Algebra - I

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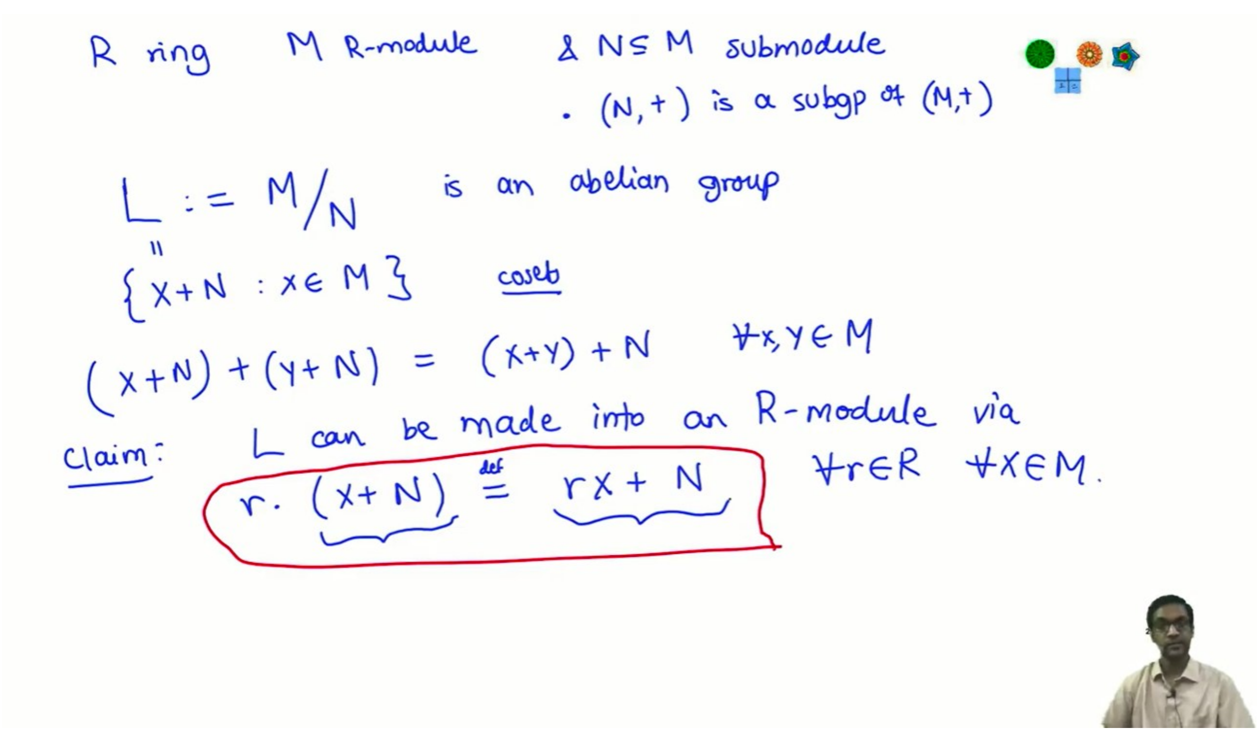
Lecture 58 [Quotient modules]

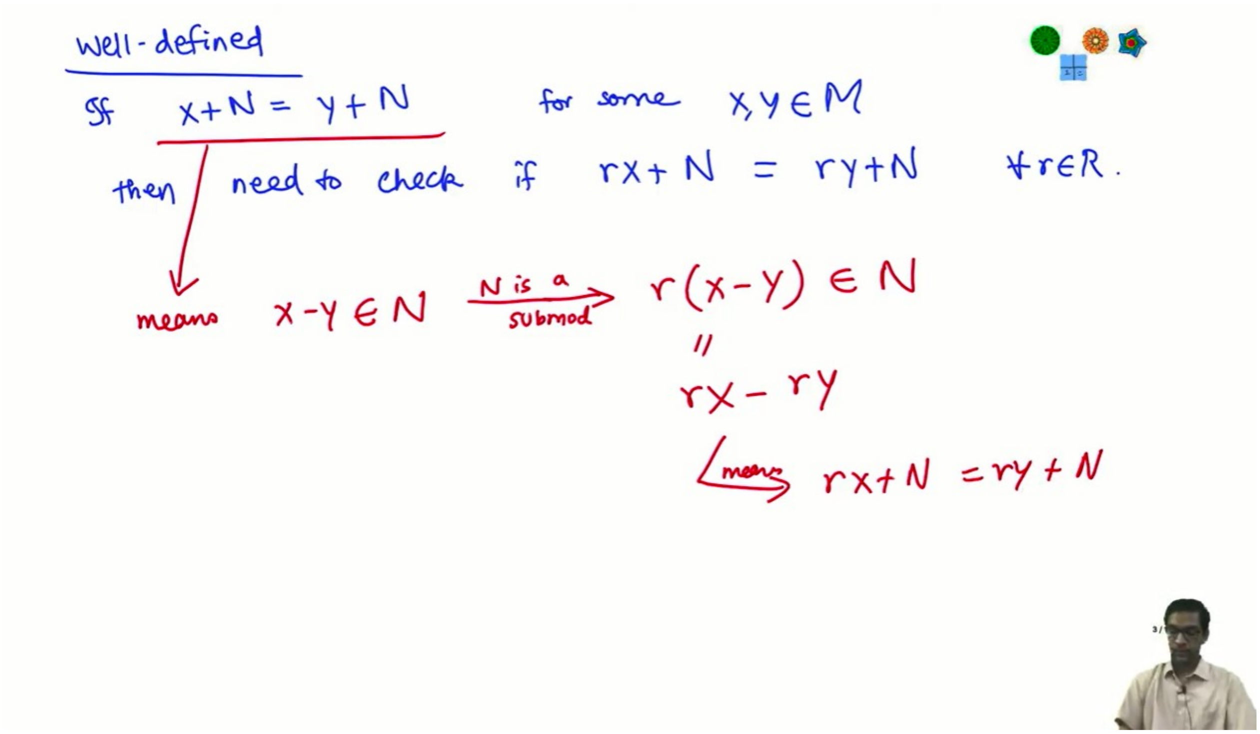
Let us talk about quotient modules. So, let us take a module R let us take a ring R and a module M over R, R is M is an R-module, and N a sub module of M ok. Now, sub module, in particular means that it is is a subgroup of the additive group, sub group of $(M,+)$. And so, we can define the quotient group.

So, recall, if I have a group and I have a normal sub subgroup I can talk about the quotient. In this case, M is an abelian group, so M by N with this addition I can talk about the quotient group, and this has is where in fact, it is an abelian group and what is the group operation I take cosets.

So, firstly, what are elements here? Elements of L look like cosets $\{x + N | x \in M\}$. So, this is space of cosets, set of cosets. And recall, addition of cosets is defined like this or the group operation is $(x + N) + (y + N)$ is just $(x + y) + N$ and this is for all x and y coming from M.

Now, what we claim is that we can do more because M is a module which means I know how to do scalar multiplication, and N is a sub module which means it is closed under the



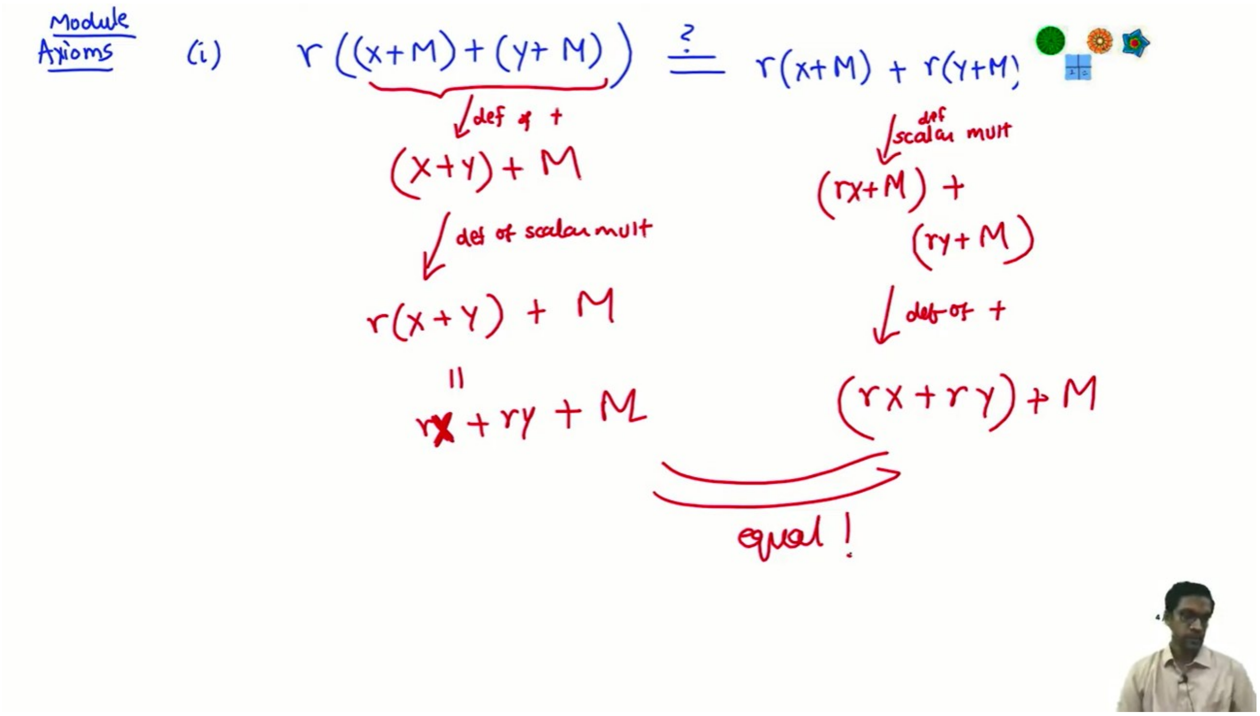


scale of multiplication. So, given this additional data we claim that we can make L into, L can be made into an R-module. So, via, so let us define the action we have in mind. So, take a ring element r, let us take a coset $x + N$ and we will define, so this is the definition that r acting on $x + N$ is just $r(x + N)$ meaning the coset of $rx + N$. And this is for all ring elements for all elements x in M. So, that is the definition of the the scalar multiplication.

And what we need to do before we even proceed further, to show that it is a ring and so it is a module and so on, is we must show that this definition is is actually well defined, ok. So, why do we need to worry about well definedness?

Because this coset $x + N$, I have chosen one particular representative x from this coset, but of course, I could have chosen a different representative. And when I do that I should ensure that my right hand side the coset that appears on the right hand side remains the same, ok. Only then, can I claim that this definition makes sense, ok. It should not depend on the representative that has been chosen. So, here is the first thing. Let us check that this definition is well defined. So, what should we do? So, well definedness means checking the following.

Suppose two different representatives, two different elements x and y give rise to the same coset, ok, then we need to check, need to check if the right hand sides that we claimed, so $rx + N = ry + N$ and this should be true for no matter which ring element I choose, ok. So, if this happens then this definition is well defined. So, let us complete the verification here. So, what does it mean to say that $x + N$ equals $y + N$, well this means simply that $x - y$ is an element of N, ok. Now, N remember it is a sub module, ok. So, now we use some of the hypotheses.



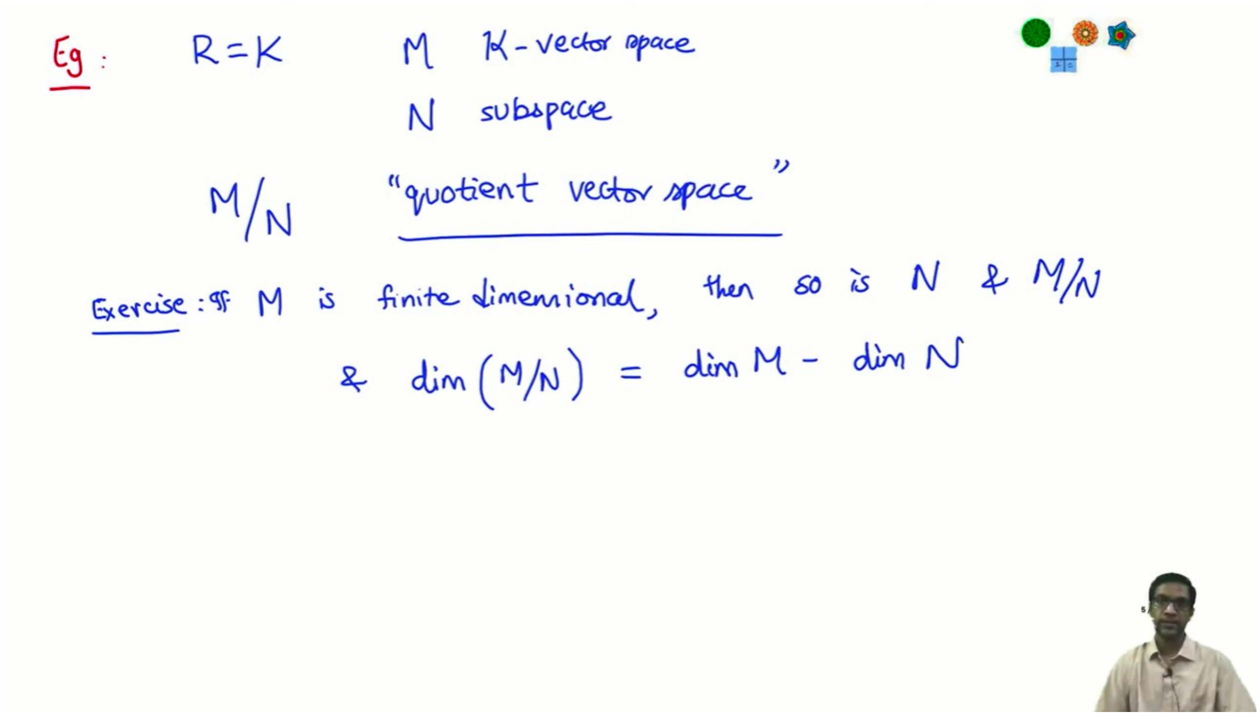
So, N is a sub module means it is closed under scalar multiplication. So, what I can do is I can multiply this element $x - y$ by the element r. So, scalar multiply it. And what I will get must again be an element of N, ok. We are almost there.

So, now we will just use what is $r(x - y)$, the axioms of modules, the second axiom, I suppose, says that rx, $r(x - y)$ is just the same as $rx - ry$ is in N and of course, this just means that the two cosets, $rx + N = ry + N$ are the same, ok. So, that completes the verification, that this definition is actually well defined, ok. Now, we will need to check the R-module axioms; you know does this make it into an R-module. So, let me check one of them, all the others are are similar. So, let us check the module axioms. So, axiom 1 for example, so, which says that, if I take a ring element r and I take the sum of two elements.

So, here the sum of two cosets, so I have to verify whether this gives me the same answer as $r((x + M) + (y + M))$ ok. So, I need to check if these two things are the same. So, again let us let us just compute each side and see what we get. So firstly, $(x + M) + (y + M)$ this is a sum of two cosets. So, this by definition, so this here I am just using the definition of addition in the the space L, in the abelian group L the addition is just you add x and y and take that $(x + y) + M$.

Now, what is $r(x + y)+$ M? Now, this is by the definition of scalar multiplication. So, this is the definition of addition in my quotient space. So, this is now by the definition of scalar multiplication, when I take r and multiply by this I am supposed to get r acting on $(x + y) + M$ ok.

Now, on the other hand what I have here the right hand side its r acting on $x + M$ is just the coset $rx + M$. So, I have to here use the scalar multiplication definition first. So, this by the definition of scalar multiplication r into $y + M$ is just $ry + M$. And now I use the definition of addition in my quotient space which is I have to add the two representatives together $(rx + ry) + M$.



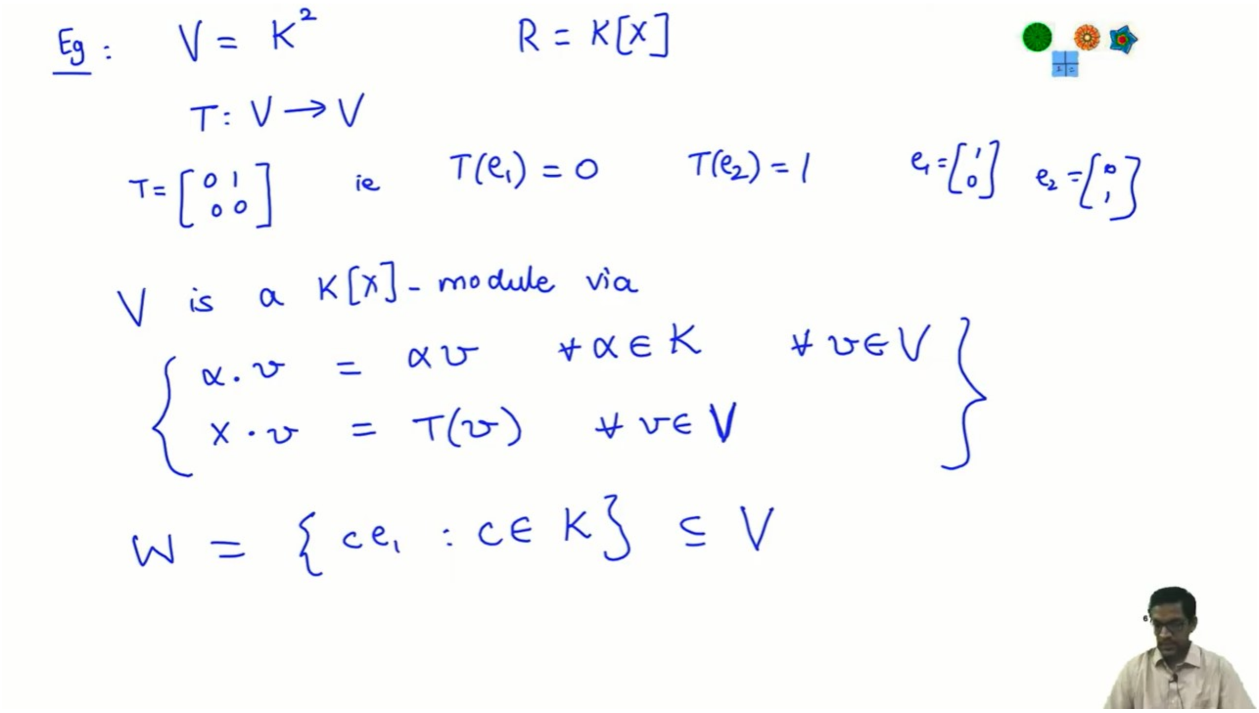
And now observe that these are both equal because of the axiom $r \cdot (x + y)$ in my module M is the same as $rx + ry + M$ ok. And so, the those two are actually equal. And all the other axioms are are similar, so you should just check all the other axioms.

So, I will leave leave that for you to check. So, what we have managed to do is show that the quotient group $\frac{M}{N}$ is in fact more than just a group it is actually an R-module, ok. Now, in the case of vector spaces this is it may be a concept that you have encountered in in linear algebra. It is usually called the quotient vector space. So, let me just make a brief remark about that This is sort of a standard example which is if r is actually a field KM and N are well M is an R-module in other words M is a vector space. So, suppose M is a K vector space, vector space over K, and N is a subspace, so if I give you these two things then we usually talk about the quotient vector space $\frac{M}{N}$. And this is exactly the notion that we have defined here, quotient vector space.

Well, how is it defined? It is as an abelian group it is just the quotient of the the abelian group M by the normal sub group N, and the the vector space structure is given by allowing scalars to act in exactly the same way that we just defined.

So, if I take a element of K, the ground field, then I make K act on the coset $x + N$ by just making it act on the representative x. So, this definition coincides with a definition which you may or may not be familiar with, the notion of a quotient vector space, ok. And this is you know, so maybe an exercise if you have seen this before, if not, we will do it during one of the the problem sessions is the following.

If M is finite dimensional, if M is finite dimensional as a vector space over K then in fact so is N any subspace is finite dimensional, of course so is the quotient and we have the following nice relationship that the dimension of the quotient as a vector space is the dimension of M minus the dimension of N.



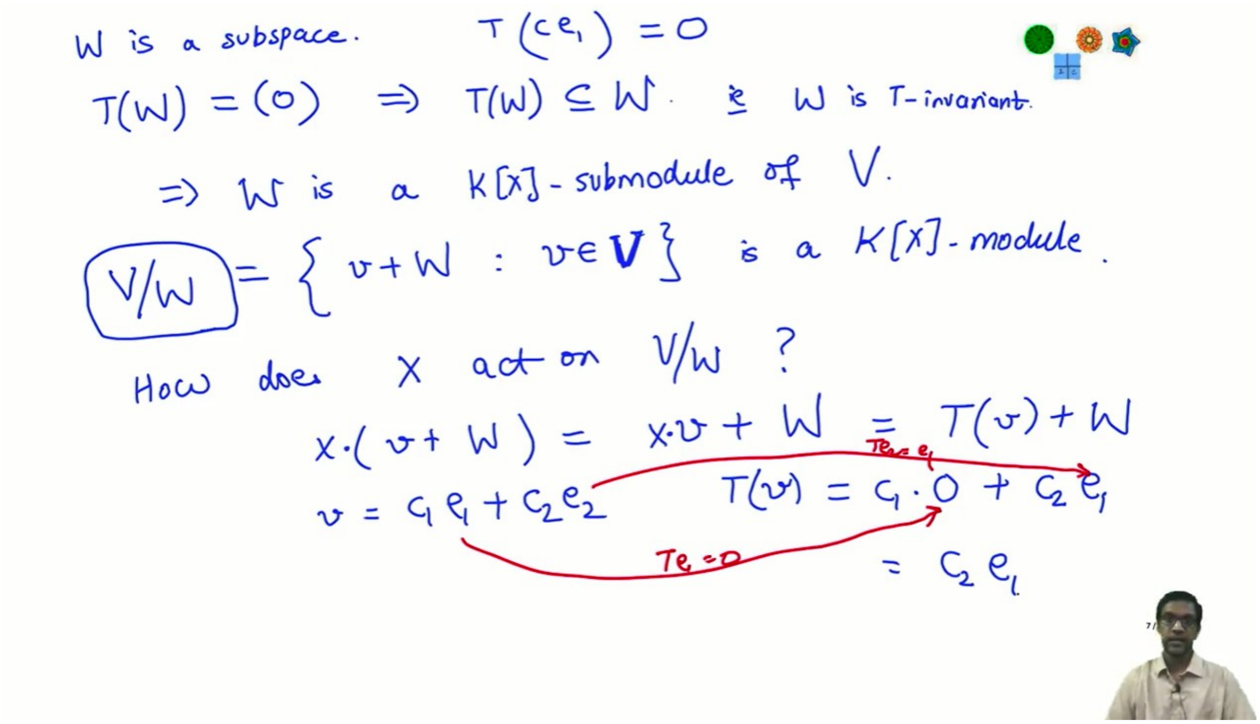
So, just try proving this using the definitions, if you have not seen the definition of a quotient space before This involves trying to find a basis an appropriate basis for the quotient vector space, ok. Anyway just something to think about regarding quotients. Now, here is a here is a slightly less trivial example is let us take the ring $R = K[X]$. So, I am going to talk about a module over this ring $K[X]$. And recall that just means I should give you a vector space together with a linear operator, ok.

So, let me take my vector space K vector space to be K^2, and I should give a linear operator on the space. So, let me just give you the linear operator T which does the following. So, the matrix of T is just; so, I will I will sort of give you the matrix of $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In other words, if you want to think in terms of basis vectors T takes the first basis vector to 0 and the second basis vector to 1, ok. That is what this this $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. mean. So, e_1 is the first basis vector $(1, 0)$, e_2 is the second basis vector $(0, 1)$.

So, I given you, I have I have fully specified a vector space and an operator, so I have fully specified the module structure of V as a module over Kx, ok. So, what is the module structure here? So, now, remember V is therefore, a $K[X]$ module, via the following definitions. The constants, the constant polynomials act as follows. So, if I just give you α acting on v, it is just the usual scalar multiplication. So, this is if α comes from K is the constants, and then the special polynomial x acts for all $v \in V$ according to the what the operator T is, ok.

The action of x is given by the action of is given by the operator T and more or less these two are all you need from this you can figure out how any polynomial acts, ok. You just have to use higher powers of T, ok. So, we have seen this before.

So, all I want to do is sort of take this particular example of T and work out what this this module looks like. In particular, this module V has a an obvious sub module. So, consider



the sub module W which is the span of just the first vector e_1 alone, ok. So, this is the subspace, so when I say span I just mean all scalar multiples with scalars in in K.

So, look at just all multiple $c \cdot e_1$, c coming from K, ok. So, this is a this is a subspace of the vector space V, and I claim that in fact, it is more it is actually a sub module for the action of Kx, ok. So, recall what is a sub module. It is just a subspace which is also invariant under the action of T.

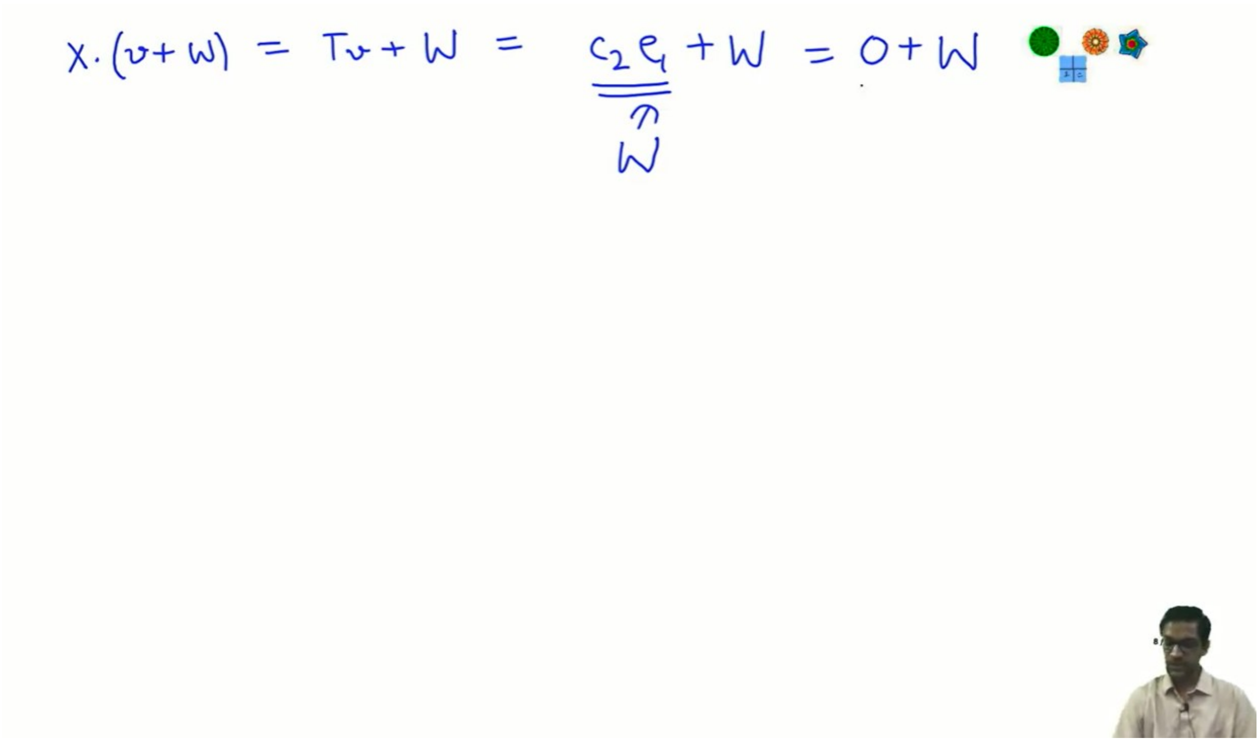
So, let us check the W is T invariant. So, note that W is definitely a subspace, W is a subspace, it is clear enough. But in fact, if I take an element of W and I act T on it. So, what is an element of W? Typical element looks like ce_1 and if I act T on it well by the definition of T that is just 0. So in fact, T of W is actually just going to give me the 0 subspace.

So, in particular it means that T of W is a subset of W ok, that is W is T invariant, ok. And from our general analysis of what sub modules look like. This means that W is actually a $K[X]$ sub module. This is a sub module, this is a $K[X]$ sub module of the module V, ok. So, we have found a module. So, just to recall what T does. So, notice this is what T was doing. $T(e_1) = 0$ and T of e_2 was sorry, it is not 1 here its e_1, $T(e_2) = e_1$, ok.

Now, let us look at what the quotient looks like. So, I have gotten a sub module W of V, so I can consider the quotient space $\frac{V}{W}$ And what does this look like? This is all elements of the form $V + W$ ok, v coming from V.

Now, let us just calculate the action of $K[X]$. So, by the general theorem or general construction in some sense the quotient space is also a $K[X]$ this should be a module, right, this has a $K[X]$ module structure. The question is; what is the what is this $K[X]$ module structure? So, a $K[X]$ module is a vector space together with a linear operator on that vector space.

So, here the vector space is this right, the quotient space whatever that is. But the question is what is the action of the linear operator that we are talking about. And recall, the linear



operator is nothing, but how x acts, ok. So, let us figure out how does this special polynomial x act on an element of $\frac{V}{W}$ So, that is my question. So, let us answer this question. How does x, the polynomial x act on $\frac{V}{W}$? Ok. So, let us compute. So, let us take x and try to make it act on coset $v + W$. By definition of the action on the quotient space, how how does a ring element act on a coset? It just acts on the representative. So, this just x acting on v, so this x acting on $v + W$ is just $x(v + W)$ ok.

Now, recall x acting on v by definition of the the operator T, x acting on v was just the original operator T acting on the vector v, ok. But now, what does v look like? v is some combination of the vectors e_1 and e_2. So, when I act T on it, Tv is just well its $c_1 \cdot Te_1$; $Te_1 = 0$, $+c_2 \cdot Te_2$ and $Te_2 = e_1$, ok. Now, I have used the definition of T here. I have used the fact that when I apply T to e_1, T acting on e_1 gives me 0. So, this is Te_1 is 0 and T acting on e_2 gives me e_1. So, Te_2 is e_1. So, I have used those two properties of T here. And what that tells me is that the final answer is just $c_2 \cdot e_1$. So, when I apply T to v, I just get a multiple of e_1, ok.

But notice, I am not looking only at Tv, I need to look at the coset $Tv + W$. So, observe my answer here is that x acting on the coset $v + W$ is $Tv + W$ but that is just $c_2 e_1 + W$ ok. But remember W itself is all multiples of e_1, right. So, this this guy actually belongs to W this is just an element of W already. In other words, this is therefore, the same as the 0 coset, ok.

If I have an element, a representative which comes from the sub group that you are quotienting by, then of course, that $+W$ that element $+W$ is the same as $0 + W$ ok. In other words, this is just the the 0 element of the module, ok. In other words, x just acts as 0, x kills all the elements of the quotient space, ok.