Algebra - I

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Lecture 56 [General constructions of submodules]

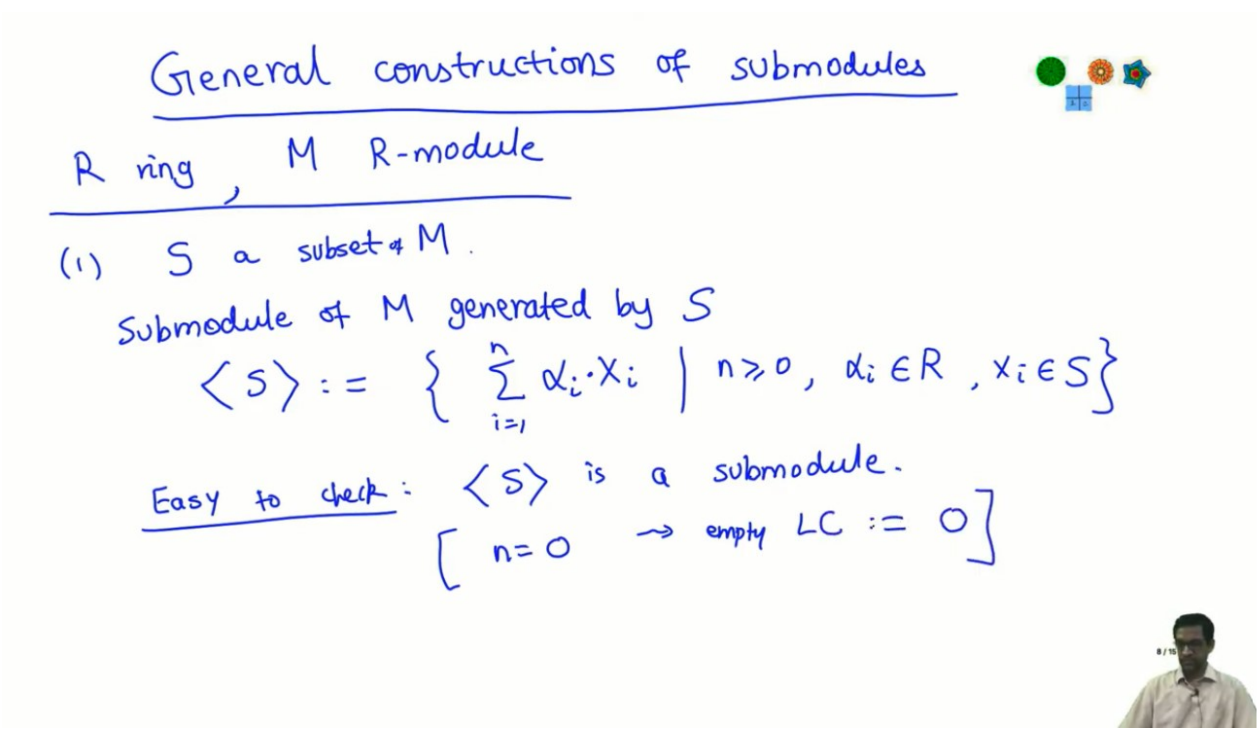
We will talk about some General constructions of submodules. So, what do I mean by that? Suppose I have a R module M. So, let me say R is a ring , M is an R module . Again that means, left R module. Now um here is one thing we can do suppose you take a subset. So, this is suppose so here is the first general sorts of construction that one can make.

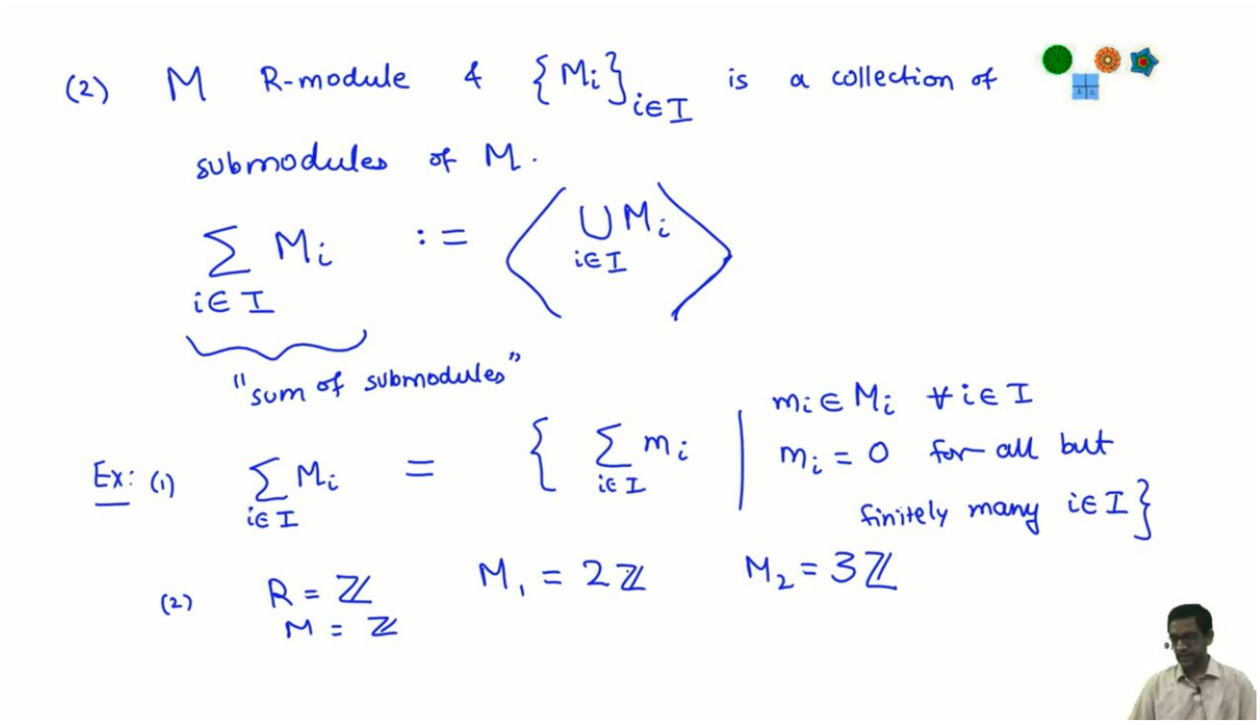
Suppose S is a subset a subset of M ok. No further properties, just a subset of M subset of M, then we can define something called the submodule generated by S. So, this is called the submodule of M generated by S. So, let us give this this notation angled bracket S which is the following.

It is the collection of all the following elements. So, I look at summation let us call it:

$$\langle S\rangle := \{\sum_{i=1}^{n} \alpha_i X_i | n \geq 0, \alpha_i \in R, X_i \in S\}$$

ok. So, this is of course well what is this? This is all finite linear combinations if you will.

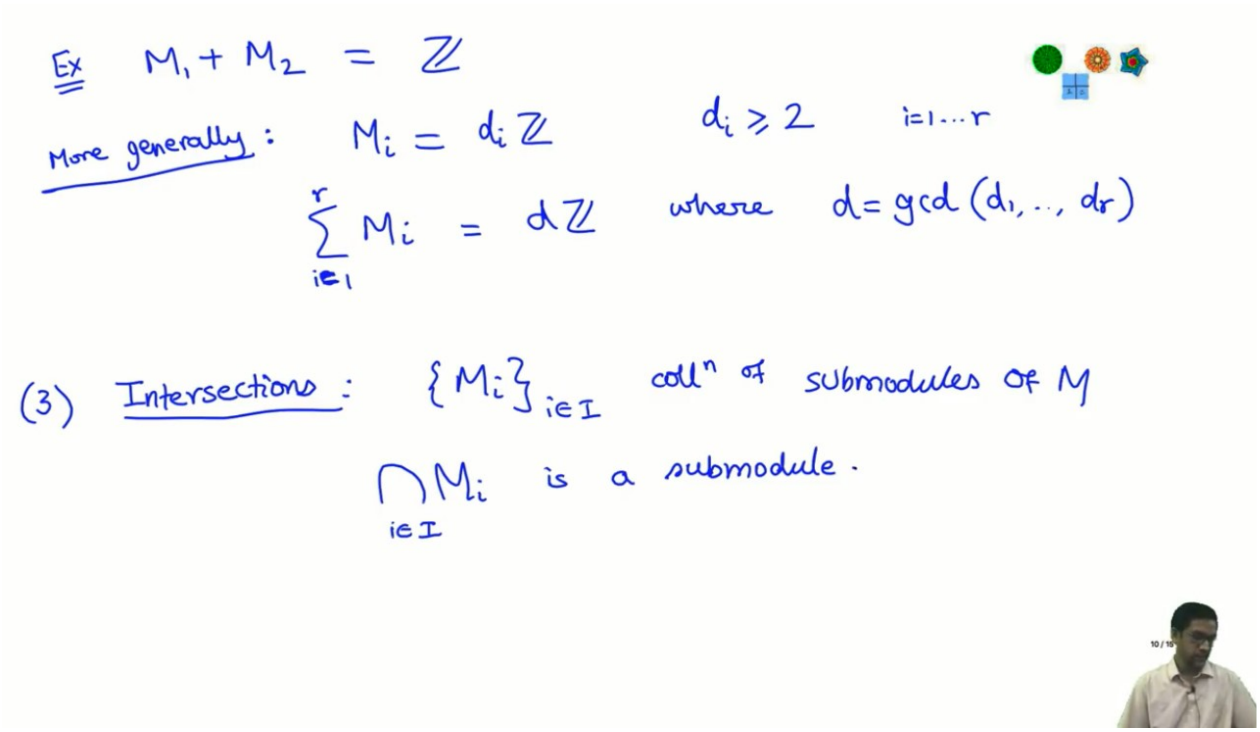




So, this this thing here what I call $\alpha_i X_i$ is the sort of the the scalar multiplication. That is given because M is a module over the ring R ok. So, the sub module is just all what we would call linear combinations, a finite linear combinations of elements of S and this is sort of what you would call as span in a vector space, ok . Now observe that ah. So, it is easy to check that this is in fact a submodule . So, here is the following fact. S is actually a submodule because I mean the the the submodule generated by S, this set here on the right hand side is closed under addition and because you you add two such terms, you again get another linear combination and if you multiply such a linear combination on the left by some element of the ring, you just get some other set of scalar, some coefficients in front that is all .

So, this is a sub module . The other little thing to check um or note is that when I say n greater than or equal to 0, so there is this one vacuous case which also we are taking into account if you wish which is if you know if you put n equal to 0, then it is an empty sum. So, we think of that as just the element 0, ok ok um ah. Now, so, I should just say that if I put if in my definition if I take n equal to 0, so this is just some little care about vacuous definition and so on. I mean you can ignore it n equal to 0 gives rise to the empty linear combination and the empty linear combination is is defined to be 0 ok.

Now, um so we have we have defined the um submodule generated by a set if your ambient module is actually a vector space over a field K, then given a set S, the the submodule generated by S is just what you would call the subspace generated by S, ok. So, that is the first general construction. The second general construction is suppose I give you M which is an R module and if I give you a collection, so um and suppose I give you a collection of submodules. So, M_i i belongs to I. So, this is some indexing set I give you a finite or infinite collection is a collection of submodules .



So, collection of submodules of M , then we can form what is called their sum. So, this is called this is another submodule it is called the sum of M_i i belongs to I. This is defined as follow as follows I take the union of all the M_i's ok.

$$\sum_{i \in I} M_i := \langle \cup_{i \in I} M_i \rangle$$

Now, this is just some subset of my ambient module and I take the submodule generated by this union, ok. So, I have just defined in definition 1. I have told you what the submodule generated by a set S. So, I just apply that definition, ok .

So, this is called the $\sum_{i \in I} M_i$. It is called the sum of the sub modules . That is a definition again. It is a submodule because it is the submodule generated by a set. So, here is a here is a little exercise. Show that the elements of the sum look like the following or prove the following equality that the set of the the $\sum_{i \in I} m_i$ comprises precisely the set of elements of the following kind ah. They look like finite sums .

So, this is I belonging to i, but the I want the sum to be finite. So, M_i should be 0 for all, but finitely many $_i$ for all, but finitely in many $i \in I$ ok and m_i's belong to capital M_i for all I . So, it is finitely supported sums if you wish or finite sums of elements from one from each M_i . So, prove that these two are the same. So, these two definitions if you will, so what we gave as the definition and this alternative description are actually the same set ok. So, that is an easy verification ah. So, let us do another example here.

So, if I take R to be the ring of integers and if I just take two submodules to be say $2\mathbb{Z}$ by which I mean all the even numbers and I take the submodule $3\mathbb{Z}$ which is all the odd numbers, ok. So, so the ring is $\mathbb{Z}$ and the module is also $\mathbb{Z}$ here. So, I am taking the ring thought of as a module over itself.

So, of course submodules as we saw before are just ideals. So, I take the even numbers and the odd and the multiples of 3, they are both sub modules here . And it is interesting to ask what is the sum of these two guys and here is the exercise. Prove that the sum of these two sub modules is in fact the whole module $\mathbb{Z}$ ok and more generally if you have, so here is ok more generally. So, you can prove this using either of the two descriptions. So, more generally if I take submodules of $\mathbb{Z}$ which are all of the following kind various multiple. So, let d_i's be um integers which are at least 2 . So, if I have a let us say a finite collection of submodules, so this is i goes from 1 to some R. So, I take these multiples of d_i's multiples of $d_i \geq 2$ and so on then their sum. So, this is a submodule of $\mathbb{Z}$, $i = 1, .., r$.

Show that this is nothing, but $d\mathbb{Z}$. It is again a submodule of the same kind where d is nothing, but the gcd of all these numbers ok. So, again this is something which is maybe something you have seen before, but just put in the language of submodules other than ideals, ok. So, that is the second general construction which is sums and the third construction is of submodules is intersections.

So, if I take an arbitrary collection as before, so if M_i's are a collection, so this is a collection of submodules of M . Then I can talk about their $\cap_{i \in I} M_i$ which is just the set of all elements which belong to all of them. The usual definition i belongs to I, M_i is also a submodule ok and again the verification is more or less straightforward. So, I will just leave that to you .