Algebra - I

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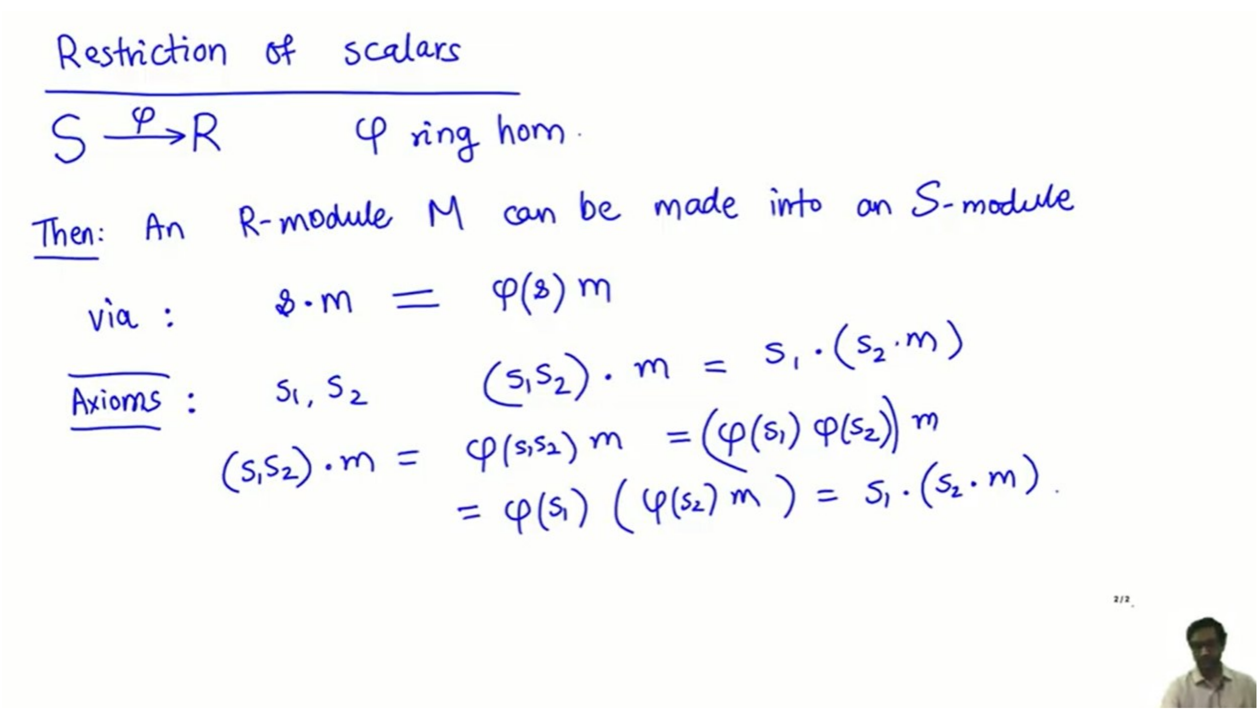
Lecture 64 [Change of ring]

We will talk about the notion of changing ring ok. So, this is called Change of ring and how to make modules over wondering into modules over another. So, here is the standard thing one usually wants to do which is called restriction of scalars . So, this is the usual sort of thing if you have a for example, a a vector space over the complex numbers is also a vector space or the real numbers right. You can restrict the scalars to the reals or in fact, to vector space over the rational numbers and so on.

So, that is one way of thinking about restriction, but the general notion is slightly slightly more general in fact. So, suppose I have a ring R ok. So, suppose I have R ring, not necessarily commutative and suppose I have another ring S. So, I am going to change from R to S here. And what do I have? I have a ring S together with a ring homomorphism from $S \to R$ ok. So, suppose I am given this set up; ϕ is a ring homomorphism from the ring R to the ring S; from the ring S to the ring R ok.

Then I claim that, if I have a module over R then an R module M can be made into an S module ok; via well what is the how to how do you do this? To do this you must define a



(Eg) $S \overset{i}{\hookrightarrow} R$ subring

(Eg) $K[X] \xrightarrow{\varphi} K$ K field

$f(x) \longrightarrow f(1)$ ring homom

Then a K-vector space M becomes a $K[X]$-module

via $f(x) \cdot m = f(1)\, m$

scalar multiplication right, M is already an additive group. Ah So, let us do the fine, I take an element s from S and I take an element m from my module and I need to define scalar multiplication $s \cdot m$. The definition is a following. Well, what can I do?

Since, I I am given a ring homomorphism, I apply the ring homomorphism to s thereby, obtaining an element of ring R and now M is an R module. So, I know how to scalar multiply m with the scalar ϕ of s which comes from R right. So, this is the definition of the scalar multiplication by elements of s and the the thing is it is easy to check the properties ok. So, one just has to check that all the axioms of module are satisfied. Let me just check one of them for you and you know you can check the other one.

So, for example, if I multiply two elements, if I take two scalars S_1 and S_2 and I look at their product S_1, S_2. So, $(S_1 S_2) \cdot m$ must give me the same answer as $S_1 \cdot (S_2 \cdot m)$ right. So, this is one of the axioms. So, let us check that this is true. So, we start with the left hand side $(S_1 S_2) \cdot m$, by definition you apply $\phi(S_1 S_2)m$ using the R module structure on m. But now ϕ is a homomorphism, this is the first place where we need to use it; this is $(\phi(S_1)\phi(S_2))m$. So, product of these two guys and m in R and its this product acting on m right. But, now the axiom the the corresponding axiom for R modules as the product of two scalars of R acting on m is just what you get by acting one after the other.

So, its $\phi(S_1)(\phi(S_2)m)$, but then that is exactly the right hand side; this by definition is $S_1 \cdot (S_2 \cdot m)$ ok and so on. So, all the other axioms involves similar verifications, we will need to use all the properties of phi; $\phi(S_1 + S_2) = \phi(S_1) + \phi(S_2)$ as well as ϕ of 1 equals 1; identity equals to identity ok. So, all of these will will be involve. So, it is easy to to check all the axioms.

So, I leave that for you to check. So, this this process by which you can make an R module into an S module is called restriction of scalars; even though it is slightly more general than the notion of just truly restricting scalars. So, the standard example of course, is the it is the canonical restriction set up. So, suppose S is a sub ring of R in which case; I have the inclusion homomorphism. So, this map ϕ is just the inclusion map, then an R module automatically becomes an S module by truly honestly restricting scalars right; by only acting by elements of S.

But this this general definition is is nice, it is useful sometime. So, let me give you an instance of the general definition as well. So, suppose S is the ring $K[X]$, the ring of polynomials and R is just the ring K the field. So, let us say K is the field in this case. So, Ext^n of scalars

$$S \xrightarrow{\varphi} R$$

Special case: $S \overset{\varphi}{\twoheadrightarrow} S/I$ $\quad I$ two-sided ideal of S

let M be an S-module.

Def: $\operatorname{ann}(M) := \{ s \in S \mid sm = 0 \ \forall m \in M \}$

Ex: $\operatorname{ann}(M)$ is a two-sided ideal of S.

I take R to be K and S to be $K[X]$ and I define a homomorphism from $K[X] \to K$, Well, what what sort of homomorphism I can take the let us call the evaluation map ok; given any polynomial I evaluated at some element of that of the field K, For example, I evaluated at 1 or 0 or other points, let us say I evaluated at 1 in this example. So, here is a map and it is very easy to check that this evaluation map is a ring homomorphism . Now, what is this mean? This means that, if I have a K module which means if I have a vector space.

So, then a a K vector space which is a module over K, a K vector space M or becomes a module over $K[X]$ or via of the action, if you take a polynomial $f(X) \cdot m$; by definition I must apply ϕ to fX and that application of ϕ is like evaluating $f(X)$ at 1. So, I evaluate I put X equals 1, I get an element of K and I scalar multiply that element of K on m fine. So, this was the definition.

So, this gives you for example, a I mean this gives you a structure of a $K[X]$ module on m and of course, we have seen the more general ways of making K vector spaces into $K[X]$ modules. All you have to do is to specify one linear operator ok and in this case this is just you know this is; this is just the choice of the linear operator identity here ok; so, anyway. So, this just to illustrate that, this general definition can give you something more interesting than just usual restriction.

Now, the the opposite problem is usually called extension of scalars which is you you have a , well I have a let say map now from S to R as before a homomorphism. But, I have an S module, M is an S module now and I want to see if I can somehow make it into an R module ok. Now this is a slightly harder problem, I mean much harder problem; you cannot do it in a very easy manner as before because the same argument as before does not work.

You if you take an element of R ah, there is no easy way of defining scalar multiplication; because I cannot convert it into an element of S by applying a homomorphism. The homomorphism goes in the wrong direction ok. So, there is this I mean, but there is a way of doing this it small tensor products and so on. So, we will not get into this right now, but I just want to talk about one special case ok and that special case is is rather common and important.

Prop^n: If I is a 2-sided ideal of S s.t. $I \subseteq \operatorname{ann} M$, then M becomes a S/I-module via:

$$(s+I)\cdot m := sm \quad \forall\, s \in S.$$

Pf: Well-definedness: $s_1 + I = s_2 + I$ for some $s_1, s_2 \in S$.

$$\Rightarrow s_1 - s_2 \in I \subseteq \operatorname{ann}(M)$$

$$\Rightarrow (s_1 - s_2)m = 0 \quad \forall m \in M$$

$$\Rightarrow s_1 m = s_2 m \quad \forall m \in M. \;\blacksquare$$

AXIOMS: Check!

So, here is one special case of extension of scalars which is suppose, I have S is a ring and I take R to be the ring $\frac{S}{I}$, a quotient of S ok; where I is some two-sided ideal. I need to pick a two-sided ideal of S. So, the question is if I if I give you an S module, can I make it into an $S \to \frac{S}{I}$ mod I module ok; can I make it into a module for the quotient? So, let M be an S module . So, the question really is when can M be made into an S by I module? And, to answer this we define the annihilator of M. So, here is the definition . So, recall you know in other context one has talked about annihilators of elements. So, this is an annihilator of a module. So, what is an annihilator of a module?

Well, this is all scalars, this all elements of S which kill every element of the modules, such that sm is 0 for all m in M ok. It acts as a 0 on the entire module, kills everything. Now, here is a little exercise, this is a ideal annihilator of M is a two-sided ideal of the ring S ok. It is a easy verification, just need to check that whether you multiply things on the left around the right; the the you know the the element continuous to be in the annihilator and also that is closed under addition.

So, it just follows from the definition. So, the annihilator of a module is a certain two-sided ideal of my ring S . And so, here is the; here is the answer to the question; when can you make M into an S, S by I module? So, if I is a 2-sided ideal ah, it is a 2-sided ideal of S such that, I is containing inside this annihilator, then M becomes S mod I module via the following definition ah. How do you define? So, take an element of $\frac{S}{I}$, what is a typical element look like? It is its coset right, let us call it S bar, the coset of s. So, maybe I just write it as a coset $s + I$. So, look at the coset of $s + I$. How can I make this act on m ? Well, there is only one obvious way to define it. I will just say this is equal to sm ok, for all $s \in S$. So, suppose I define it like this ok. So, the first thing that one has to check in all these cases is that this is well defined right.

What if I choose a different representative of my coset, will I get the same answer? Ok. So, well definedness is the first property and that is more or less the only thing to be check the others are obvious. So, suppose the same coset has two different representative; suppose $S_1 + I = S_2 + I$ for some $S_1, S_2 \in S$ ok. What does this mean? This act as the difference $S_1 - S_2$ is in I ok. This is another way of saying that $S_1 - S_2$ is in I ok. Now, let us check that I will get the same answer whether I apply S_1 or S_2 or whether I choose S_1 or S_2 as my representative ok.

So, observe $S_1 - S_2$ is in I, but I is contained inside the annihilator of M ok. What does it mean? It means that $S_1 - S_2$ annihilates every element of the module m is 0 for all mM. What does that mean? It says $S_1 m$ is the same answer as $S_2 m$ for all $m \in M$ ok and that is exactly what we wanted to prove right. So, that completes the proof of well definedness, because if you had chosen S_1 as a representative you would have got an S_1 m; if you have chosen S_2 you have got an $S_2 m$ but, those two are the same answer always ok. So, this is well defined and then the other axioms are are more or less obvious. So, I I will leave the other axioms for you to check of a module, if I take a sum of two cosets then the right hand side becomes the sum you know or if I take a product, it it gives you successive action and so on.

Everything follows because, you know its finally, only depends on the representative ok. So, check that the other axioms also were ok. And so, this is a this is a very important and useful proposition. An example of its use is when you have $K[X]$ modules. So, suppose V is a K vector space which I make into a $K[X]$ module ah. So, I recall how do you make this into a $K[X]$ module? You pick a linear operator, you fix some linear operator on V and you make V into a $K[X]$ module by saying that ah; how how do I make a polynomial fX at on vector v?

I just substitute the operator T in place of X, it gives me a new linear operator and it is that operator acting on v ok. So, this was the definition for all V . So, I made V into a $K[X]$ module. And now the question is well what is you know the the whole business of annihilators and so on. So, let us just compute the annihilator of this module. So, here is an interesting. So, what is the annihilator of this module V? Thought of as a $K[X]$ module; so, I am I am doing everything for $K[X]$. So, what does the annihilator of V or $K[X]$ mean? It means it is all those elements; so, all those polynomials in $K[X]$ which annihilate the whole which annihilate the whole module. In other words, which means when I plug in the operator T, it just gives me 0 on every vector. So, this is 0 for all vectors in V.

In other words, the operator fT is actually the 0 operator right. So, this this property here can be rephrased to say that, f of T is just the 0 operator ok. Now observe that; so, we already said the annihilator is an ideal ok and $K[X]$ is principle ideal domain, which means that this must be a principle ideal right. So, this must be the ideal generated by a single polynomial, it is called that polynomial $m_T(X)$; because it depends on my choice of T. And, well what what are the properties defining properties of this polynomial?

Firstly, this polynomial annihilates X ; sorry annihilates T, I plug in T I just get the 0 operator. And, because it is the generator ideal, it means that among all the polynomials which annihilate T; this is the one of the smallest degree for example, ok. And, if you sort of recall your linear algebra, this is exactly what is termed the minimal polynomial of the linear operator T ok. It is exactly this, it is the smallest well it is the smallest degree in monic polynomial. You can also normalize it to have leading coefficient 1. So, the smallest degree monic polynomial which annihilates the the operator T, that is exactly the minimal polynomial. So, that is the the annihilator of V taught of as a $K[X]$ module. So in fact, (Eg) V K-vector space. Fix $T : V \to V$ op^r

V $K[X]$-module by $f(x) \cdot v = f(T)\, v \quad \forall v \in V$

$$\operatorname{ann}_{K[X]} V = \{ f(x) \in K[X] \mid \cancel{f(T)\, v = 0} \ \forall v \in V \}$$
$$f(T) = 0$$

$$= (m_T(x))$$ "minimal polynomial of T"

V is a $\dfrac{K[X]}{(m_T(x))}$

what this implies is that, if I make V into a $K[X]$ module using my operator T then in fact, it is its more than just the $K[X]$ module.

V is actually a module over $K[X]$ modulo, any ideal which is contained in the annihilator ok. In particular, it is its a module over $K[X]$ modulo the ideal $m_T(X)$ ok and $m_T(X)$ is the minimal polynomial of T. So so, in some sense things like diagnosability in many other properties of you know linear algebraic properties sort of come from the the structure of this ring. So, in you know we will probably do this in one of the example. So, you probably seen things like, if the minimal polynomial factors into distinct linear factors then the the operator is diagonalizable and so on ok. So, we will consider things like that in the problem sessions .