Algebra - I
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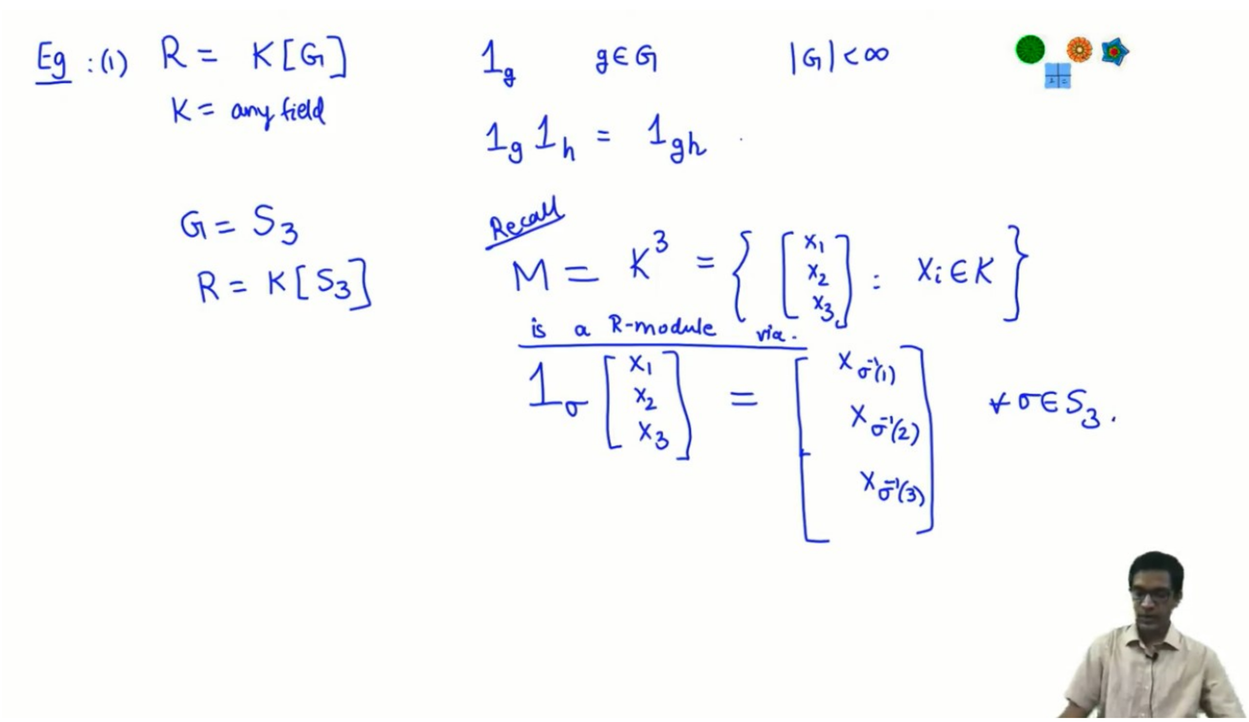
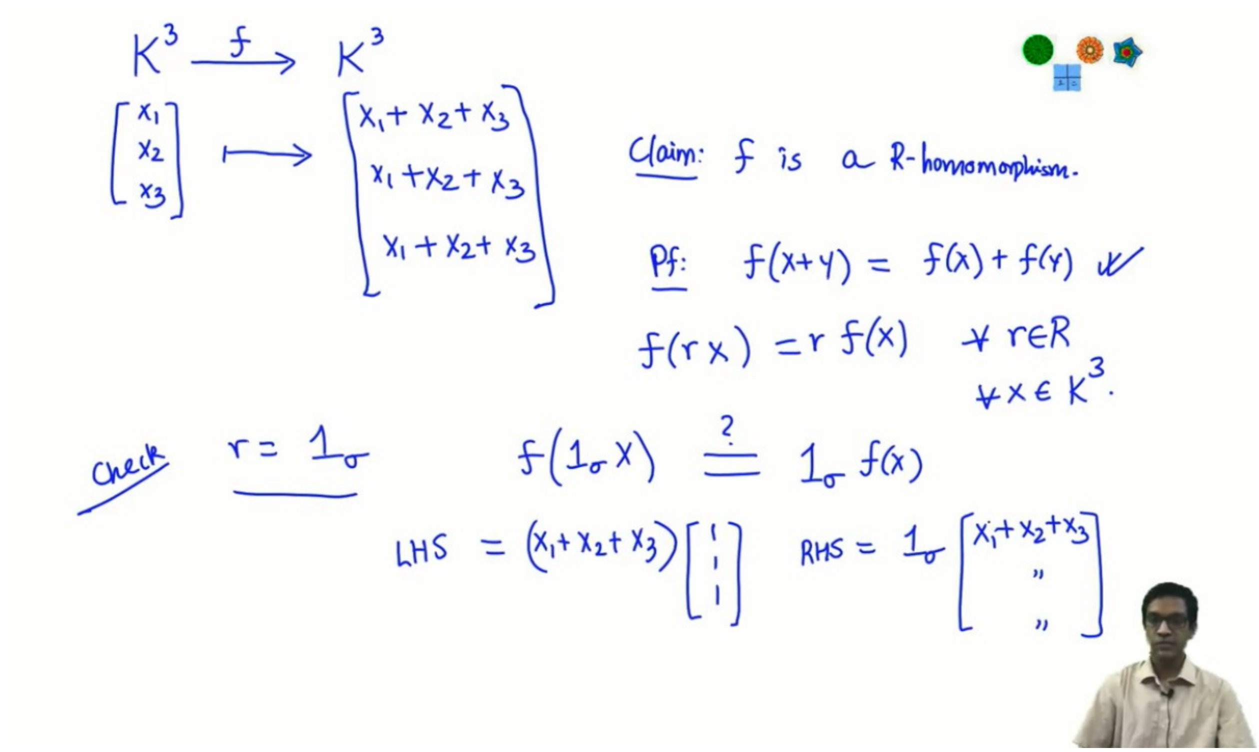
Lecture 60 [More examples of homomorphisms]

Let us talk about a few more examples of homomorphisms . So, we saw few examples already. So, here is an example with the ring R being the group algebra ok. So, recall that the group algebra of a finite group or the group ring of a finite group over a field K is well, what was it? As a vector space, it had elements which look like 1_g So, these were the basis elements, as g runs over the the group G. So, recall here, I am taking G to be a finite group. So, these form a basis of this space thought of as vector space over K.

But what is interesting here really is the ring structure. So, we take 1_g multiplied by 1_h we said is $1_g h$ ok. So, look back on on the lectures on the group ring and so on to recall the definition and let us do a specific case. Let us take G to be the group S_3 , the symmetric group and K to be any field. So, R is $K[S_3]$, this case K is just any field here. So, recall, we had talked about certain module for this group ring. What was that?

The module M was as an underlying abelian group, it was just a space K^3 comprising column vectors $(x_1,\ x_2,\ x_3)$; where, x_i has come from K ok and so, that is the underlying additive structure, that is the abelian group. In fact, it is a it is also vector space; but that





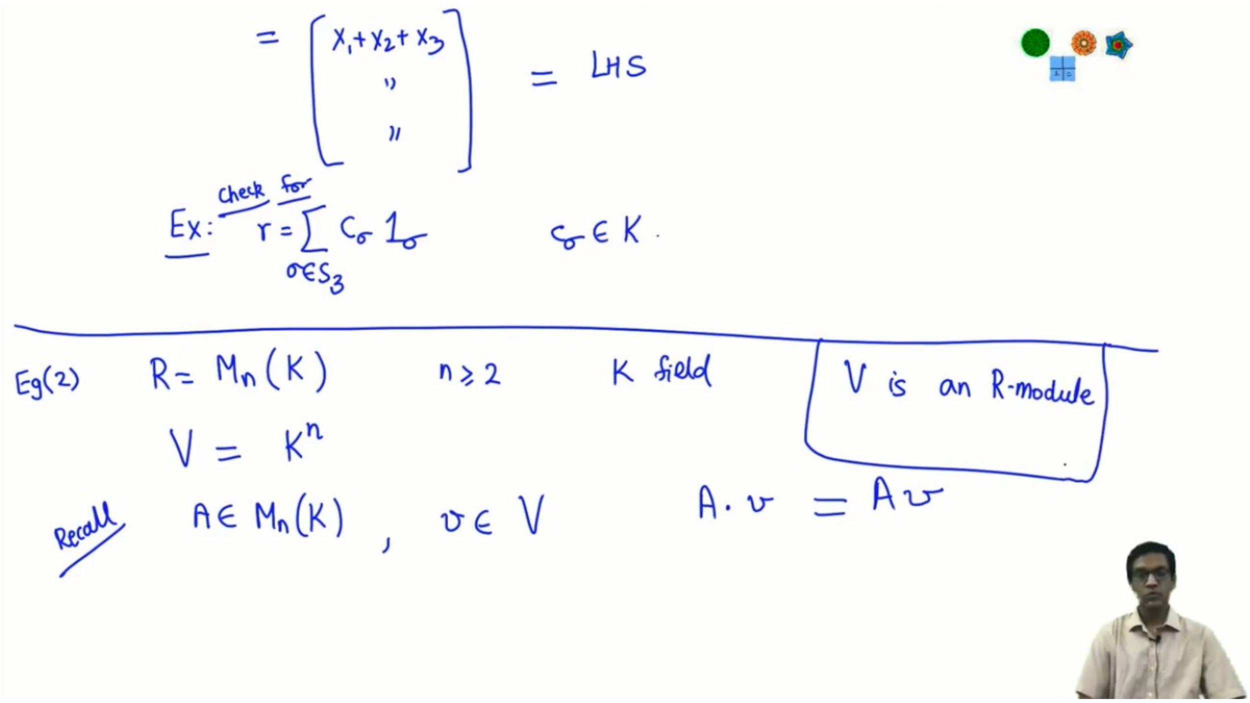
is at the moment we think of it as a as an abelian group. And the action or the these scalar multiplication by these basis elements 1_σ, we had said is given by the following formula, we can make this into. So, recall M. So, recall the following that M is R module and via the following definition that the the elements of the basis elements 1_σ's, they act on this as follows. It maps it to a permutation of the three co-ordinates and we have said you need to use σ inverses there ok.

And this is for all $\sigma \in S_3$ ok. So, again look back on that lecture to recall how this became an R module and so on. It is a left R module. Now, what we have going to do is to define homomorphism of this space to itself ok.

So, let me take this space this module K^3 and let me define for you a homomorphism of K^3 to itself and the the the map is the following. To look at (x_1, x_2, x_3); the vector map to $(x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$ in each of the three coordinates . So, it just sums the three coordinates and places that value in each of them ok. So, I claim that this map is actually a homomorphism ok. So, let us check this. So, claim f is R homomorphism; where, R is the group ring. Let us prove it. What do we need to do? We need to check that this is it satisfies the two axioms.

So, the additivity is easy. It is just you know easy to check if I take a sum of two vectors, then the the answer will be again a sum. So, this is straight forward. I will leave it for you to check; easy. But the key property we need to check really is the the compatibility under scalar multiplication that I can pull the r out for all x in K^3 ok. Why is this possible? Well, let us take R to be these special elements, these basis elements 1_σ and let us check it for those ok, it turns out to enough to for those elements.

So, let us check for R equals 1_σ. So, on the one hand, I have $f(1_\sigma x)$. So, my question is this is this the same as $1_\sigma f(x)$ ok. So, I need to compute both sides and check that they



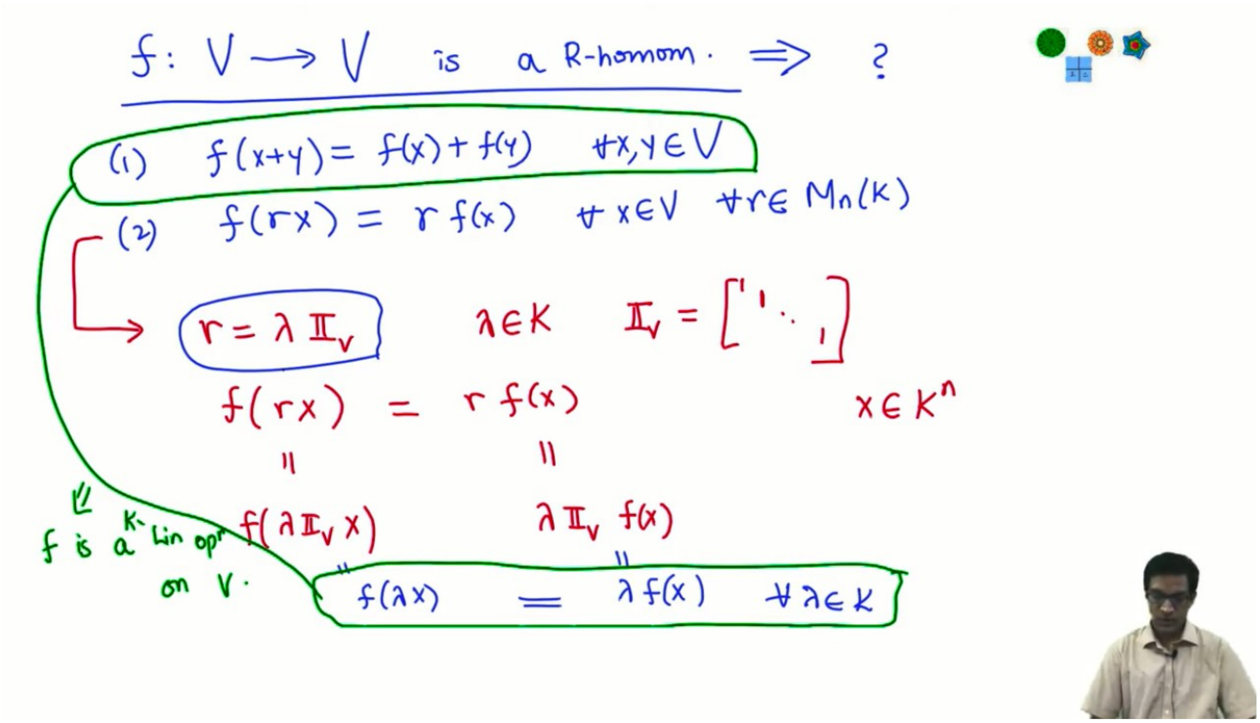
are actually equal. So, let us do that. So, let us first compute the left-hand side of this, this equation.

So, $f(1_\sigma x)$. So, what is this? Well, first I need to to operate 1_σ on x ok and recall, what 1_σ does. So, here is the action of 1_σ; when 1_σ acts on x_1, x_2, x_3. It just permutes the three coordinates x_1, x_2, x_3 in some way according to σ. It is it is still the same three guys, but they occur in some other order ok. But observe that when I apply f to that answer what is f doing? f is just summing up the three coordinates and giving me the total. Now, if I permute the three coordinates and then, I sum them up, well that is the same as the sum is the same as the the sum which I get without permuting the coordinates right.

It is the same answer anyway. So, the left hand side is well it is $x_{\sigma^{-1}(1)}$,$x_{\sigma^{-1}(2)}$,$x_{\sigma^{-1}(3)}$ in each component; but that is the same as just saying it is the sum x_1, x_2, x_3. Well, I let me just say multiplied by the vector $1, 11$ that is the same as this functional ok.

So, it is it is I can compute like this and now, observe that what about the right hand side? So, ok, I have computed the left hand side. The right hand side is I am supposed to take 1_σ and I am supposed to act it on the vector $x_1 + x_2 + x_3$ in each component; $x_1 + x_2 + x_3$ ok in each component. Now, what does 1_σ do to this vector? Well, it again permutes the 3-3 components, but all three are equal to each other. So, when I permute the three, I do not get anything new; I just get the same same answer again ok.

So, observe because the three components are the same, it just gives me the same answer once more. It is just $x_1 + x_2 + x_3$ on each components ok and so, that is exactly equal to the left hand side ok; observe the left hand side was also the same thing ok. So, we have checked it for things of the form 1_σ and again I leave it as an exercise for you to check that its true more generally if I take my ring element to be sort out the more general form, a linear combination. So, this is over all $\sum_{\sigma \in S_3} c_\sigma 1_\sigma$ are all field elements ok.



Recall that is a typical element of my group algebra or group ring and you know check that the same check for such ring elements as well, the most general ring elements that the the left hand side and right hand side will actually be equal ok. So, that is the first example for this lecture, homomorphism of a module over a group ring ok. Now, here is a second example, let us look at another non-abelian group, the group of n cross I am sorry non abelian ring. The ring of $n \times n$ matrices over a field K. So, let n be at least 2 K is any field .

So, this is the matrix ring and recall again from one of our last lectures that we constructed a module for this ring and the module was just again K^n which is all column vectors of size n and what was the action? To sort of the most obvious one; how does the ring R act on vectors from K^n. So, if I take an element of R. So, in this case elements of R are nothing but matrices . So, if I take $A \in M_n(K)$ and $v \in V$; then, how do I scalar multiply v with A?

Well, the answer is you just multiply that matrix with that vector, it will again give you a column vector that is the answer ok . So, this is this is I am just recalling for you the the scalar multiplication on this module V ok by elements of R . So, again V is an R module ok. So, again look back on the the previous lectures to to recall precisely how this became a module and so on. But for now, I am interested in trying to figure out what homomorphisms look like ok.

So, what do I want to do? I want to understand if $f : V \to V$ is R homomorphism ok, then what can I say about f? What does that tell me about f ok? In other words, can I somehow characterize all homomorphism from this module to itself ok, ok. So, let us again see what is the consequences of this being a homomorphism would be.

So, observe for a start that I can do the following, let me take A. So, . So, what is what is a homomorphism let us again recall its two properties; $f(x + y) = f(x) + f(y)$; $fx + fy$ for all $x, y \in V$ and f of property 2, $f(rx) = rf(x)$ for all x in my ring which in this case $M_n(K)$. So, let me try to understand what are all the various consequences of these two definitions.

So, in particular let me start with this axiom 2 and put take a special case, let me take my R ring element to be the I_V matrix ok or more generally, let me take it to be λI_V, where λ is a is an element of the K and I_V just denotes the the I_V matrix ok. So, $\lambda \cdot I_V$ is the scalar diagonal matrix with all λs along the diagonal. So, suppose I take this particular ring element and see what it implies, so I have $f(rx)$; it should be $rf(x)$.

Now, what is $f(rx)$? Well, $f(rx)$ by definition is I have to act $\lambda \cdot I_V$ on x. So, this is $f(\lambda \cdot I_V x)$ multiplied with the the vector x. So, recall x now is a column vector here ok. The right hand side is well again I have to multiply the λI_V with the column vector fx ok. So, what does that mean? Well, this guy here is just if I multiply I_V with Xi get back x. So, this is just f of the column vector x multiplied by this scalar λ. In the right hand side is again $I_V \cdot fx$ will just give me fx again. So, this is nothing but the scalar $\lambda \cdot fx$ ok. So, what do I conclude?

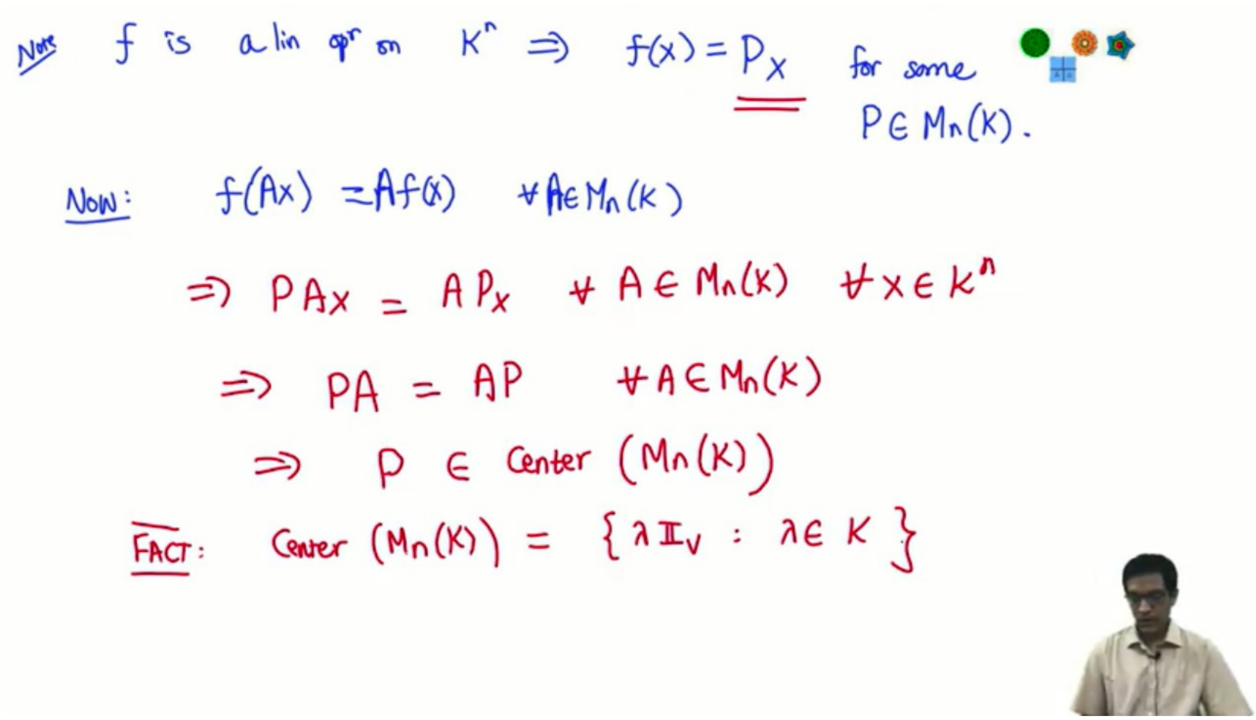
Well, for these special matrices are the scalar matrices, I conclude that $f(\lambda x)$ must give me the same answer as $\lambda(fx)$ and this should be true for all scalars λ from the the base field K ok and this should sort of remind you of something that we we did last time for another example . So, we now look at the additivity property that we have that was from first axiom. Together with this special property, we have concluded and what do these two things tell us? It tells us that if you think of V only as vector space over K, then f is a linear transformation. So, these two properties tell you that f is nothing but a linear operator on V ok; thought of as a vector space. So, let me say, let me call it K linear operator; I am only thinking of it as a K vector space ok.

So, first I have concluded its a linear operator ok good, but again I have to use the full force of my hypothesis. I know that $f(rx) = rf(x)$ for all r's for all matrices, thus far I have only used R equals scalar matrices to make my first conclusion. So, let us let us do it a little bit more generally. Let us use all r's. So, first before that observe f is a linear transformation. So, what does the linear transformation or a linear operator mean? So, observe first note f is a linear operator on the space of column vectors; just means that f is given by multiplication by some matrix ok. So, let us call it the matrix P may be ok. If you wish, this is just a matrix of this linear operator with respect to the standard basis of Kn. So, any any linear operator always has a matrix and in this case, I take the standard basis I compute the matrix of f with respect to the standard basis; then of course, the f evaluated on x is nothing but the matrix of f multiplied by the column vector x ok. So, any linear operator is nothing but multiplication by a matrix P. Now, the question is what are the possible values of P right. If I tell you what P is, I know what f is.

So, let me try and figure out if f is a homomorphism of R modules, what does it tell me about this matrix P ok? So, now, let us use the full force of the hypothesis f of rx is rf x, this is known in MnK ok which means what? So, let me take a let me instead of R, let me call it A ok . So, it is may be better notation because psychologically I think of it as a matrix. So, f of Ax should be a acting on f of x for all matrices A in MnK.

Now, what is this mean? I have just told you what f is right, f is just multiplication by P. So, this means in particular that if I take Ax and I multiplied it by P should give me the same answer as $A \cdot Px$ for all A in MnK and for all column vectors x in K power n ok.

Now, what is that mean? Well, it saying that the the matrix PA and the matrix AP ok, they give me the same answer when I multiply it by x for all $x \in K^n$ ok. This means in particular that the matrix PA has to be the same as the matrix AP right. When I multiply it by every vector, I get the same answer; that means, the two matrices are actually equal.



One way of seeing it is by multiplying it by the the standard basis vectors ok, that give you each column of the product ok. So, I have concluded that P must commute with this matrix A, but it must commute with A for every $A \in M_n(K)$. In other words, this matrix P is extremely special, it commutes with every matrix. So, this belongs to what is called the center of this ring ok.

The center of a ring is just set of elements of the ring which commute with all elements of the ring ok. So, I conclude that this matrix P must actually belong to the center of this ring ok and here is a little fact which maybe we will prove during one of the tutorials is that the center of the ring of matrices.

In other words, the set of matrices which commute with all matrices is nothing but the scalar matrices. These are the only ones which commute with all matrices ok, set of all λ in K ok. So, what this means is that this matrix P is actually a scalar matrix . So, what does that what does this? I mean follow moment accept this as a fact. So, what conclusion do we make? We conclude then that the the matrix P therefore, must look like sum scalar matrix. Therefore, in particular this homomorphism f which looks like multiplication by P is just nothing but $\lambda \cdot I_V(x)$.

So, this is just $f(x)$ this homomorphism is just this very simple trivial homomorphism which is scaling by λ ok. So, what we concluded is that the only homomorphisms from $K^n \to K^n$ are this scaling homomorphisms ok, the maps which send a vector x to some multiple of x ok. And that multiple λ is of course some fix. So, for some fixed λ for $\lambda \in K$. So, if I choose different λ's of course, I get different scaling operators ok. So, those are exactly the set of all homomorphisms. So, we have made the following conclusions. So, set of all $f : K^n \to K^n$ such that f is R linear or is a homomorphism, homomorphism of $M_n(K)$ modules .

$$\Rightarrow \quad P = \lambda \mathbb{I}_V \quad \Rightarrow \quad f(x) = Px = \lambda \mathbb{I}_V x$$

$$\boxed{f_\lambda(x) = \lambda x} \quad \text{for } \lambda \in K$$

$$\left\{ f : K^n \to K^n \mid f \text{ is a hom. of } M_n(K)\text{-modules} \right\}$$

$$= \left\{ f_\lambda \mid \lambda \in K \right\}$$

So, I want think of this as K^n as $M_n(K)$ module, the set of all homomorphisms is just the set of all scaling maps. So, this is just set of all ah. So, maybe we should call this scaling map something. So, I fix a λ. So, maybe I call this f_λ, we are scaling by λ map.

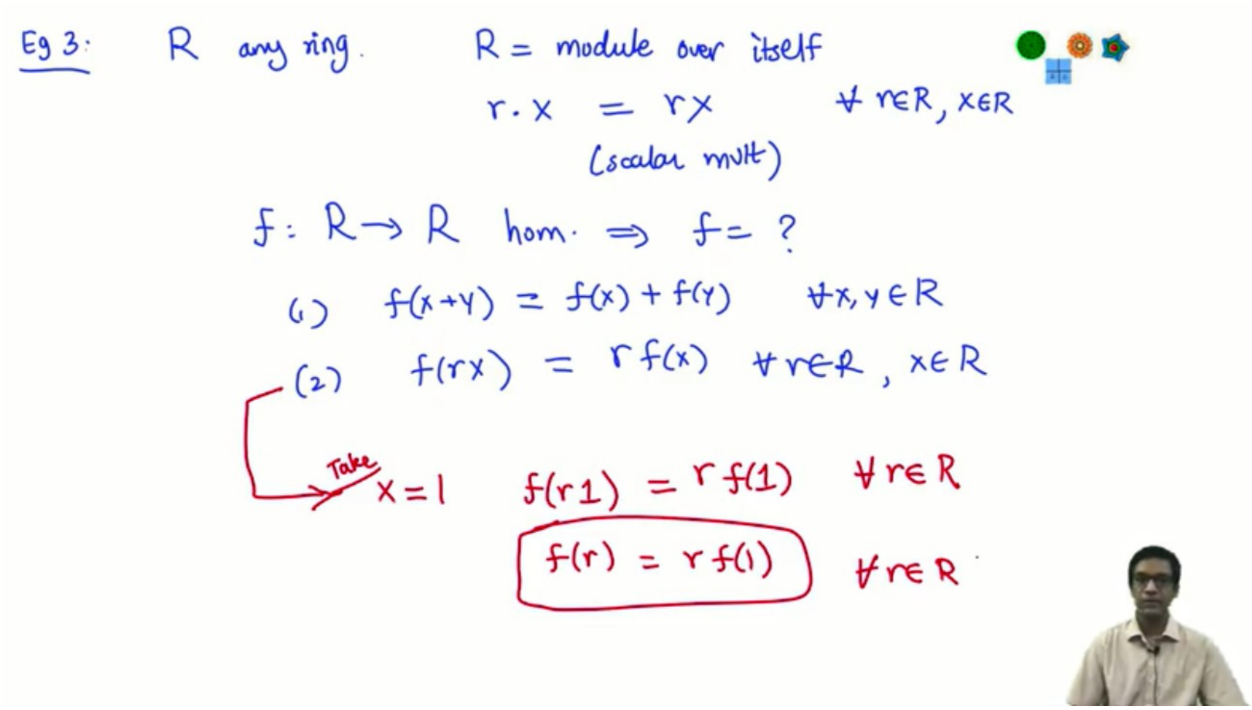
So, this is just the set of all f_λ, where λ ranges over all elements of K . So, exactly the set of scaling scaling map maps ok. So, that was a second example of what homomorphisms look like for this particular module over the matrix ring ok. Now, here is another interesting example, slightly more general which is that suppose R is any ring ok and recall that I can think of R as a module over itself, R is a module over itself with respect to this left multiplication action. So, how does a ring element; what does the scalar multiplication on x? It just multiplies rx ok.

So, this is the scalar multiplication operation. So, this is the scalar multiplication ok. In addition is of course, the usual addition of the ring now let us ask when you think of R as module over itself in this manner what are homomorphisms from $R \to R$.

So, I want to know if f is a homomorphism, then what can I conclude about f? What sorts of possibilities are there? So, let us again plug in to the definition a homomorphism means $f(x+y) = f(x) + f(y)$ for all xy in R; property 2 says $f(rx) = rf(x); x \in R$ ok. Now again, like we did before. So, let us use the second property that is the one which usually gives us something interesting non trivial . So, let me take, in this case let me just take x to be the identity element of the the ring ok. So, take x to just be the element 1 of the ring R and now, let us see what I get from that.

So, I get $f(r) \cdot 1$ should just be $r \cdot f(1)$ ok and this should be true for all ring elements $r \in R$. I fixed x to be 1 here. Now, what is this mean? Well, this just means the left hand side is f of R; right hand side is $r \cdot f(1)$. So, this is what I conclude. I have actually figured out what value my f has on the ring element R. The value is just $r \cdot f(1)$ ok. So, $f(1)$ is some particular element of my my ring here. So, let us call $f(1)$ as something.

So, let a be the element $f(1)$. So, we will just give this a name let us call it a. So, then observe that this homomorphism f of r is just this following map which is R mapping to $r \cdot a$. So, this is some· what is called the right multiplication by a. This is just the map which is right multiplication by the element a ok. So, the the the beautiful thing here is that the homomorphisms from $R \to R$ when you think of R as a left module over itself.



So, remember this is I am thinking of it as a left module over itself via the scalar multiplication, then the homomorphisms from $R \to R$ turn out to be right multiplication maps. It is all of this form where each element is a is multiplied on the right by some ring element; whereas, if you now type the same thing where sorry let me leave this as a little exercise for you.

Recall I can also think of R as a right module over itself, as a right R-module via the following thing if I take a ring element x and I think of right action of r, this is just multiplying on the right. So, this is a scalar multiplication for all $x \in R$ for all $r \in R$ ok, when I multiply the element x by R scalar multiply, the answer is just x multiplied by R on the right ok. This makes R into a right R-module, the addition is just still the usual addition. Then, what do homomorphisms look like?

Then, $f : R \to R$ homomorphisms are exactly of the following kind, it just takes ring element $f(r)$ and multiplies on the left by some element a of the ring $f(r)$ is this for some fixed element a in the ring R ok. So, this is the "left multiplication by 'a' map" ok. So, when you think of R as a left module over itself the homomorphisms turn out to be the right multiplication maps and when you think of it as a right module over itself, the homomorphisms turn out to be left multiplication maps ok and this is a this is an interesting structure that comes up here and that something that we will take up again later on ok.

Let $a = f(1) \in R$

Then $\underline{f(r) = ra}$ = "right mult by 'a'"

<u>Ex:</u> R right R-module via $x \cdot r = xr$ $\forall x \in R$, $\forall r \in R$

$f : R \to R$ hom $\Rightarrow$ $f(r) = ar$ $\forall r \in R$ <u>for some $a \in R$</u>

"left mult by 'a'"