Algebra - I
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Lecture 67 [Determinants]

Let us talk about determinants ok. Here of course, the catch here is we are not talking about determinants for matrices over a field, but rather over general rings ok. So, recall ah. So, let me make the following assumption that R, let R be a commutative ring now; let R be a commutative ring.

Recall that means, in our notation if I look at what we called R op, the identity map, it sends every element to itself. This is an isomorphism of $R \to R^{op}$right because the um operation in R^{op} is just the product in the reverse order; but since R is commutative $ab = ba$ ok. All the way of saying it is therefore, that the identity map from $R \to R^{op}$ is an isomorphism ok. So, of course, why why do I want to bring in R^{op} and so on?

So, recall from last time, we talked about how to think about endomorphisms of free modules. So, a homomorphism from $R^n \to R^n$ is really given by an $n \times n$ matrix and this matrix, we said should correctly be thought of as an element of R op. But since R is commutative, I can replace R^{op} by R right because of this fact that I just said. This is true for R commutative ok.

And recall how the map was defined? Given a homomorphism ϕ to that homomorphism, we could pick the standard basis of R^n; what we call the e_i's and you just apply ϕ to e_i and

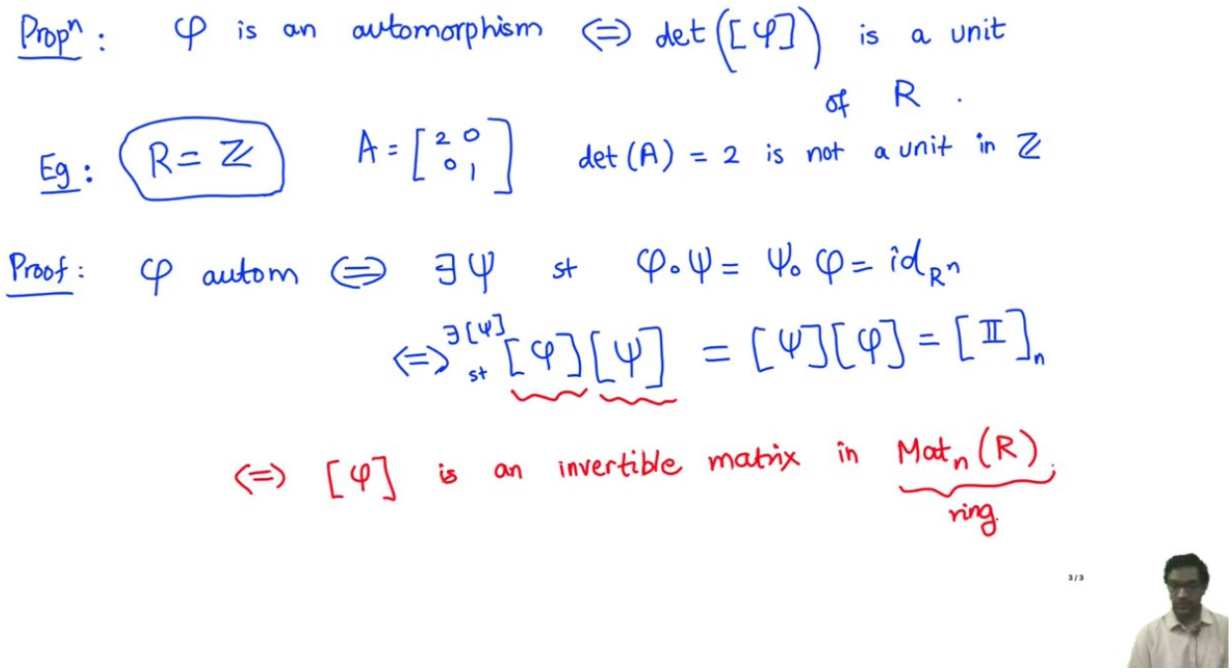
Let R be a commutative ring $\qquad R \xrightarrow{\sim} R^{op}$, $a \to a$

Recall: $\mathrm{End}(R^n) \xrightarrow{\sim} \mathrm{Mat}_n(R)$ for R commutative

$\varphi \longrightarrow [\varphi]$

Automorphism of R^n: $\varphi \in \mathrm{Aut}(R^n)$ if $\exists\, \psi \in \mathrm{End}(R^n)$

st $\varphi \circ \psi = \psi \circ \varphi = id_{R^n}$.



write out the the entries. I mean write the answer as a linear combination of the e_i's again and write the entries in the ith column right. So, this we call the matrix of the homomorphism ϕ ok. So, now what we want to do is to talk about automorphisms ok. So, what is an automorphism? Remember that just means it is an invertible endomorphism .

So, an automorphism of R^n just means it is a map ϕ which also admits an inverse ok. So, ϕ is said to be an automorphism, if there exist a map ψ which is again an endomorphism of R^n such that $\phi \circ \psi$ is the same as $\psi \circ \phi$ and this is equal to the identity map of R^n ok.

So, the usual notion of automorphisms. It is an isomorphism from the module R^n to itself ok. Now, here is the um important thing that if I have you know, so the question now really becomes every endomorphism is associated to a matrix; but what can I say about automorphisms, what do matrices of automorphisms look like ok? And so, here is the here is the main proposition which answers that question and we will say a little bit more about this. So, the proposition says that ϕ is an automorphism . So, same notation as before . So, map from $R^n \to R^n$ if and only if, the matrix of ϕ has the following property.

Look at the matrix of $\phi^n \times n$ matrix, if only if the determinant of that matrix; well, what is the determinant? Well, we will come to that in a minute. If only if, the determinant is a unit of the ring R ok. So, recall the unit means it is an invertible element of the ring R ok. So, let me before actually getting into the proof and explaining more about this, let us just do an example. Suppose, I take the ring R to be $\mathbb{Z}$ ok and suppose, I take a matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, very simple diagonal matrix. Well, the determinant means the usual thing, you sort of know how to take determinants of matrices um by using the formula, you expand along rows or columns and so on.

So, in this case if I have $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, the determinant is just 2 times 1 which is 2, but observe if I think of this as being a matrix with values over the underlying ring $\mathbb{Z}$, then 2 is not a unit; it is not invertible in $\mathbb{Z}$ ok. It is of course, invertible if you think of it as an element of $\mathbb{Q}$ the rational numbers or R the real numbers or complex numbers and so on, but it is not a unit in in $\mathbb{Z}$ ok.

So, the point is you know so let let us just try and so, this is what I mean by saying that the determinant must be a unit of R. So, let us first so you know as a preparatory step to proving this proposition. Let us first convert this statement about automorphisms into the corresponding statement about its matrix. So, observe ϕ is an automorphism means there exists an inverse ψ, this is an automorphism the same as saying there exists $\psi : R^n \to R^n$ such that as we said $\phi \circ \psi = \psi \circ \phi$ is the identity operator on R n; but now, we also looked at what does this mean in terms of the matrices.

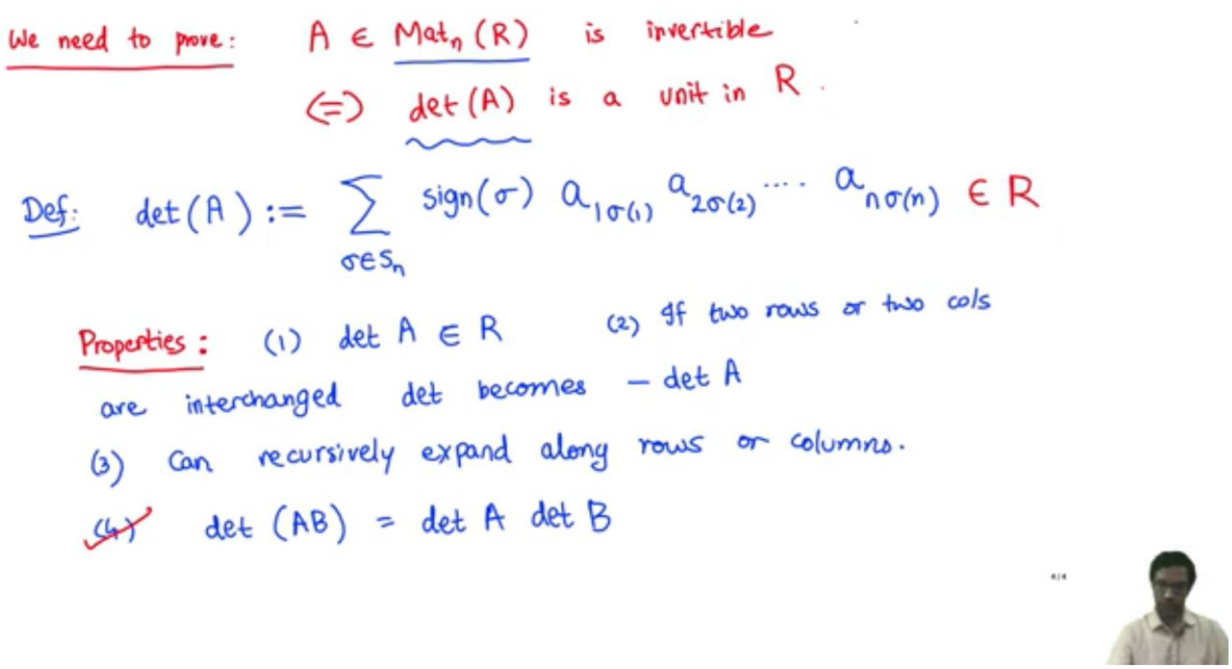
So, I said that the map which associates to each homomorphism, it is matrix that map is a ring isomorphism right. In other words, composition of maps goes to product of matrices. I mean if you ever over a non commutative ring you would have to take R op; but over a commutative ring, it is just the same as R itself. So, $\phi \circ \psi$. So, let me take the matrices on both sides this is the product of the two matrices of ϕ and ψ. On the other side, it is the product of the matrices; ψ and ϕ and the matrix of the identity operator is just the usual n times and identity matrix ok with 1s on the diagonal. So, remember 1 is now here an element of the ring R ok. So, I think of the the multiplicative unit of the ring R ok. So, this is this is the $n \times n$ identity.

So, now what does this mean? And, and conversely, if suppose I could find corresponding to the matrix of phi, if I could find another matrix such that you know the product gave me identity in both directions. Then, I could just take ψ to be the the homomorphism; the the endomorphism whose matrix is that given matrix ok.

In other words, it is it is sort of easy to see because everything is you know you can identify homomorphisms with I am sorry, you can identify endomorphisms with matrices and corrosion matrices. So, I can sort of reverse this, this chain of equations ok; if not if there exists a matrix of ψ such that ok. So, what is this really mean? It says that to understand when ϕ is an automorphism, it is enough to understand when the matrix of ϕ has an inverse ok. So, this this matrix here is the inverse of this matrix ϕ ok.

So, therefore, ϕ is an automorphism is the same as saying the matrix of ϕ is an invertible matrix ok. It is got a matrix inverse; but now, where is this happening. This is in the space of all $n \times n$ matrices with entries in the ring R ok. So, recall that this itself is ring; the ring of matrices with entries in in any ring R itself is a ring and we are saying that this matrix ϕ has to be an invertible matrix in that ring, to be an invertible element of that ring ok. So, what we need to therefore, do is really the following proposition.

So, what we need to prove is really the following fact that when is a matrix invertible? So, a matrix A in the ring of $n \times n$ matrices is invertible, if and only if it is determinant is a unit ok. So, this is really what our proposition amounts to. I have just converted everything from homomorphisms to matrices ok. So, first thing is what is the definition of the determinant. Well, it is it is the most obvious definition. So, let just convince ourselves that even if if A is a matrix with entries in some commutative ring R, I can still make sense of the determinant of A . So, how do we define the determinant? Well, we have the usual definition, where we expand along the rows and so on.

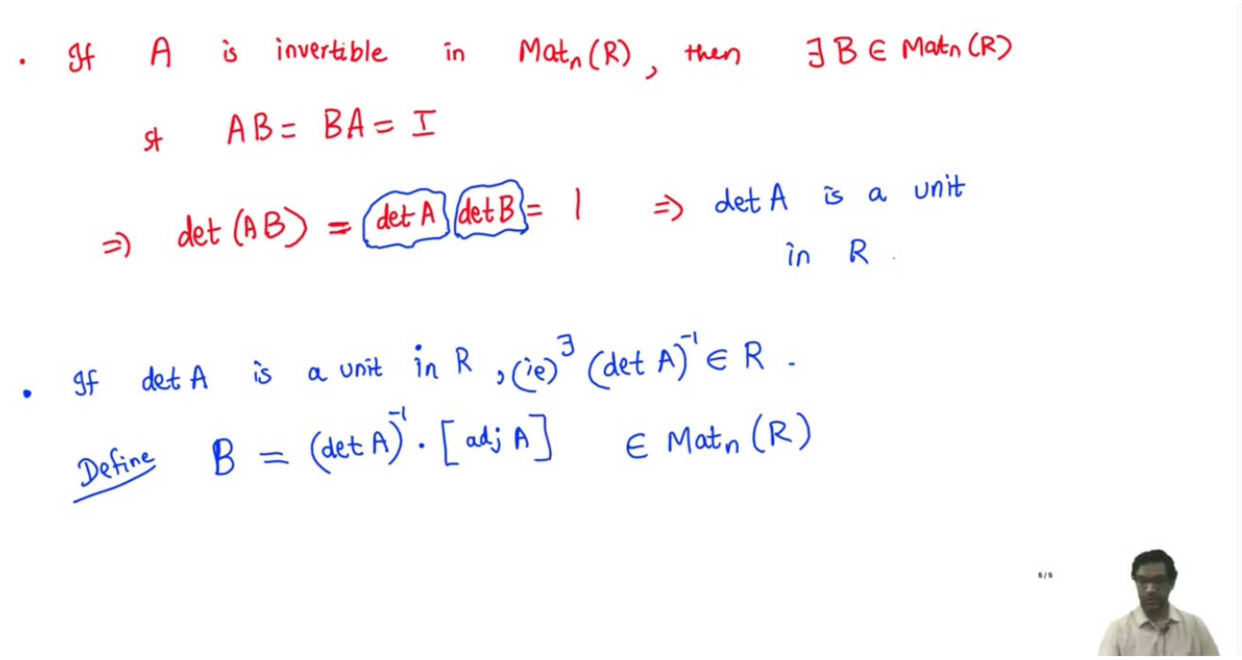


But in one shot, if one sort of wanted to write it here is how we we usually write this. We say take all permutations of S_n; permutations in s n permutations of 1 through n and do the following, let us look at the ok; I will call this epsilon of σ is a sign. So, I will just write this as a sign. So, remember, we have the sign of a permutation which is +1, if it is an even permutation and −1, if it is an odd permutation. So, I take the sign of of σ and then, $a_{1\sigma(1)}$. So, first row σ 1th column, take the element in the second row σ 2th column and so on. So, I take this product $a_{n\sigma(n)}$ ok and this is sum overall $\sigma(n)$ If this is the usual definition of the determinant in terms of the expansion along rows and so on.

$$Det(A) := \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}....a_{n\sigma(n)}$$

So, what what this amounts to is like saying if I have say 3 × 3 matrix. So, I I take this from the first row, maybe this from the second, this from the third or you know I could take this from the first row, I could take this from the second row, this from the third row and so on. So, I I sort of run through all the possibilities of choosing one entry from each row and one and each column. So, I have to choose n exact entries. I take their product and then, I multiply by the sign. There is always a sign involved in the determinant expression and this sum is exactly the determinant.

So, observe that this makes perfect sense over any commutative ring R ok. So, this firstly, the answer lies in R because it is a product of elements from R and the commutatively of of the ring is required so that I do not have to you know keep track of the order in which I mean you could conceivably define this for non-commutative rings; but it would not have the nice properties that you you expect of it ok. So, the commutative ring at least the nice thing is that you do not need to keep track of the the order in which you you multiply right. You take something from the first first row and then, you take something from the second row or or you do it in the other order ok.



So, this will ensure that the nice properties of the determinants remain. For example, if you interchange two rows the the value of the determinant becomes a negative ok. So, things like that we want to happen that will only happen if you assume that the ring R is commutative ok. So, we only define it in that case ok great. So, observe that the order is immaterial and the definition makes sense, it gives me an element of R and the other main reason for doing this of course, it has all the usual properties that we expect of a determinant ok. So, for example, so I have already said it belongs to R. So, just reiterate that the determinant of A is an element of the underlying ring R.

Second property if you interchange two rows or columns, if two rows or two columns are interchanged, the value becomes negative; minus of the original determinant ok that is the second property. Property three, if I um you know I can do this recursive expansion along rows or columns can recursively expand. So, this is just the usual way in which we compute it and that is just the another way of saying that it is the same definition. And the fourth property which is arguably the most important, which says that if I take the product of two matrices, the determinant of the product is just the product of the determinants ok. This is this is not so easy to prove from the definition ok, it requires a little work may be using the elementary row operations and so on and this is something we will we will come to later.

So, for now, just think of this as a black box. So, just accept this formula. It works um in some sense for the same reason that it works over a field. It is just a formal property of of multiplying two things out and so on ok. So, we we will give a proof of this during the you know maybe one of the problem sessions ok. So, let us let us go back to what we want to prove. So, we really want to say that the determinant is a good way of detecting when a matrix is invertible ok. If the determinant turns out to be a unit in the underlying ring, then the matrix is invertible and conversely. So, let us prove this prove this statement.

So, first observe that if a is invertible . So, that is the easier half of the implication. So, if A is invertible in the matrix ring, what does that mean? There exists a matrix B such that $AB = BA = I$ equals identity and now, we use the the property of determinants that determinant of $det(AB) = \det A \ \det B = 1$ and the determinant of the identity is just 1; it just the product of the diagonal values and so, this implies that. So, now recall everything is in the ring R; all these action is taking place in the inside the ring R. This is a ring element; this is a ring element right. So, what that means is that the ring element $\det A$ has an inverse in the ring, there exists an element in the ring such that their product is 1 ok.

So, this is exactly saying that determinant of A is a unit in the ring R ok. So, that that is this one-half of the proposition ok. Now, let us prove the converse that if the determinant is invertible in the ring, then the matrix A has an inverse matrix ok. So, let us do the converse now. So, if determinant of A is a unit ok. So, what does that mean? In other words, I can talk about an element called $\det A^{-1} \in R$ ok, there exists an element which is a multiplicative inverse of the determinant ok.

Now, given this I need to produce an inverse for an inverse matrix for A and this is I mean if you sort of just think about it for a minute, the clue for how to do this really comes from the formula for the inverse that one uses in the usual case of fields, which is how does one construct or um compute the inverse of a matrix A Well let us let me call that B, this is what we usually do. We say it is one by the determinant times what is called the adjoint of the matrix A right, the the transpose of the matrix of cofactors of A So, this is nothing but so our our usual formula. So, let me put it within brackets over a field sorry within code.

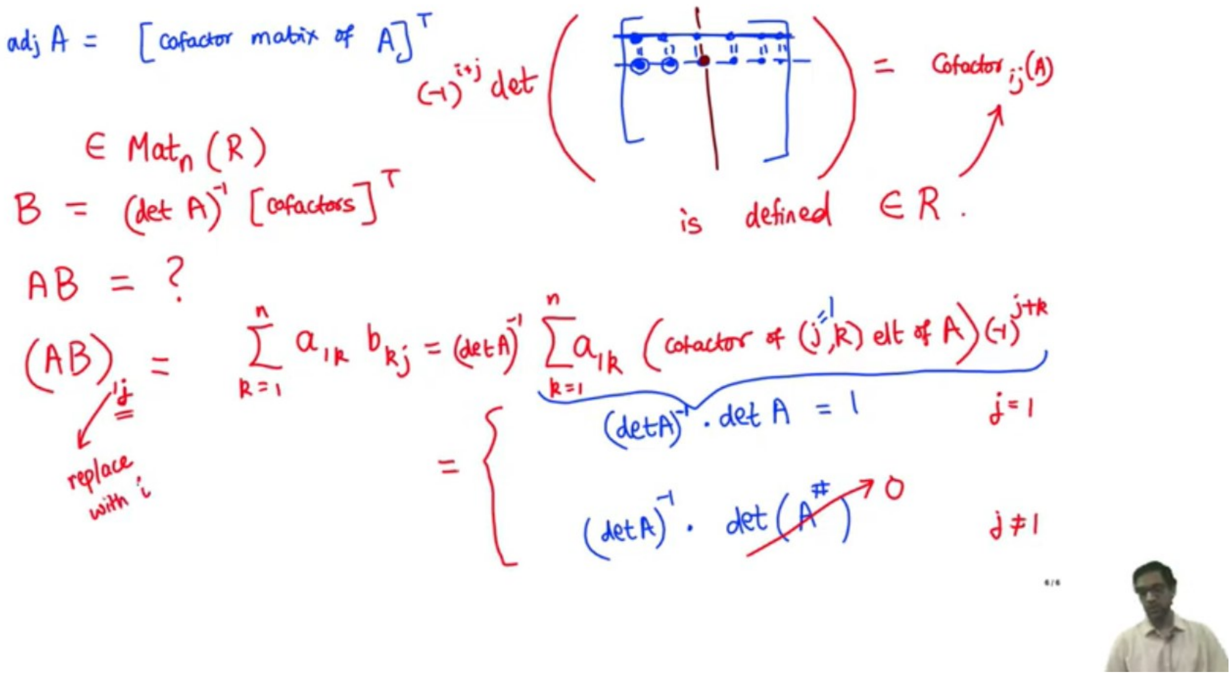
So, we would write it like this one by $\det A$ times the matrix called adjoint of A which is transpose of the cofactor matrix right. This is the usual formula. Now, what we have to I mean this over a field in general, but what we have to realize is that this formula actually makes perfect sense over any ring ok, provided I can make sense of 1 by $\det A$ ok. So, that is exactly what I have assumed that $\det A^{-1}$ is an element of R ok. So, let us try just you know as a guess, just try making the same definition. So, we will say B. Let us define B like this and hope that it has the right properties ok.

So, now let me say B should be an element of where should be lie, it should be an element it should be in $n \times n$ matrix with entries in R. Well, I will do the following. I will use $\det A$ it is a ring element. The inverse of $\det A$ is again it is assumed to be assumed to exist element of R. So, I will take this element of R and I multiply it by the adjoint matrix.

Now, what is the adjoint matrix? Does that make make sense over any ring R? Well, it is the matrix the transpose of the cofactor matrix right; the adjoint of a matrix is defined like this. This is you take the cofactor matrix of A and then, take the transpose. Now, what does that mean? So, question now reduces to does the cofactor matrix make sense right or does it make sense to define cofactors now, when you are over an arbitrary ring. Well, what is the cofactor of an element? Suppose, I have a matrix A and what is the cofactor of the ijth element?

Well, I just look at so ith row jjth column, the cofactor of this element is obtained by deleting the ith row and the jth column; looking at the remaining matrix that is $n-\ 1\ \times n-$ 1, taking it is determinant and then, multiplying it by a sign right. So, that is exactly the the ijth cofactor. So, recall that cofactor of the ijth element of A is nothing but determinant of this matrix in which you delete the ith row jth column and then, you multiplied by a sign right. That is $-\ 1$ to the $i + j$

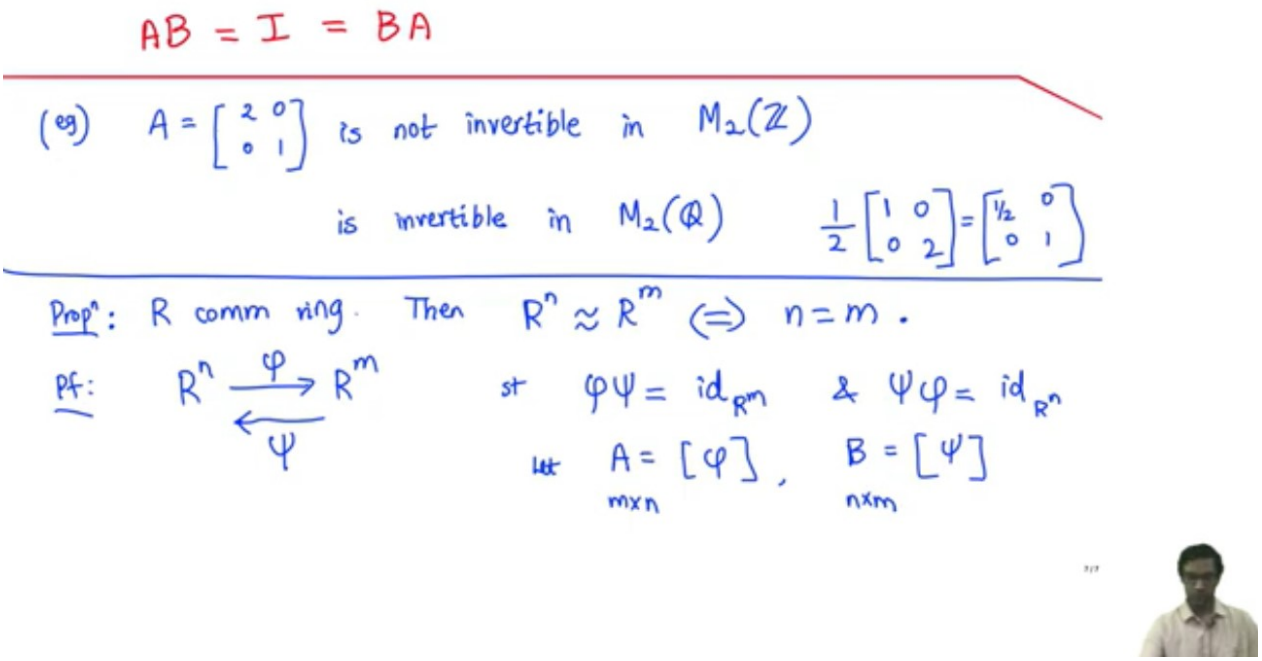
So, this this entire definition makes perfect sense even if A has elements from a ring R right because all I am doing finally, is computing the determinant of some $n-\ 1\ \times n-\ 1$ matrix and I can compute determinants of matrices or or any ring of any size. So, this makes sense.



So, this is well-defined right. This is in fact, defined it is an element of R again. Sorry, not yeah; yeah the cofactor ij is an element of R ok, the cofactor matrix means i for every ij I put the cofactor of ij ok. So, this entire definition makes perfect sense and. So, what do we get finally? The adjoint of A is in fact a well-defined matrix, defined in the same manner with elements in R ok and now, the question really is if I take the product of A and B, then do I get do I get the identity matrix right and similarly, the product of B and A ok. Now, let just do one computation; you know just check one of the rows and all the other rows are similar, what do I get when I multiply AB, you know if I look at some say the first row of the answer. So, 1 comma jth element here what does it look like? Well, it is by definition $(AB)_{1j} = \sum_{k=1}^{n} a_{1k} b_{kj}$ ok. This is a sum over k .

So, what is b? b remember is the transpose of the cofactor matrix multiplied by the determinant. So, what is bkj now? So, I need to say this is k goes from 1 to n a_1k So, what is the B matrix? B is nothing but $\det A^{-1}$ cofactor transpose . So, the k jth element here is the cofactor of the j kth element. So, this is this into cofactor of the j comma kth element of A ok. Because of the transpose and then, there is a sign that is $(-1)^{j+k}$ ok and then there is a of course, a $\det A^{-1}$. So, remember all these is multiplied by a $\det A^{-1}$ and since any way everything is a commutative ring, I can multiply things in any any order I want.

So, I will just put the $\det A^{-1}$ outside . So, this determinant of A^{-1}, this is a sum over k ok. Now, now comes the the important observation that a_{1k} cofactor of (jk) ok. So, if I put j equals 1, so remember j and k here; so, so sorry j is j is variable, I can I can choose j as 1. If I put j equals 1, so when I take $j = 1$ versus $j \neq 1$. So, two cases. If $j = 1$, this is just going to be the cofactor of the 1 comma kth element of A right. So, let me just for the moment think of it. If $j = 1$, then this is the cofactor of the 1 kth element. This is a_{1k} right and I am summing over k So, what does that mean? It is like saying that I take this this matrix. So, just draw this matrix here again . So, suppose this is my matrix A I am running through the first row the entries of the first row right; a_{1k} as k varies a_{11} a_{12} a_{13} and so on.



I am multiplying each element by its corresponding cofactor because this is exactly the cofactor of that element and the cofactor just means, I delete that row and column and look at what is left. But that is exactly how you compute the determinant of a matrix by expanding among the first row ok. So, when I form, when I take j equals 1, this sum here turns out to be exactly the the answer is exactly determinant of A because of how the determinant can be computed by expanding along a row and outside, remember I have $\det A^{-1}$ right. So, $\det A^{-1} \cdot \det A$ So, this answer is just a_1 ok. Now, what about the other guy? Suppose, j is not 1; what does that mean? It is like saying I take the the entries of the first row ok, these these blue dots here and I multiply them with the the cofactors of some other row ok, j is different from 1.

So, for example, let us say j is 2. So, I I multiply these blue dots with the cofactors of the brown dots ok. So, I I I do it along two different rows. Now, the point is that expansion is um well what what does it mean to um take the cofactor of of the brown dots. Every time, I compute the cofactor of a brown dot, I will have to delete that second row and the corresponding column right, that is how the cofactors of the elements in the second row are computed ok. So, the point is that this answer now for if j is 2 here, this answer will not depend on what the those actual values of the brown dots are ok. So, what I am concluding?

So, this is the this is the main argument. I am multiplying the values of the blue dots here with the cofactors of the brown dots here ok. Now, since every time, I compute the cofactor of a brown dot, I have to delete the row there. So, I will have to delete the second row each time and only the remaining elements will will be part of the calculation. So, this final answer here does not depend on what the entries of the second row R. It is the same answer for all all possible choices of second row ok.

Now, that is a that is a very useful thing to know because here is what I can do then, since the answer does not depend on the second row, I can choose a convenient I can make a convenient choice of second row . So, let me do the following. I will just make the entries of the second row equal to those of the first row ok. So, let us make this choice I I know modify my matrix, I take the new matrix whose second row entries coincide with the entries of the first row. By the argument, we just gave this this sum here does not depend on. So, it will it will give me the same answer, even after I have done this modification ok.

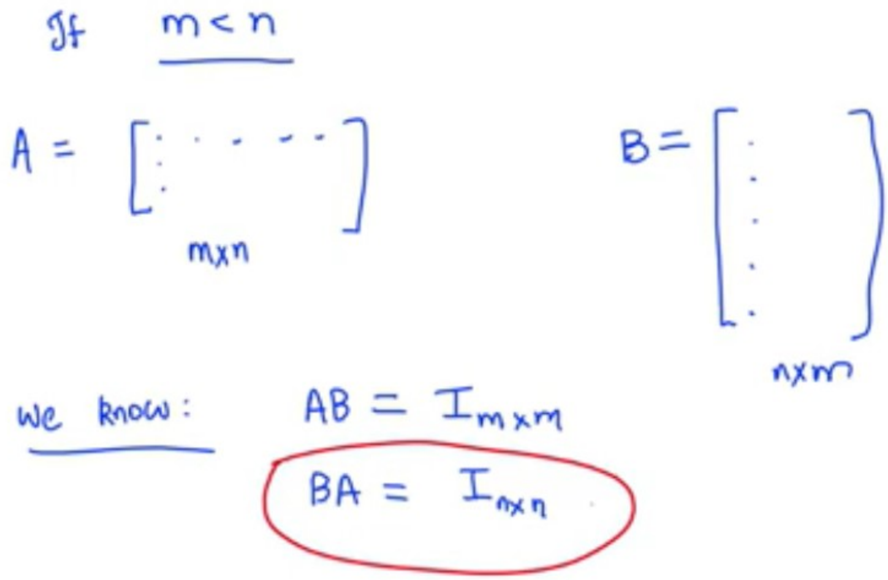
Now, let us see whether it makes sense. So, now, what am I doing? I am computing the values of the the blue dots above with the cofactors of of the corresponding blue dots below; but since I have chosen these numbers to be the same ok, each dot above is equal to the value below it. Now, this sum has a meaning. I can make sense out of this sum. This sum is just the determinant of this new modified matrix when expanded along the second row ok because I made both blue dot value as the same. So, I can think of it as I I run over the second row, take each blue dot multiply it by its cofactor; take a blue dot, multiply by its cofactor and so on and when I make you know do that summation that is exactly the determinant, when I expand along the second row. So, this is therefore, going to give me determinant of A^{-1} that is still outside times determinant of let me call this A hash this modified matrix ok.

Now, what was this modified matrix? It was obtained by making the entries of the first row and the second row the same ok,; but now we know the answer to this because when two rows are identical, then the determinant must be 0 right that follows from the fact that when I interchange two rows, the value becomes negative. So, in this case therefore, I have obtained that this is 0 ok. So, what is that give us we have finally proved that this product AB when you look at it is 1 comma jth element, it is 1, if j is 1 and 0 otherwise ok. So, you you can repeat the same same sort of argument replacing this one with any other row i you can replace with any row i and then a similar sort of argument works. So, what we have really shown is that only when j equals I, you will get 1 and if j is not equal to i you will get 0 . So, this argument actually shows that the product AB is identity and you can just repeat the same thing in the other order ok, that corresponds to expanding along columns rather than expanding along rows. So, you just have to repeat the same argument.

So, in in some sense what this does is really copies the same proofs that work in the case of fields by realizing that you know you really are not using the property of fields anywhere. you It will work perfectly well over any ring any commutative ring provided that that 1 by determinant A that that is the only catch provided that term makes sense, then you are ok ok. So, this is a very important sort of statement that a matrix over a ring R is invertible, if only if the determinant of the matrix is a unit in the ring ok. In particular, that example that we gave initially this matrix A which is 2 0 0 1, the determinant is not a unit in said. This matrix is not invertible in $M_2(\mathbb{Z})$ ok. So, here is our little corollary that if I take this.

um So, observe this example that I gave in the beginning which is the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is not invertible in the ring of matrices with entries in $\mathbb{Z}$ ok. It is I mean it is invertible, if you think of the ring R as being the rational numbers or $\mathbb{Q}$ and so on. And in some sense, you can see you know it is invertible over $\mathbb{Q}$ it is invertible as an element of $M_2(\mathbb{Q})$ And in fact, if you if you sort of use the usual way to compute the inverse, it is 1 by determinant of A into the transpose of the matrix of cofactors which in this case is this. So, here is the inverse ok the by the usual formula for computing and you can notice that well these entries are not in Z.

So, this is half $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ that is the inverse right. So, it it sort of ties up that you can invert it over Q; but the inverse will not have integer entries, it is going to have some



denominators ok. So, this is this sort of theorem is very important. And let us end this with a very important proposition which is a corollary of all these analysis with determinants and so on which says that if I have a free module R^n ok, so this is all again everything we have said is only when R is a commutative ring .

So, if R is a commutative ring, then the free module $R^n \cong R^m$, if and only if $n = m$ ok. So, I mean this is of course one of the statements we would like to have that the vector spaces have this nice notion of dimension and a vector space say over the complex number c_n and c_m are two different vector spaces, if n and m are different right; the dimension is n and we we would like to prove something similar. It turns out we can really only prove it in the case of commutative rings. It is not two in general and for commutative rings the proof sort of proceeds via determinants. So, let us prove; let us give a proof of this um . So, what does it mean to say $R^n \cong R^m$ are isomorphic? It means that there is isomorphism between them. In other words, there is a homomorphism ϕ which admits an inverse.

So, in other words, there are maps in both directions such that phi, ψ is the identity ah. So, ϕ, ψ is the identity on R^m and $\psi \circ \phi$ the the other order is the identity on R^n ok and again, as we have been doing throughout, it is best to convert everything to matrices. So, let us give the matrices names. Let A denote the matrix of phi; B denote the matrix of ψ ok. So, one of them is so let us see what the sizes are. This is an $m \times n$ matrix ok; $A_{m \times n}$ and $B_{n \times m}$ ok. So, we need to prove that m and n are equal. So, we will we will just make an assumption. So, suppose m is the smaller the two ok, we can repeat the same argument for the other case. Suppose, m is smaller than n ok. Now, if $m < n$ so what was A? A was the smaller. So, let us say let us go up there m is the smaller number. So that means, that A has fewer rows, more columns. B has um more rows and fewer columns.

So, what does A look like therefore? A looks like this a is $m \times n$. So, it is got some m rows, but lots more columns. It is a rectangle of this kind. B is sort of the opposite, it is got lots more columns and fewer rows; sorry, lots more rows and fewer columns ok. So, this is $m \times n$; this was $n \times m$ ok. So, now the claim is that I mean what we are saying really is that when you multiply these two matrices AB, you will get $m \times n$ matrix, $m \times m$ matrix that is also the identity and BA is also the identity right. So, that is really what these these two equations mean in terms of homomorphisms. The same thing here implies that. So, we know the following that when I multiply AB, it should give me a $m \times m$ matrix.

So, it is the identity $m \times m$ and when I multiply BA, it should give me the identity $n \times n$ ok. The claim is that this really cannot happen when you multiply two such rectangular matrices, both cannot give you the identity ok. Specifically, we claim that this can never happen ok; the larger one can never be the identity matrix. So, let us lets prove that. So, claim is this this cannot happen ok. So, claim BA cannot be the identity ok. Remember n is a larger number. So, the larger identity matrix that can never happen ok. So, let us try and prove this, um maybe we will just do it sort of by example with of with sizes. So, let me just take B to be 3×3 for example. So, B has I mean you can make this into a general argument 2.

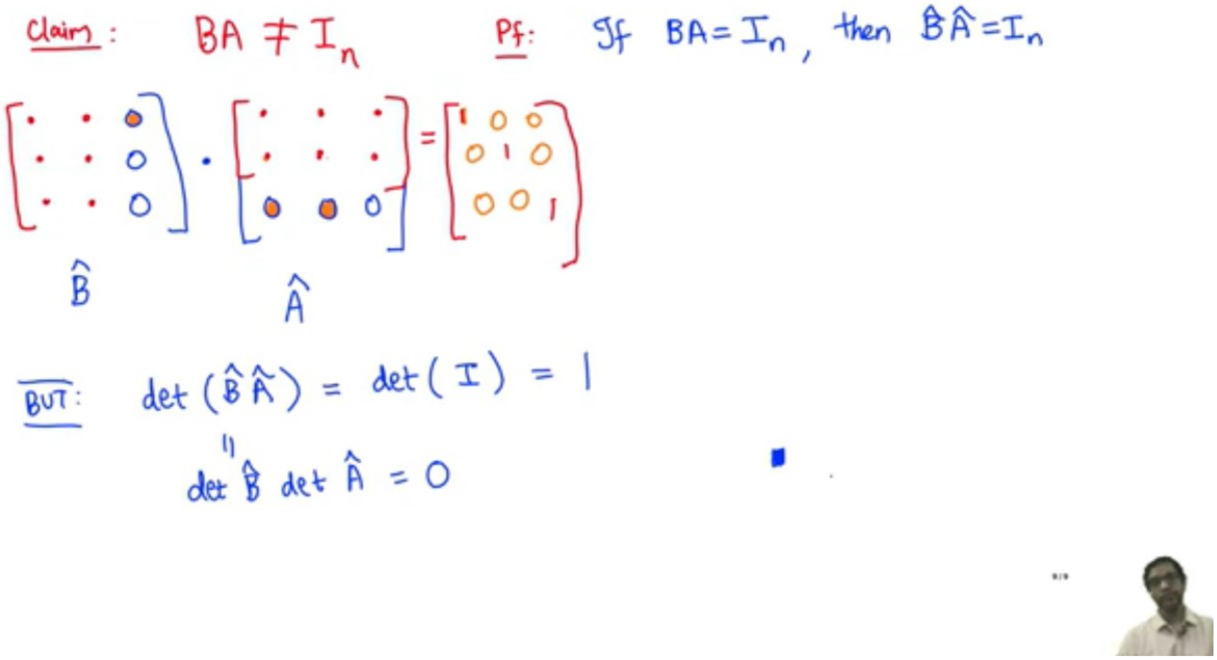
So, suppose B looks like this; B is say 3×2 ok and then, I multiply it by A which is 2×3; 3 columns and ok. Now, I I claim that this product cannot give me the identity right. So, let us see I want this suppose it gave me the identity, where is the contradiction. So, suppose I got this . So, they are all 1s ok. Suppose, I had this where is the contradiction. So, let us do the following. So, let us observe. So, let let me just move this over . So, let us say this is the answer.

Now, let us let us do something to this matrix; let us make B and A somewhat bigger ok. So, I am going to make instead of B, I will put one additional row ok. So, here is let us do the following. So, let us move these over. So, make some more space for B. So, I am going to augment B now as follows. So, I will put an additional. So, I am going to make both of them into square matrices. So, I have augmented B ok. So, this new augmented matrix I will call $\widehat{B}$ and $\widehat{A}$ I will augment by just putting zeros on the bottom ok, this is A hat ok. So, let us let us see I I claim that the same equation holds.

If BA is identity, so if BA is identity, then I claim that $\widehat{B}\ \widehat{A}$ is also identity; where, $\widehat{B}$ and $\widehat{A}$ are just the augmented matrices ok. So, I made both $n \times n$ Why is this? Well, just observe what the the definition of um you know how how would you have done matrix multiplication. Well, you take the entries in the first row of B and multiply them by the entries in the first column right. Now, that multiplication what will it do? What is the what is the difference when you went from $B \to \widehat{B}$ and $A \to \widehat{A}$ hat? Well, it has that you know that multiplication involves one extra term which is the 0 into the 0 right. These two blue zeros are extra; they were in there initially.

But they do not contribute anything to the sum right, it it is just a 0 anyway. So, it is only the original two red dots multiplying with the original two red dots that matters the what I have added extra is just a 0. So, the answer here does not change, it is still a_1 ok. Similarly, look at the second for example, the next entry the first row of B is multiplied by the entries of the second column of A hat and the only additional term I am introducing is this 0 times this 0 ok and that is again no contribution. So, it is only the original red into red + red into red that matters.

So, originally the answer was a 0 right because B into A is identity. Therefore, the the same answer holds. It is still identity; I mean still a 0 and so on. So, you you just try doing this to each of these um to to each of you know these 9 multiplications, you will have to do.



And you will notice that really you are you are not adding anything new. It is just the same answer as before ok with the 0 tagged along ok. So that means, that if BA is identity, then the augmented matrices B hat A hat also give product identity, but observe that both B hat and A hat have determinant 0 right. Now, I am in in the the square matrix case and I can compute determinants.

Now, observe $\det(\widehat{B}\widehat{A})$ must actually be the determinant of the identity matrix which is 1. But we know that this is $\det\widehat{B}$ $\det\widehat{A}$, well both of which are 0 because we explicitly put one 0 row or 0 column into both ok. ah So, that is that is the end of the proof, that is a contradiction right; you cannot have BA equals identity means that you cannot have m less than n Observe by the way, in this case that AB could be identity. If you try doing the same proof with AB, you won't be able to construct a contradiction ok.

It is only BA that um breaks down ok. So, that is the end of the proof because we have shown that m less than n gives rise to a contradiction. If n is less than m, then you you do the other order. Then, you show that AB cannot give you the identity ok. So, this is an important statement. So, what this finally, this proposition proves is that R^m and R^n can be isomorphic, if and only if n and m are the same ok and this involves the assumption that R is a commutative.