Algebra - I
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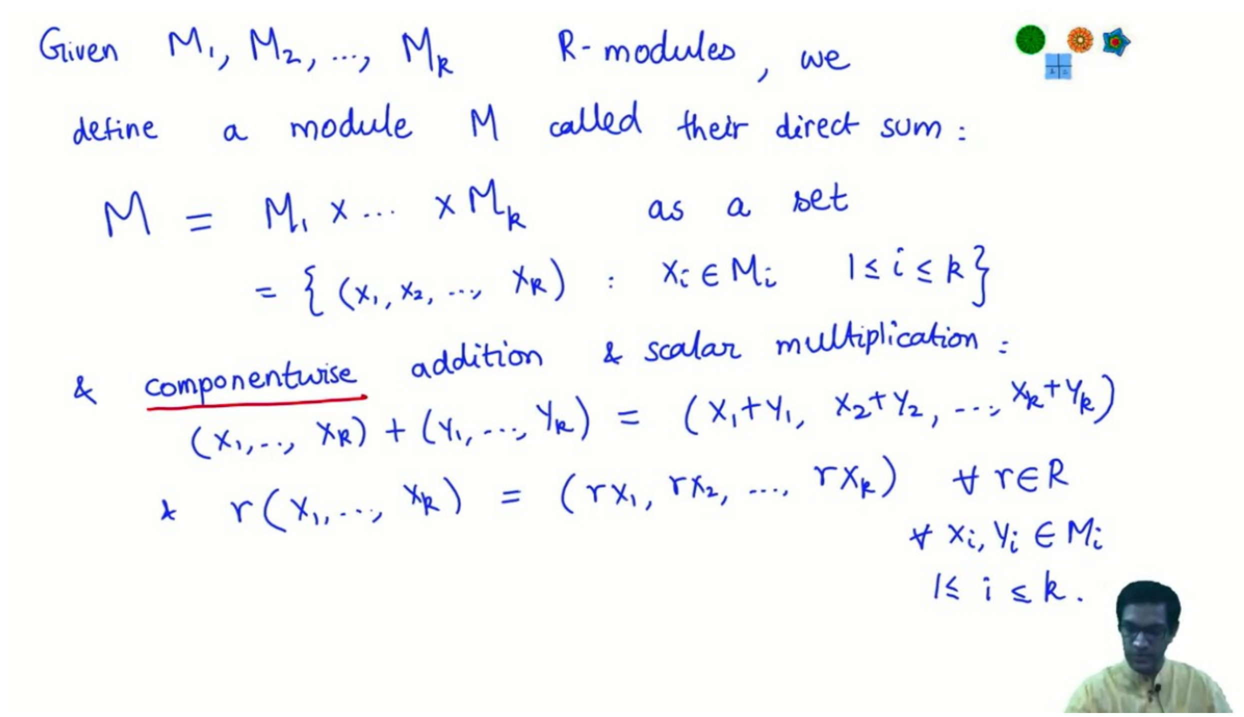
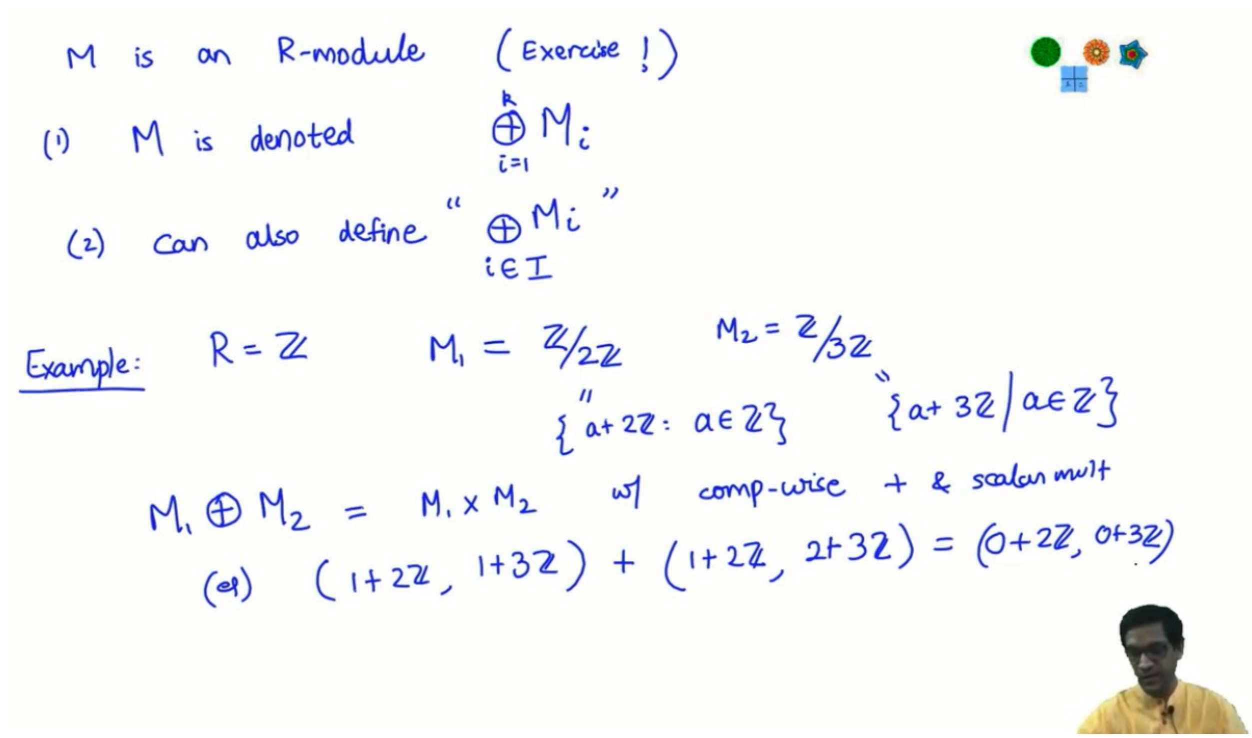
Lecture 62 [Direct sum of modules]

Let us start talking about direct sums of modules. Given a collection, so let us say given for now a finite collection M_1, M_2,..., M_k. Let us assume they are all R-modules for some ring r not necessarily commutative, we define a module called their direct sum. So, we define a module M called their direct sum, M called their direct sum as follows as follows. We first need to specify what M is as a set. So, purely as a set, it is just the cross product of these as a set.

In other words, the elements of M look like tuples, order tuples X_1, $X_2, ..., X_k$; where, each X_i is an element of the corresponding M_i, i runs from 1 to k and we need to also specify how this becomes an R-module. This becomes a module via component wise. So, this is the set and component wise addition and scalar multiplication. So, what this means is when you want to add $(X_1, X_2, ..., X_k) + (Y_1, Y_2, ..., Y_k)$, just add each component dot dot dot $(X_1 + Y_1, ..., X_k + Y_k)$ on the one hand and if you want to multiply by the scalar r, $r(X_1, X_2, ..., X_k) = (rX_1, rX_2, ..., rX_k)$ and all these hold for all r running over elements of the ring and for all X_i, Y_i elements of M_i, i from 1 to k ok.





So, this is what we mean by component wise addition and multiplication. So, this is a this is a keyword here ok. Now, it is very easy to check that this makes M into a module. So, I leave that as an exercise. At the end of the day, every axiom that you need to check will hold because it holds in each component ok.

So, first observation here, easy observation is that M is an R-module under these definitions ok. So, exercise ok. Now, this guy is called the direct sum. So, here are some some remarks. So, M is called the direct sum of the M_i's and is usually, denoted by the symbol direct sum. So, this is the usual notation for the direct sum direct sum $\oplus_{i \in I} M_i$, i ranging from 1 to k ok. M is usually denoted like this ok. Another short remark here, one can actually define direct sums even for infinite collections of modules, not necessarily for finite collections. So, I will just state that quickly can also define something called the direct sum of an infinite collection.

So, now i ranges over a potentially infinite indexing set I, one can also define this notion; but the key thing to remember is that this turns out not to equal the product as a set ok. So, if you have a infinite collection of modules, the direct sum is well it is actually a certain subset of the product ok.

More precisely, it is the it is that subset of the product in which only finitely many of the entries are allowed to be non-zero ok. So, anyway that is something that we won't need to we won't encounter in this course. So, for now, we will not worry about that ok ok. So, let us just do an example of direct sum of modules. So, easiest example is to take a billion groups.

If I take my ring to be the integers, I can take any any abelian group is automatically $\mathbb{Z}$ module. So, I can take the cyclic group of order 2 and the cyclic group of order 3 ok. So, this is all cosets. So, what does this comprise? This is all cosets of the form $a + 2\mathbb{Z}$ right; a running out the integer. So, really there are two cosets; the odd and the even numbers.

This comprises all the cosets congruent to 0, 1 and 2 modulo 3 and of course, these are both R-modules. So, what is the the direct sum now mean? It just means I take $M_1 \oplus M_2$ which is sort of the same notation as this, is well as a set it is $M_1 \times M_2$ with component wise operations right. So, with component wise comp wise + and scalar multiplication. So, for example, if I took an element from $M_1 \oplus M_2$. So, let us take the element, what we would call 1 bar which is $(1 + 2\mathbb{Z}, 1 + 3\mathbb{Z})$. So, here is an element of of $M_1 \oplus M_2$. So, how do I perform addition?

So, let us say I want to add to this 2 + 3 $\mathbb{Z}$, then all I have to do is just add each component separately and the answer is well it is 2 + 2 $\mathbb{Z}$ which is the same as the 0 coset and 3 + 3 $\mathbb{Z}$ which is again the 0 coset. So, here is the addition for example ok. So, scalar multiplication similarly. So, this is a $M_1 \oplus M_2$. So, well it is some it is a set with six elements.

And in fact, you actually have encountered this before, when studying groups and the Chinese remainder theorem. So, for example, here is a good way to understand what this direct sum is. It is actually this isomorphic to the the cyclic group of order six ok as $\mathbb{Z}$ modules. Therefore, $\frac{\mathbb{Z}}{6\mathbb{Z}}$, I claim is isomorphic.

So, there is an isomorphism to this direct sum that we just constructed and what is isomorphism? So, like I said you have already encountered this, so recall what is what was called the Chinese remainder theorem, in the context of a abelian groups and the map is the following given any coset $a + 6\mathbb{Z}$, we just map it to the corresponding $(a + 2\mathbb{Z}, a + 3\mathbb{Z})$.

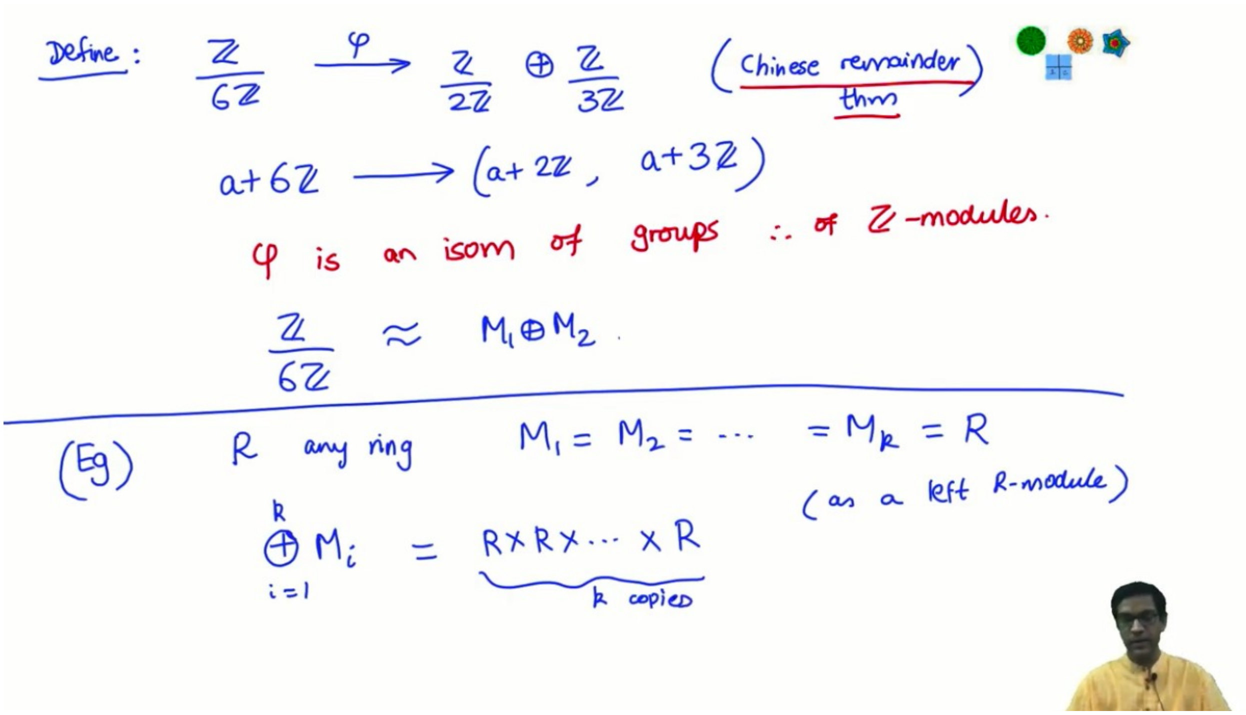
So, given a number a modulo 6; you just map it to the congruence class of a modulo 2 and a modulo 3 and recall that this map was an isomorphism ok. So, I am I am sort of referring you back to these to this lecture, the Chinese remainder theorem. So, recall ϕ was an isomorphism. So, is an isomorphism of groups. Well, they both abelian groups in this case. So, they are in fact, an isomorphism of $\mathbb{Z}$ modules in our case. Therefore, of an isomorphism of $\mathbb{Z}$ modules . So, in this case, the direct sum is nothing but the hm the the well, we can identify it with the $\mathbb{Z}$ module $\frac{\mathbb{Z}}{6\mathbb{Z}}$ ok.

So, hm. So, conclusion $\mathbb{Z}$ mod $6\mathbb{Z}$ is actually isomorphic to the direct sum ok. Now, let us do another example. So, this is an example with $\mathbb{Z}$ modules. Here is a sort of a general example. So, if R is any ring, so recall you can think of R as a modulo over itself by the left multiplication.

Now, let us just do the following. Let us take $M_1 = M_2 = ... = M_k = R$ to all be just R ok as a left modulo over itself . So, I just think of all the all all the M_i s as just being R itself. So, what I am trying to do now to form the direct sum is it is like I am taking k copies of R. So, what is the direct sum now become?

So, the direct sum of the $\sum_{i=1}^{k} M_i$. Well by definition, this is just as a set its just $M_1 \times M_2 \times ... \times M_k$. So, in other words, it is k copies of the ring R itself. So, this is k copies ok. So, as I said, I am just forming the cross product of R with itself k times. So, this set is what we usually denote as R to the k right. So, this is the the the direct sum of R with itself k times is as I said after the k with component wise addition and scalar multiplication ok. Ah So, this is an again an example of a of a direct sum. Now, let me look at direct sums in relation to homomorphisms.

So, there is there are some obvious homomorphisms that are important. So, suppose, I again have the same setting, I have M_1 through M_k which are R-modules, then observe that



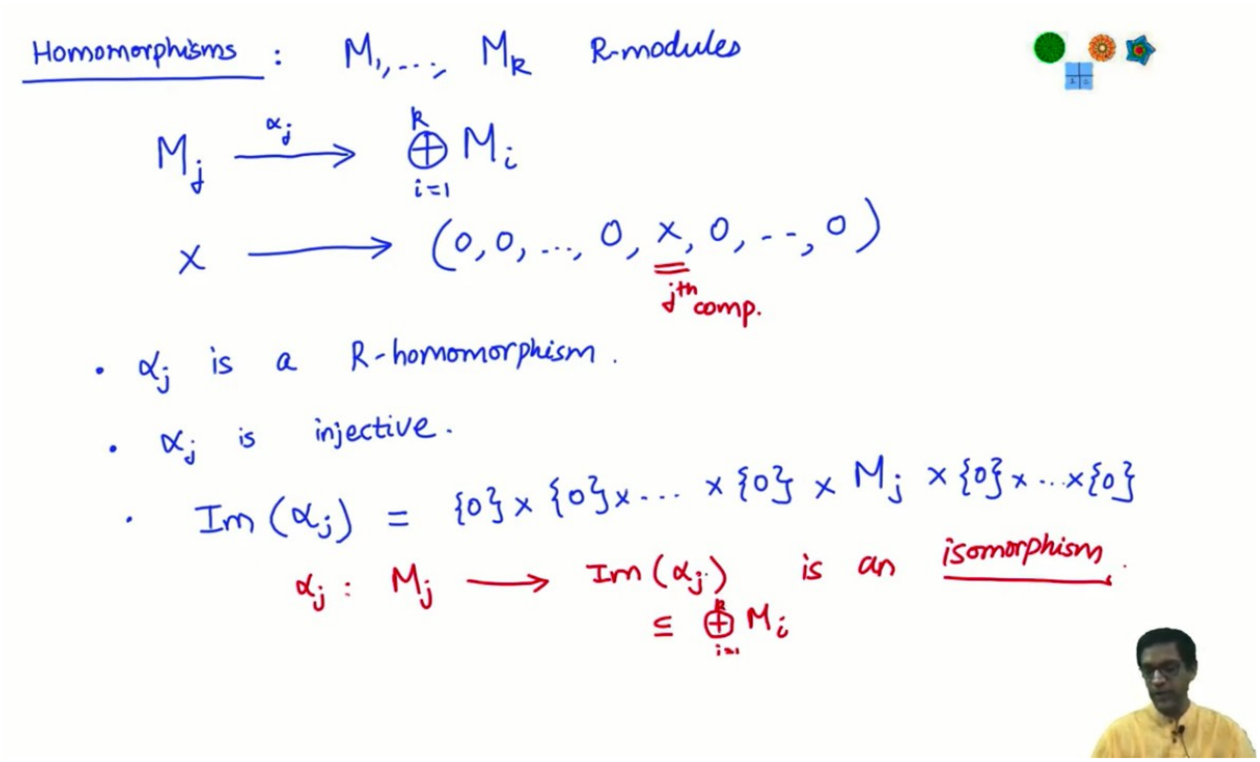
I have a map from each M_j . So, let me take any any one of these guys and from that, I have a map to the direct sum ok. What is this map?

So, let us give this map a name, let me call it α_j. This map does the following. If I have an element X_i in the module M_j , it maps it to the ordered tuple which is 0 in all the other components except in the in the jth component alone. So, this is now the jth component. It takes the value X ok. So, this is the the map from M_j to the direct sum. So, this is a well-defined map and as is clear because the right hand side has all component wise operations. It is clear that α_j is a homomorphism, is R-module homomorphism. Because addition goes to addition $X + Y$ goes to $X + Y$ on the right side and similarly, when you multiply by R on the left side, it goes to multiplication by R on the right side ok.

So, α_j is in fact, a homomorphism and what is more α_j is an injective map; α_j is injective. In other words, it is a one-to-one map, that is again clear because X maps to the sort of the odd tuple with X_in the jth component right. So, two different things cannot map to the same thing on the right side ok.

So, it is clearly an injective map and ah. So, if you look at the image of α_j. So, consider the image of α_j. So, what is the image? It is well one way of saying it is you you you just take all tuples in which so this is can be written like this 0 cross 0 in all the components. The jth component alone I take the whole module M_j ok. So, that is the image of α_j. Recall the image of a homomorphism is always is a sub module. So, what is this map α_j do? It maps M_j injunctively right. It is a one-to-one map and instead of taking the entire direct sum on the right side, let me just look at this image that I described for you image of α_j which is this set.

So, consider α_j from M_j to the image of α_j, clearly this map is an isomorphism now . So, why is it an isomorphism? It was already injective and since, I have restricted on the right



hand side to just the image, it is also surjective ok. So, α_j from M_j to image of α_j is in fact an isomorphism of modules.

So, what that means is that in some sense say and recall that image of α_j is a subset of the direct sum right. This is a sub of the direct sum M_i ok. So, the image of α_j is a copy, is an isomorphic copy of M_j ok. So, what we have inside the direct sum really is a copy of each of these, each of these modules M_i ok and so, sometimes one draws the following picture.

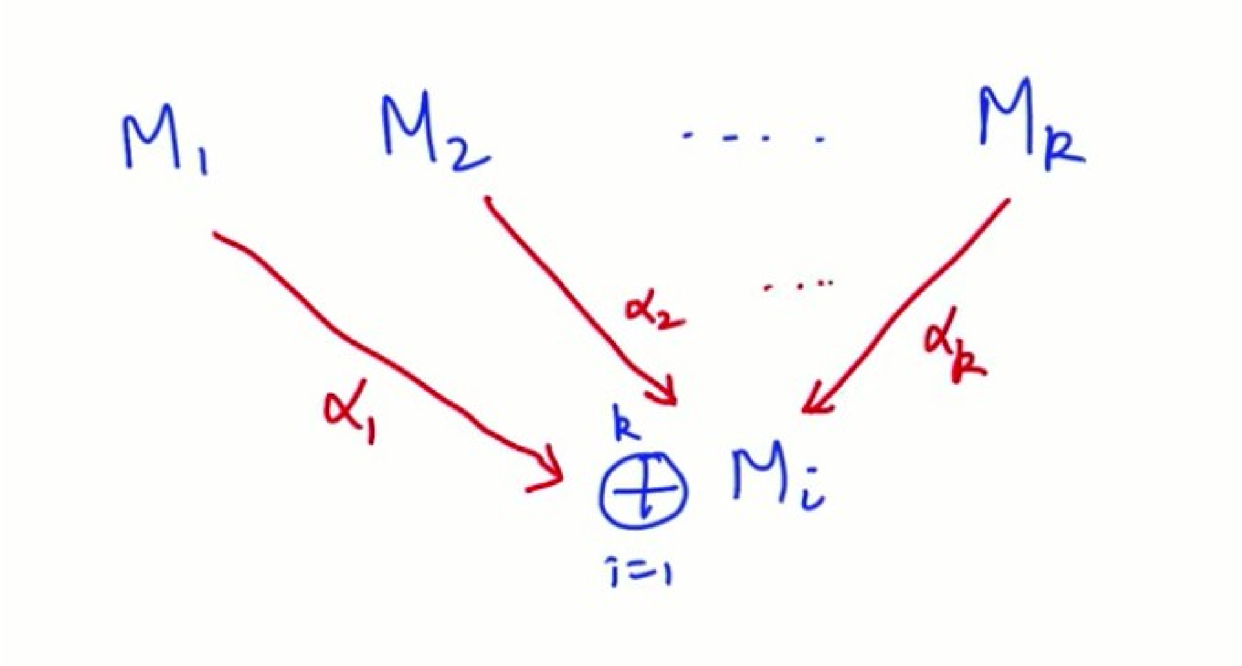
So, I have M_1, M_2, M_3 and so on M_k, the direct sum of these modules M_i, it is there. Now, from each of these modules, I have this map what I am calling α_j right our α_1 from M_1 to this α_2, from M_2 to this and α_k from M_k to this. So, I have all these homomorphisms and each homomorphism, does the following.

ah It takes the corresponding M_i isomorphically to a copy of itself which sits inside the direct sum ok and the way it sits inside is what we described right here. Its zeros everywhere else and only in the jth component, you you keep M_j as it is ok. So, this is one way of understanding what the the direct sum really is.

Now, let us talk about another sense in which the word direct sum appears this notion. And this is sometimes called the just to distinguish it from the earlier one, we sometimes call this the internal direct sum ok. So, this notion is is usually called internal direct sum.

So, I am just putting the internal within parenthesis right now because soon, we will see that for all practical purposes, this internal direct sum is really the same as a direct sum in some sense ok. So, we will make that precise. But for now, let us keep this this word internal to remind us that this is different from what I have just described ok.

Now, what is this internal direct sum; what is the context; so, what do I need for this? So, I need a module M . So, suppose suppose M is an R-module. So, suppose M is an R-module and I have a collection of sub modules of M ok and M_i is a sub module of M, for all i from



1 to k ok. So, I have k different k I have k sub modules of my module M. Now, definition we say that M is the internal direct sum, we say that M is the internal direct sum of the M_i of this collection of sub modules M_i, i goes from 1 to k. If the following happens, if each element m belonging to M can be written as a sum can be written as sum $m = m_1 + m_2 + ... + m_k$, m_i is coming from the corresponding modules M_i for all i in a unique manner ok.

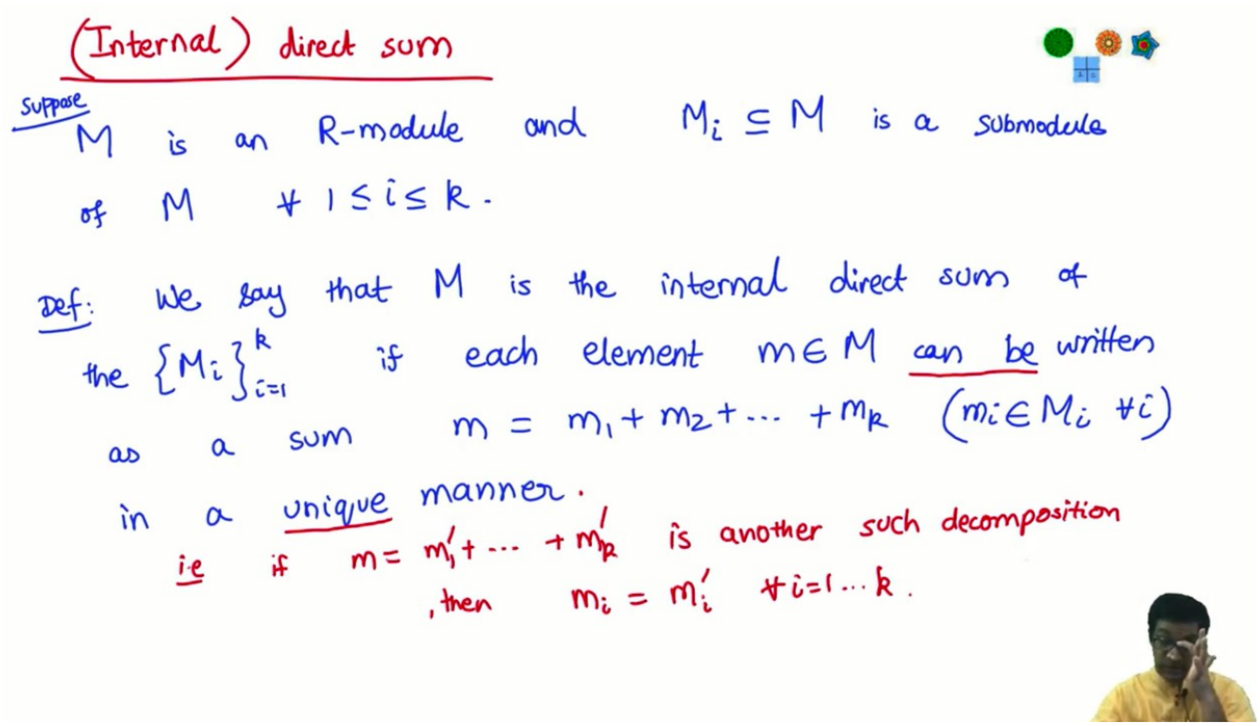
So, there are two parts to this to this definition; on the one hand, you can write it. In other words, there exists such a way of writing M and there is a uniqueness part also that this way of writing must be unique ok. So, what is this unique mean? So, let me just explain that a little bit more unique manner means if I can find another such decomposition $m_1' + m_2' + ... + m_k'$ is another such decomposition of m or way of writing it into a sum; then, in other words, with m_i's all coming from the sub module capital Mi, then the M_i and the m_i's have to be the same, for all i from 1 to k ok.

So, this is the definition, we say that M is an internal direct sum. If every element admits decomposition into a sum of M_i's in a unique manner ok. Now, let us now reconcile these two notions of internal direct sum and what I initially called direct sum?

So, now, observe that these modules M_i's, so when I define these modules M_i's I mean when I gave you the modules M_i's, I said they are all sub modules of some ambient module M right. So, I have an M that is my module and inside M, I have these sub modules.

So, that is M_1 well. So, any two sub modules at least intersect in the origin, I mean in in the 0 element. So, let us say let us draw a better picture. So, let us say that is M_1, maybe this is M_2, that is M_3 and so on. So, I give you a bunch of sub modules; but a sub module can of course be thought of as some module in its own right.

Forget the fact that there is an ambient module module which contains it, just think of each of the M_i's as just being modules in their own right ok. So, view the M_i's as modules in their own right and when you do that, we can then form their direct sum ok. So, this is the direct sum that I talked about at first; sometimes called the external direct sum as suppose



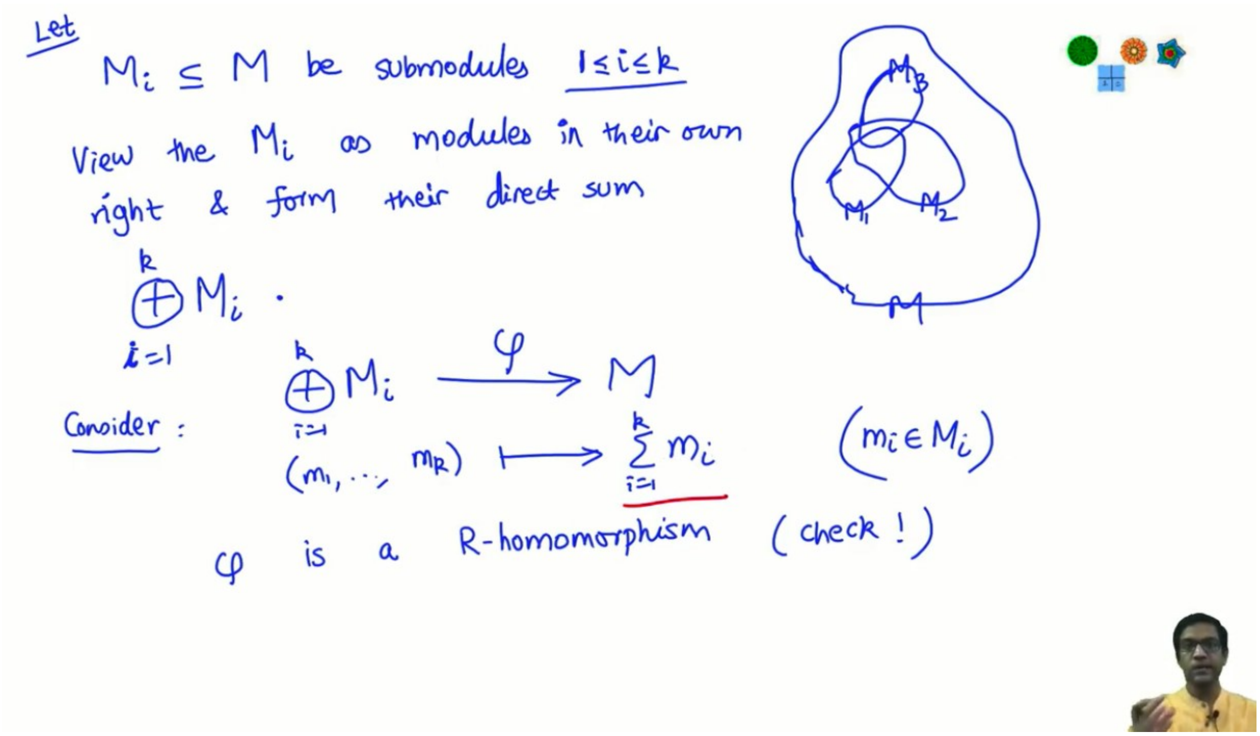
to the intern. So, let us form that direct sum; i equals 1 to k ok. So, recall what is this as a set, this is very different from the elements of this external direct sum are not going to be elements of M or anything ok.

They are in fact you know tuples of elements right. Remember, this is just as a set its the cross product $M_1 \times M_2 \times M_3 \times .... \times M_k$. So, this is some some new set. It is it is a module in its own right again the direct sum ok. So, we can form this this new object direct sum; but what is interesting here is the following, that now consider the the following.

So, so maybe to to phrase this properly, let me just say let M_i be sub. So, let me say b sub modules ok, i from 1 to k . So, this is a collection of k sub modules of an ambient module, I can view those as modules in their own right and form the direct sum. Now, from this direct sum, there is always a homomorphism that I can define to my module M ok. So, consider the map ϕ from this direct sum. So, what is this this map going to be? Well, a typical element here looks like this; M_1, M_2,...,M_k. So, what what I will do under this map ϕ is to just send it to the sum of these elements ok.

So, this is for all M is coming from M_i ok. Now, this map, I can always define whenever I have a collection of sub modules and it is very easy to see again that this map ϕ is in fact a homomorphism. So, ϕ is R-homomorphism; where, R is the ring over which everything is a module. Check this is again from the fact that the direct sum has all component wise operations ok.

So, ϕ is a R homomorphism that is very easy. Now, the key point here is that observe that if these sub modules M_i, if if the ambient module M is an internal direct sum of these sub modules. Then, what is what does it mean? Well, it means the following. If you go back to this definition if M is the internal direct sum, then every element of M can be written as a sum of of M_i's ok.

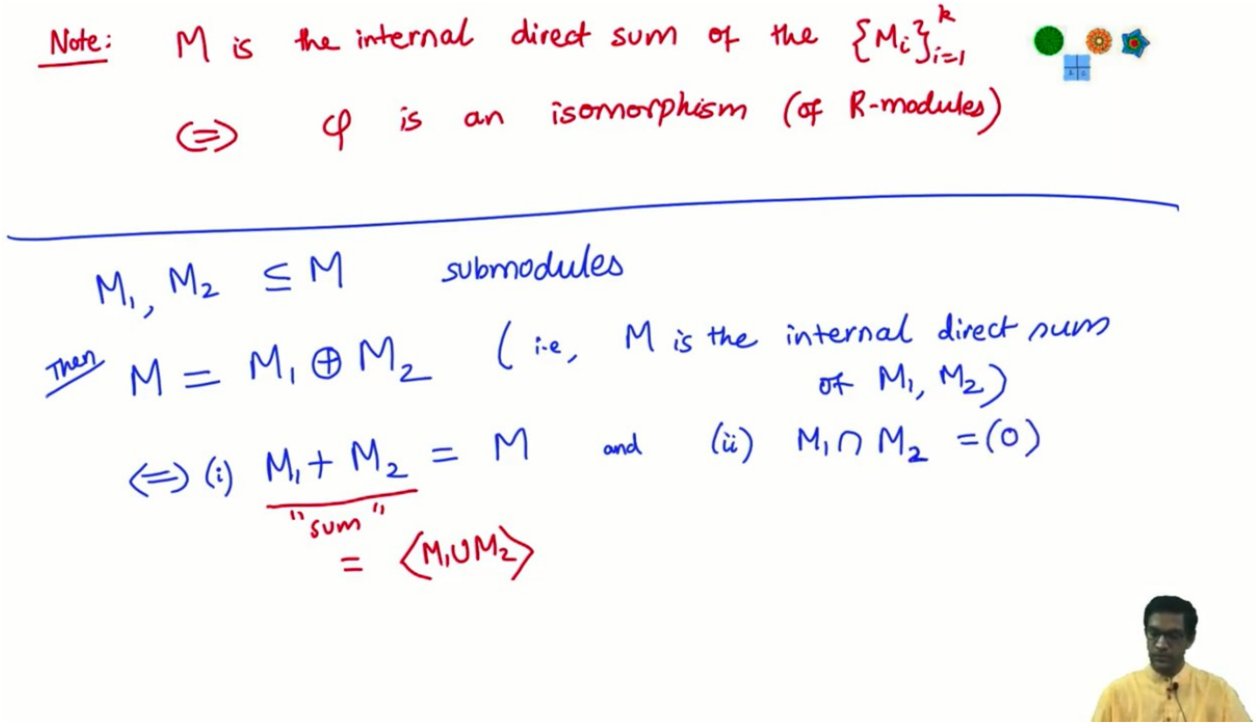


What does that mean? It means that this map ϕ which I have just defined, this map is surjected right because every element of M has one such decomposition ok; every M can be written like this. Now, so, ϕ is surjective is really the first part of the definition and the second thing says everything has a unique such decomposition ok. What does that mean? Well, that just means that ϕ is injective; ϕ is a one-to-one map right because given any element on on the right, I can only find a single tuple M_1, M_2,..., M_k such that M can be written as their sum ok. So, what this means is that more or less by definition observe that note .

M is the internal direct sum of these sub modules M_i, 1 to k is the same as saying that this map ϕ which I just defined is an isomorphism; ϕ is an isomorphism; isomorphism of R-modules. Of course, everything is over over some ring R ok. So, what this means is that this notion of internal direct sum right, if M is the internal direct sum. In some sense, it is isomorphic to the external direct sum of the M_i's ok. So, that is sort of why one after a point stops making this distinction between internal and external; it is in some sense, it is it is really the same notion ok. Now, let us look at this internal direct sum or direct sum.

Now, in the special case of where you only have two sub modules ok. So, this is you know this is something which comes up very frequently. So, let us just take two sub modules. So, let me look at suppose M_1, M_2 are both sub modules of M; sub modules, M is an R-module as before. Then, let us look at $M_1 \oplus M_2$. So, what does it mean to say that M is the internal direct sum of $M_1 \oplus M_2$ ok. So, M is the internal direct sum. So, we will now denote it like this. M equals $M_1 \oplus M_2$; in other words, it is the internal direct sum.

So, this is now not a we will just use the same notation for the you know for both direct sums internal or external i.e in other words, M is the internal direct sum of these two sub

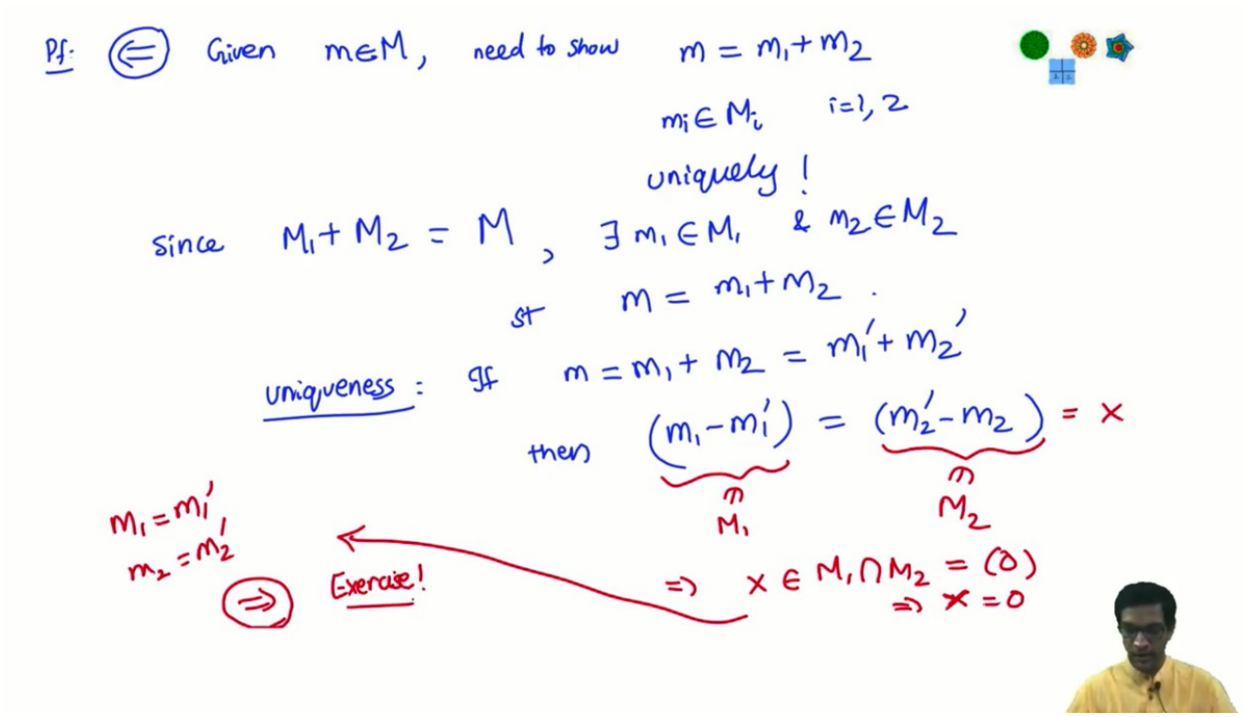


modules ok. So, then observe, then note the following M is the internal direct sum can be written as follows, rephrased as follows if and only if $M_1 + M_2$.

So, two things should happen one the sum of these module should equal M ok. So, recall, we talked about this general construction of sub modules. This is the usual sum of sub modules ok and what was sum? Sum was supposed to be the following, you take the union of those two guys M_1 and M_2 and look at the smallest sub module, the sub module of M generated by the union ok, that was the definition of the sum and we had back when we talked about this construction, we also said the sum is you know what are elements of the sum, they are nothing but things of the form M ones little $m_1 + m_2$; where, m_1 comes from capital M_1 and m_2 comes from capital M_2 ok. So, go back and look at the the video on. Sums of sub modules. So, here here here are two conditions; number 1, that the sum the ordinary sum of these two guys should be M and property 2, the intersection of these two should be just the zero-sub module . So, I claim that if these two properties are satisfied or yeah, if these two properties are satisfied, then M is just the internal direct sum of M_1 and M_2 ok and conversely. So, let us prove one direction, the other is sort of similar.

So, let me prove the the reverse direction. If I know that $M_1 + M_2$ is M and M_1 intersection M_2 is 0, why does it imply that capital M is the internal direct sum of m_1 and m_2. So, let us prove the converse. So, we need to show that given any element $m \in M$ need to show that m can be written as a sum $m_1 + m_2$. m_1 belonging to so m_i belonging to M_i in a unique fashion ok. So, let us prove this number 1, m can be written like this because so firstly, observe since just the ordinary sum of M_1 and M_2 is all of M. This means that there exists $m_1 \in M_1$ and $m_2 \in M_2$ such that m can be written as their sum ok.

This is again from like I said from that video on sums of sub modules. Now, let us prove the uniqueness that comes from the second part of the hypothesis ok. So, let us show uniqueness. So, if I can find another such decomposition, so let us say I have one such and another one,



then let us do the following let us subtract, let us pull the m_1' to one side and m_2 to the other side. So, this is the equation that we now get and now, we make the observation that the left hand side is an element of M_1, the right hand side being a difference of two elements of M_2 is an element of M_2 ok. So, here is an element which lives. So, whatever you call this element something X maybe, now this element X lives both in M_1 and M_2 which means that X belongs to their intersection; but which by hypothesis is just 0 ok.

So, this means X_is 0 and X_is 0 automatically means that these two guys are equal to each other right. So, what is the conclusion from here? It is completed this means that $m_1 = m_1'$,$m_2 = m_2'$ ok. So, that completes the verification of the the converse. So, let me just say this is similar. So, let me just leave this as an exercise ok. So, this is the case of two modules, two sub modules; M_1 and M_2 ok. Now, this is sometimes called pair of complementary sub modules, when you have two modules which whose direct sum is the is the whole ok and we will we will talk about them in a later video.