to u r.

So, a corollary of this is if M and N well rather a special case, if M and N are finitely generated R modules; then, M plus N is also finitely generated. Of course, this may be much easier to prove than as a deducing it as a corollary of the previous theorem, you could just take a generating set of M a generating set of N and take their union and that would be a generating set of M direct sum N.

ah But this is I will just explain the proof using this, how can we think of M plus N as a module with some sub module and some quotient module. The point is M is a sub module of M plus N and M plus N mod M is isomorphic to N. So, this sub module is finitely generated; this quotient is finitely generated. So, by the previous theorem, M direct sum N is also finitely generated. It is a slightly different proof..

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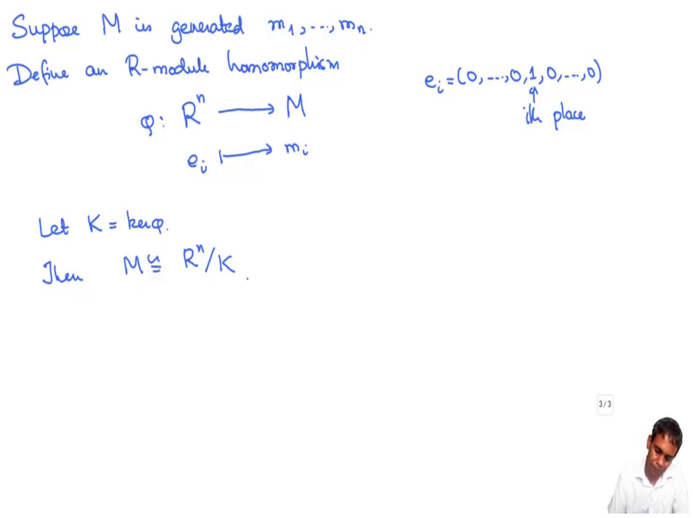

And one observation about finitely generated R modules; so, suppose our module M is generated by m 1, m 2, m n; then define an R-module homomorphism. So, this R n itself R to the power n which is R direct sum R direct sum R n times this is an R-module and we define an R-module homomorphism, this to M by taking.

So, in this you let e i denote the element which is 0 at all places except for a 1 at the unit at the ith place ok; a coordinate vector; so, to speak by taking e i to m i. And what we have is that we will call this map phi and let K be the kernel of phi, then M is isomorphic to R n mod K as in R module.

Because whenever you have surjective homomorphism of R modules, then the target R-module is a quotient of the domain modulo its kernel. A very useful condition in studying finitely generated R modules is that of Noetherianness, you have already seen the Noetherian condition for rings. Remember a ring is said to be Noetherian, if every increasing chain of ideas eventually stabilizes. So, we can use the same idea for modules. So, let us define Noetherian modules.

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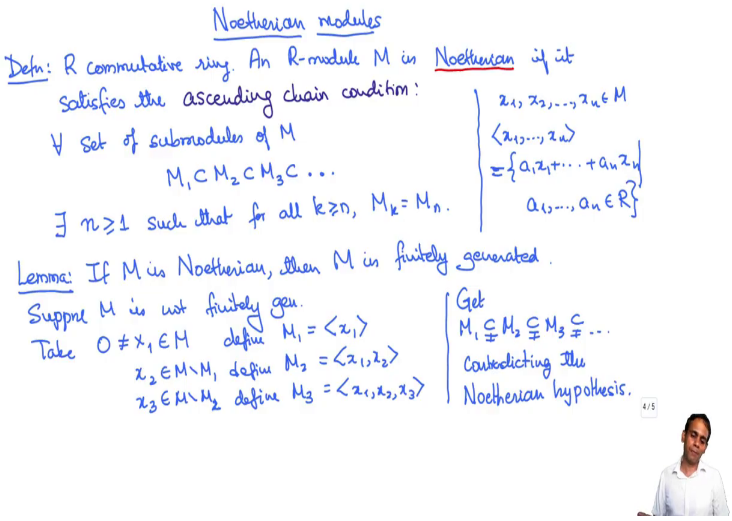

So, again R is a commutative ring and an R-module here M is Noetherian. It is said to be Noetherian, if it satisfies what is called the ascending chain condition; ascending chain condition. What is this condition? For every increasing sequence of modules, so for every set of sub module of M, maybe I should say sequence of sub module of M, see it starts with M 1 which is contained in a sub module M 2 which is contained in the sub module M 3 and this goes on indefinitely, there exist a number N greater than or equal to 1 such that for all k greater than or equal to n, M k is equal to M n.

So, what we are saying is that this ascending chain of modules eventually stabilizes. So, you cannot keep getting larger and larger and larger sub modules indefinitely and the relation to finite generation is quite straightforward to start with. We can show an easy lemma that if M is Noetherian, then M is finitely generated.

Well, the way to prove this is by contradiction. If M is not finitely generated, then it is quite straightforward to construct an ascending chain that never stabilizes ok. So, suppose M is not finitely generated ok. So, obviously that means that M is not the zero-module. So, there exist some non zero element x 1 in M ok.

So, yeah; so, let me introduce some notation over here. So, given set of elements x 1, x 2, x n in M, I will define this angular brackets x 1, x 2, x n to be the set of all elements of the form a 1 x 1 plus; oops, I do not need the summation here; a n x n; where, a 1 up to a n belongs to R.

This is called the sub module of M generated by x 1, x 2, x n is the smallest sub module of M that contains all the elements x 1, x 2 up to x n ok. So, now, since M is not the trivial module, I can take a non-zero element in M and let us define M 1 to be the sub module generated by x 1. So, it is just all elements of form a times x 1, where a runs over R.

Now, M 1 cannot be equal to M because M 1 is finitely generated and M is not finitely generated. So, then I can take x 2 belongs to M minus M 1 and now, let us define M 2 to be the module generated by x 1 and x 2. Now, M 2 is strictly larger than M 1 because x 2 is not in M 1. Now, M 2 cannot be equal to M.

So, I can define take x 3 in M minus M 1. M 2 cannot be a M because M is not finitely generated, but M 2 here is generated by two elements. So, now, you get the idea right? So, now, I take x 3 in M minus M 2 and I define M 3 to be the module sub module generated by x 1, x 2, x 3.

Proceeding in this manner, I get M 1 properly contained in M 2 properly contained in M 3. So, on an infinite chain, this process will never stop because at each stage, I cannot have that m n is equal to M because M subscript N will always be generated by N elements will be a finitely generated R module. So, we get this strictly increasing chain of R modules which never stabilizes. So, this contradicts the Noetherian hypothesis ok. So, that is the first relationship between Noetherianness and finite generation ok.

So, Noetherian modules are finitely generated; is that all there is to it? Is the Noetherian condition just the same as being finitely generated? The answer is no. The Noetherian condition is actually much stronger than the condition of being finitely generated and one nice thing about the Noetherian condition is that it goes easily from modules to sub modules ok.

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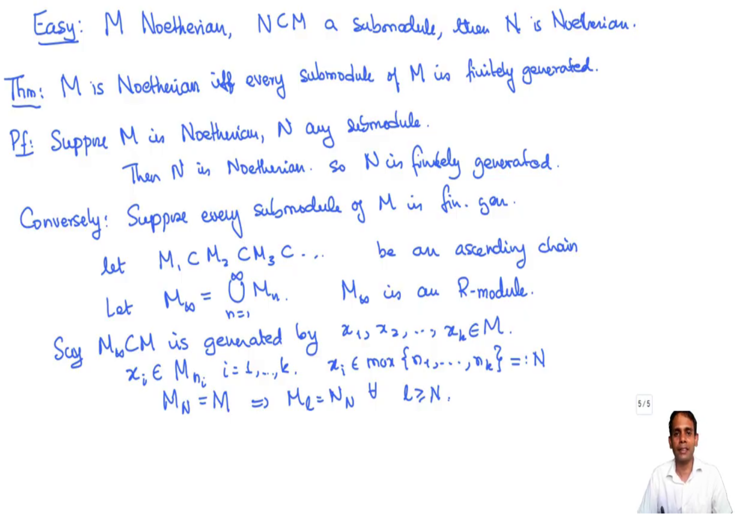

So, here is something that is easy to see that if M is Noetherian N a sub module, then N is Noetherian that is the very nature of this Noetherian condition. Because so, we want to show that N is Noetherian. So, if you take sub modules an ascending chain of sub modules of N, they will also be an ascending chain of sub modules of M. So, they will stabilize. So, a sub module of a Noetherian module is Noetherian.. However, it is not always true that a sub module of a finitely generated module is finitely generated.

So, this is where the Noetherian condition becomes very helpful. So, let us prove the stronger result about Noetherian and finite generation. So, theorem and this is actually a characterization of Noetherianness. An R-module M is Noetherian if and only if every sub module of M is finitely generated ok. So, not only is M finitely generated every sub module of M should also be finitely generated. Proof, if M is Noetherian and N is any sub module; so, let us prove that you know the sub modules are finitely generated.

So, suppose M is Noetherian and N is any sub module, then of course, N is Noetherian, but we have just seen that every Noetherian R-module is finitely generated. So, N is finitely generated ok. So, this side is easy; let us try the other side conversely. So, we have to show that if every sub module is finitely generated, then M is Noetherian.

So, let us start with an ascending chain. So, suppose every sub module of M is finitely generated and let us start with the chain M 1, M 2, M 3 be an ascending chain. Now, what you can do is you can take M infinity to be defined to be just the union n goes from 1 to infinity of M n.

And it is not difficult to show that M infinity is an R-module and all this is happening inside M right. It is the union inside M of M 1, M 2. These are sub modules in M ok. So, it is a sub module of M and so, M infinity is a sub module of M; M infinity is finitely generated.

So, let us say M infinity sub module of M is generated by some elements x 1, x 2, x k of M ok. So, each of these elements must be in M infinity. So, x i belongs to M infinity which means that it is in one of these M n. So, x i belongs to let us say M n i for i equals 1 to k and just means that x i belongs to the max of n 1, n 2, n k. Let us call this number N.

So, all the xi's belong to M capital N, but then these xi's generate M. So, that means that M capital N is equal to M. It cannot be any smaller because these elements x 1, x 2, x k, they generate M capital; they generate M and they are all in M n. So, M n must be equal to M because that is the smallest R-module containing x 1, x 2, x k which means that M k is equal to M n for all k greater than or equal to N; maybe I should not call this N; maybe I should call this l, for all l greater than or equal to N.

So, here we have characterized Noetherian modules in terms of finite generation. An R-module is Noetherian if and only if every sub module is finitely generated. We saw earlier that if M is in R-module and it has a sub module N, such that the sub module N is finitely generated and also, M mod N is finitely generated, then the module M itself must be finitely generated. Now, I claim that the same property holds for a Noetherian modules as well.

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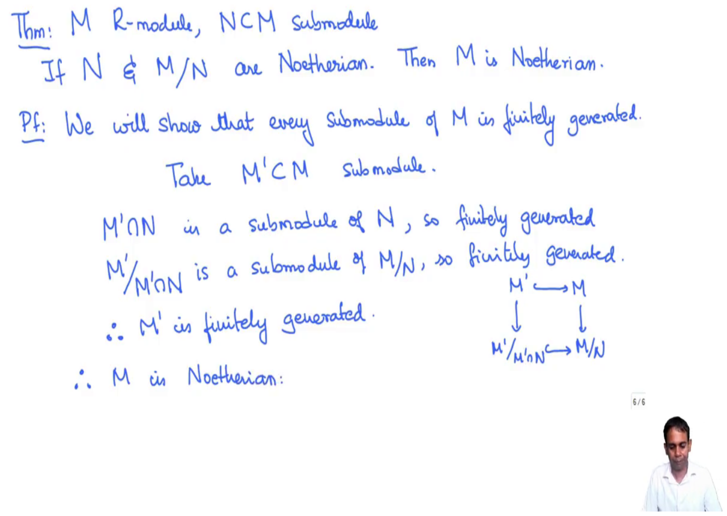

The theorem is that. So, you have this commutative ring R and suppose M is Noetherian; no, M is an R-module, any R-module N is a sub module of M and suppose we know that N and M mod N are Noetherian; then, I claim that M is Noetherian.

In fact, this theorem will follow easily from the earlier result that I stated about finite generation and the characterization of Noetherian modules, as those modules for which every sub module is finitely generated. So, proof is we will just show that every sub module of M is finitely generated.

So, yeah; suppose M prime is a sub module of M, then we will try to find a sub module of M prime which is finitely generated and such that M prime modulo that sub module is finitely generated. So, take M prime intersect N this is a sub module of M prime and this is finitely generated because it is a sub module also of M N; N is Noetherian.

So, M prime intersect N is finitely generated. And now, I want to take M prime modulo N, but N is not a sub module of M prime. So, I have to take M prime modulo M prime intersect N. Now, the inclusion map of M prime into M; so, you have M prime goes into M, it is just a sub module.

So, I just take the inclusion map and I have a M mod N the quotient map and I have the quotient map M prime mod M prime intersect N. And it is not hard to show that this is an injective map that is if something in M prime goes to 0 and M mod N, then it must line M prime intersect N. Well, that basically I have said it. So, the only things in M prime that go to 0 in M mod N are those which are in N. So, they are in M prime intersect N. So, this is a sub module of M mod N. So, it is also finitely generated.

So, this M prime has a sub module which is finitely generated and the quotient by that sub module is also finitely generated. Therefore, M prime is finitely generated. This means that M is Noetherian because every sub module is finitely generated. So, we have been proving all these technical results about Noetherian modules and finite generation.

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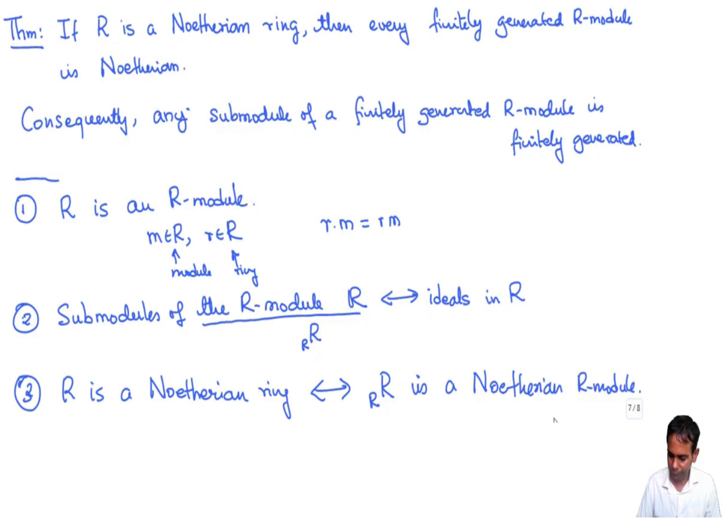

Now, it is time to harvest the fruits of our hard work. The first result that we can prove is that in some sense, the main result that we will be proving from all this is that if R is a Noetherian ring; then, every finitely generated R-module is Noetherian.

In particular, it will follow that if you have an sub module of a finitely generated R-module, then that will again be finitely generated. Any sub module of a finitely generated R-module is finitely generated. See for Noetherianness, it is very easy to see that if a module is Noetherian, then every sub module is Noetherian that is the Noetherian condition is like that.

But for finitely finite generation that is not at all obvious. Why would a sub module of a finitely generated module be finitely generated and that is not even true in general; but when R is a Noetherian ring this always holds and Noetherian ring abound right?

We have already seen lots of examples of Noetherian rings, all our Euclidean domains are Noetherian, all our principle ideal domains are Noetherian in fact, a rings of polynomials infinitely many variables are Noetherian. Noetherian rings are everywhere and this very general theorem gives us a very strong result. So, how do we prove this? So, the proof is now very simple.

So, first point is what is the relation between R being a Noetherian ring and R modules. So, the first point is that. So, suppose R is a ring, then R can be thought of as an R-module itself. How? Well, you just if give an M in R and r in R. So, this is the module version of R and this is the ring R, then you just define r dot m to be rm, the product in r itself and you can check that r is in R-module this is sometimes called the regular R-module ok. And then, what are the sub modules of R? Sub modules of the R-module R.

So, maybe I will use it is very awkward to keep saying the R-module R. So, I will just this thing I will write as R and on the left I will put a small R saying that this R is acting on R. So, R is in left R module. Sub modules of the R-module R, these correspond to ideals in R. Here, we are working with commutative ring. So, all ideals are two-sided ideals and so, R being Noetherian ring is the same as R as a left R-module is a Noetherian R-module ok.

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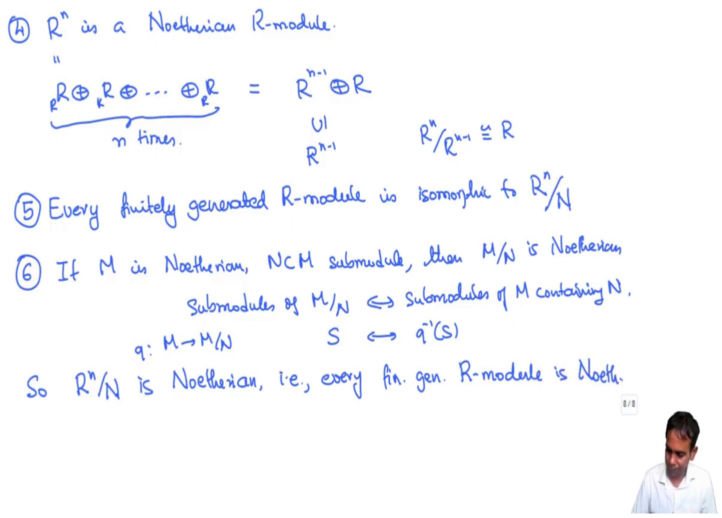

Now, I claim that R to the n is a Noetherian R-module. So, this is just R as an R-module direct sum R as an R-module direct sum R as an R-module. This direct sum is taken n times and why is this true? Well, you can prove it by induction on R. This is R to the N minus 1 direct sum R, this has a sub module R n minus 1 and the quotient is R.

So, this contains R n minus 1 as a sub module and R n mod R n minus 1 is isomorphic to R. So, by this theorem that we proved that if you have an R-module which has a sub module that is Noetherian and the quotient with respect to that sub module is Noetherian, then the R-module is Noetherian.

So, it follows that R to the n is a Noetherian R-module. Now, finally, every finitely generated R-module is of the form is isomorphic to R n mod N for some sub module N of R n, we already saw this when we talked about finitely generated modules.

You take if there are little n generators, then you just take a homomorphism from R n to your module which take the ith generator to the ith copy in R; to the ith take the ith coordinate vector in R n to the ith generator in your module and you will get an isomorphism with R n mod N.

Now, this R n is Noetherian. So, let us just we need a small result here, if M is Noetherian and N is a sub module of M; then, M mod N is Noetherian. This is easy because sub modules of M mod N are the same as sub modules of M containing N ok. So, if you have an increasing chain of sub modules of M mod N, then you just take their pre images in M that is an increasing chain of sub modules containing N.

Just if you have a sub module here S, then you just take it to a q inverse s; where, q is the quotient map from M to M mod N. So, Noetherian condition for M mod N is a special case of the Noetherian condition for M; just applied to sub modules containing N.

So, now we know that R is Noetherian, we know that R n is Noetherian and we know that R n mod N is Noetherian. So, we conclude that R n mod N is Noetherian every, but every finitely generated R-module is the form R n minus N. So, every finitely generated R-module is Noetherian. So, what we have is any sub module of a finitely generated R-module is finitely generated. So, that is the main theorem of this lecture.