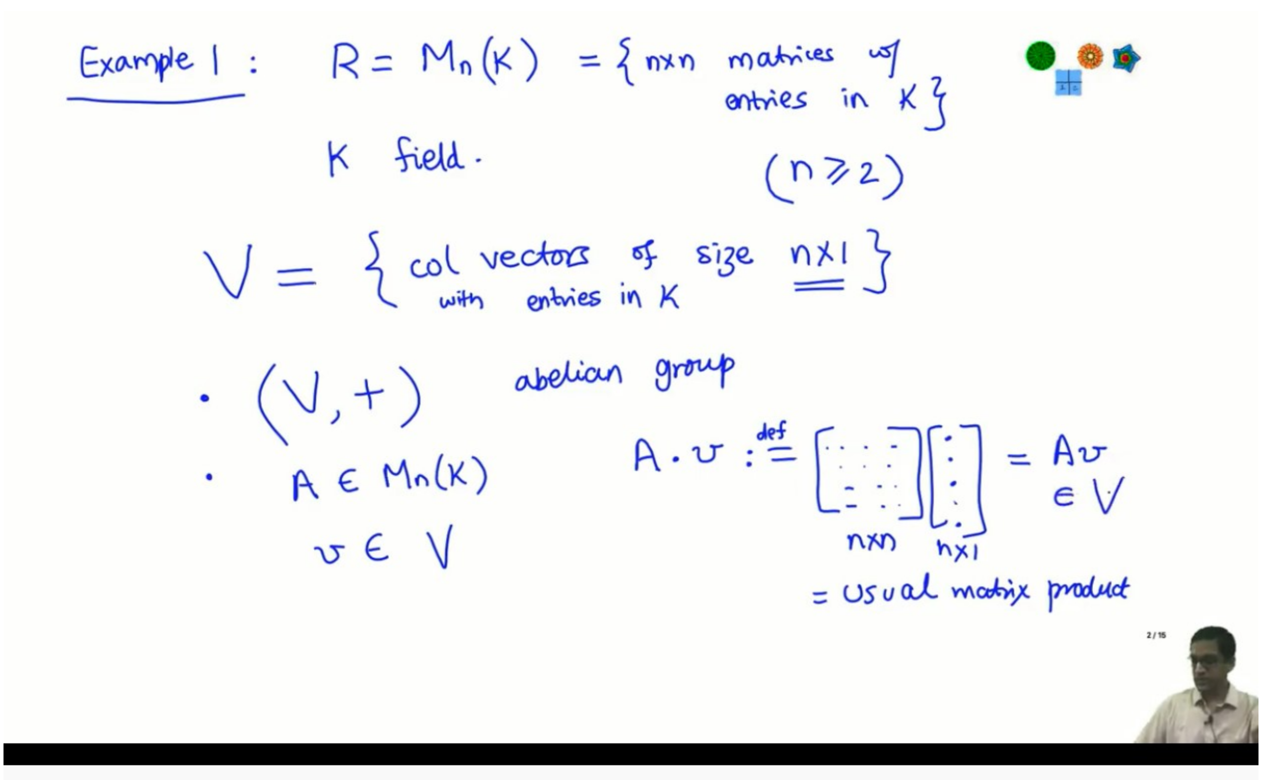
Algebra - I
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

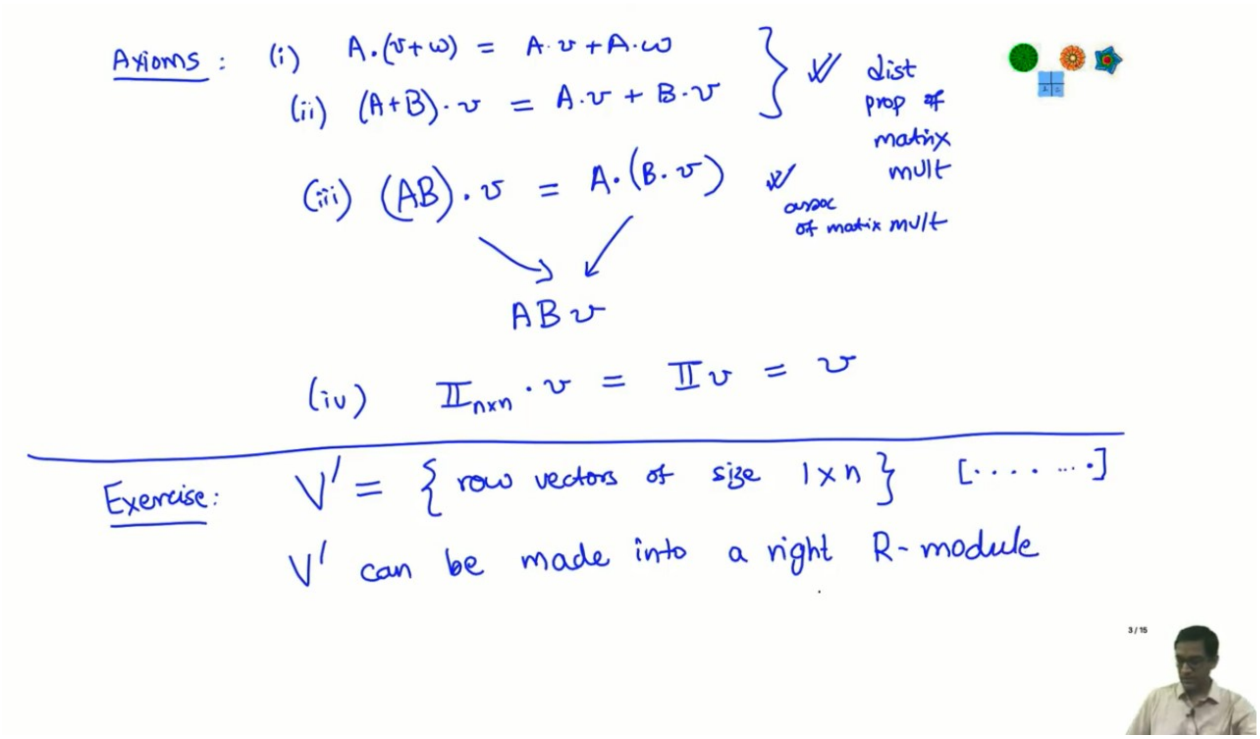
Lecture 54 [Modules: More Examples]

Today, we will talk about ah more examples of Modules. So, far we have seen modules over the ring z it is a abelian groups, modules over the ring K x, which are basically vector spaces together with a linear operator. Now, ah let me consider some non commutative rings this time.

And, when I say module of course, I will always mean left module, but today we will also talk about some right modules ok. So, let us just take example 1, which is the ring example 1 is the ring $R = M_n(K)$ of $n \times n$ matrices with entries in K . So, this is $n \times n$ matrices with entries in the field K . So, here K is a field, entries in K.

So, let me fix a field and of course, if $n \geq 2$ this is not a commutative ring. If, n is 1 this is just the field K itself ok. And, where modules over K are just vector spaces so, I want to look at the same situation for n greater than or equal to 2. And, let me give you some examples of modules . So, here is ah V is the set of all column vectors of size $n \times 1$.





So, let us take the set of all column vectors of size $n \times 1$ and here ah this is still with entries in K ok. So, this is with ah entries in K, the base field ok. So, uh what are column vectors? I I mean they are just matrices of size $n \times n$ ok. Now, this is uh uh well this is firstly, it is an abelian group. So, I want to claim, that I can make this into a module over the ring R ok as follows .

So, observe already I have ah addition of vectors addition of column vectors. So, this is an abelian group definitely, but what I want to do is to define my matrix multiplication . So, let me take a matrix $A \in M_n(K)$ in my ring R, and I should take an element v from my module ok so, I will take a element $v \in V$. It is a column vector, now I have to tell you how to define the scalar multiplication of A with the element V of the module ok. Now, there is only one obvious thing we can do here A being an $n \times n$ matrix . So, A looks like this; this is some $n \times n$ matrix and v is an $n \times 1$ matrix or a column vector right. So, the most obvious thing you can do is to just multiply the two of them ok.

So, Av is just defined so, this is the definition here is the definition. Define A be to be just the product of the $n \times n$ matrix A with the $n \times 1$ matrix V ok. So, usual matrix product if you wish. Now, ah the result is of course, an $n \times 1$ matrix again or a column vector ok . So, what is this give you so, let us just denote it as Av the usual matrix product. The answer of course, is again a column vector of size $n \times 1$. So, ah A matrix of size $n \times 1$; so, that is again an element of v. So, this is the definition of of the scalar multiplication and it is easy to check all the axioms. So, for example, what are the axioms? Ah. So, I claim that this makes V into ah R module.

So, what were the axioms there was distributivity which says if I take $A \cdot (v + w) = A \cdot v + A \cdot w$ or if I change A, look at $(A + B) \cdot v = A \cdot v + B \cdot v$. Now, observe both these are just they follow from the usual distributivity property for matrix multiplication ok. So, these are both ok, because they come from the distributive property of matrix multiplication. Now, axiom iii was if I take a product of 2 matrices A and B and I scalar multiplied with v it should give me the answer of repeated A acting on B acting on v. But, again this is just the associativity of matrix multiplication . So, this just comes from the associativity of matrix multiplication .

Because, both sides are just equal to you know both both sides are just the product take the product of the 3 matrices AB and v ok. And, finally, the identity so, the the matrix ring m n are has the identity matrix, $I_{n\times n}$. And, the identity matrix acting on a vector v is by definition they are they are usual matrix product, but of course, if you multiply identity matrix with anything you get back to it ok .

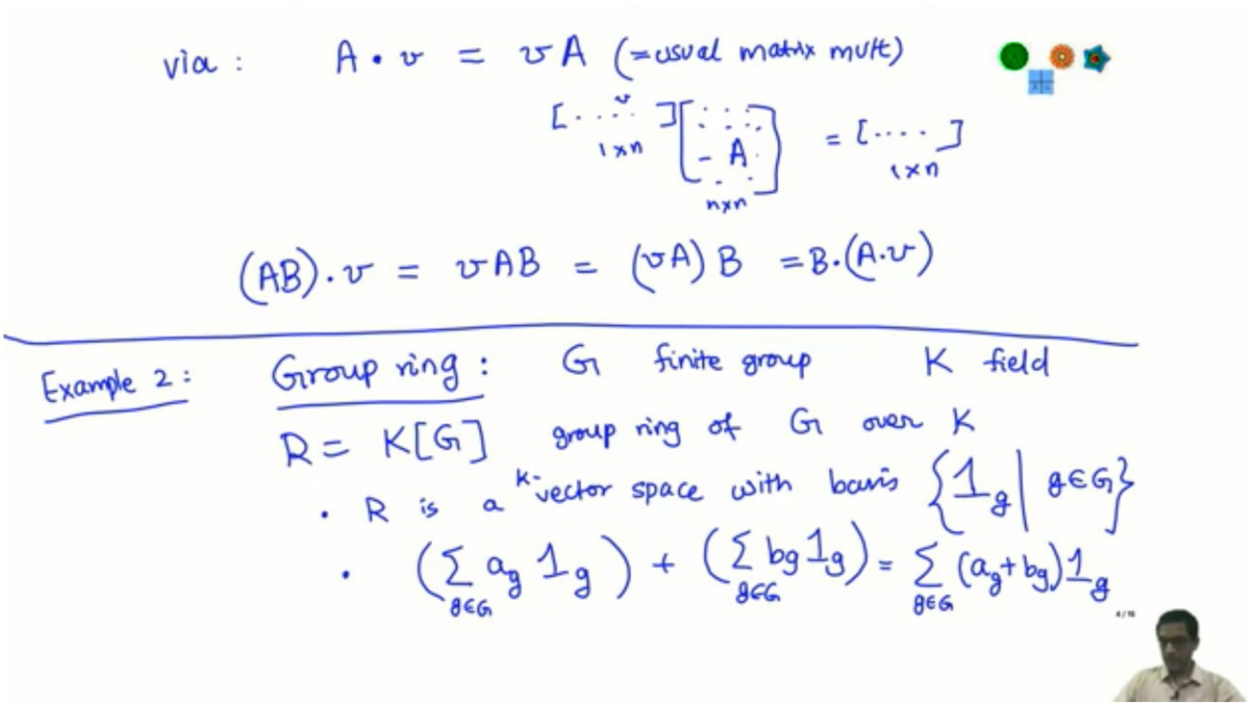
So, all the axioms are satisfied. So, here is an example of a left module. So, this is now the space V of $n \times 1$ column vectors is a left module over the ring M_n ah M_n of K ok. And, in fact, we can just tweak this example just a little bit to construct a right module. So, here is an exercise for you. So, instead of V let us take V' to be the set of all row vectors of size n ok, row vectors of size n by which I mean ah matrices of size 1 cross n ok . So, I mean I should either call them row vectors of size n or matrices of size 1 cross n so, so anyway. So, this is so, you know what these look like a typical element is a row vector like this. That is a typical element of V' and now ah the claim is that V' can be made into a right module over this ring R of matrices yeah.

How? Via the following operation I take a matrix A and I should tell you how to do this scalar multiplication, well the definition is this just look at ah the the row vector v . So, now v is a $1 \times n$ matrix and I multiply it on the right by the matrix A, which is an $n \times m$. So, this is V and this is my matrix A ok.

I multiply it on the right by A. And, the answer is again the usual I mean by this I just mean the usual matrix multiplication of these 2 of V with A; and, the answer again is a row vector remember $1 \times n$ into $n \times n$, the answer will again be $A1 \times n$ matrix right, it will again be a row ok . Now, this is a right module not a left module, because if we take a product of 2 matrices AB and I try to act it upon B according to this definition. Then, this is just the ah the matrix product v into A into B, but that is just the same as what you would get if you first multiplied v with A and then multiplied the answer with B . So, in other words this is what you would get if you took B you first acted on it by A or scalar multiplied it by A and then scalar multiplied it by v ok .

So, that is exactly the definition of a right module . So, row vectors with this ah definition form a right module, column vectors with the usual matrix multiplication form a left module ok. So, ah time for another example. So, let us take example 2. Now, I will take the the ring R to be a group ring. So, recall from the lectures on rings, that the group ring of a finite group. So, let us take G to be a finite group. It is group ring is defined as follows it is the um well it is it is firstly, it is a vector space . So, I need a field as well. So, K is a field. So, R which is called the group ring of the group G over the field K right $K[G]$. So, this is the group ring of the group G over the field K was defined as follows ah. Firstly, it is a vector space. So, how do you define R, R is a vector space, it is a vector space over the field K, it is a K vector space .

So, you start with the K vector space with basis given by some elements . So, we were ah labeling them as $\{1_g | g \in G\}$. So, I take these finitely many basis elements 1_g as g runs over so this is the set, this is the basis ok . And, ah of course, that is just in some sense I only told you how to do addition now. So, what is the addition here? If I take so, what is a

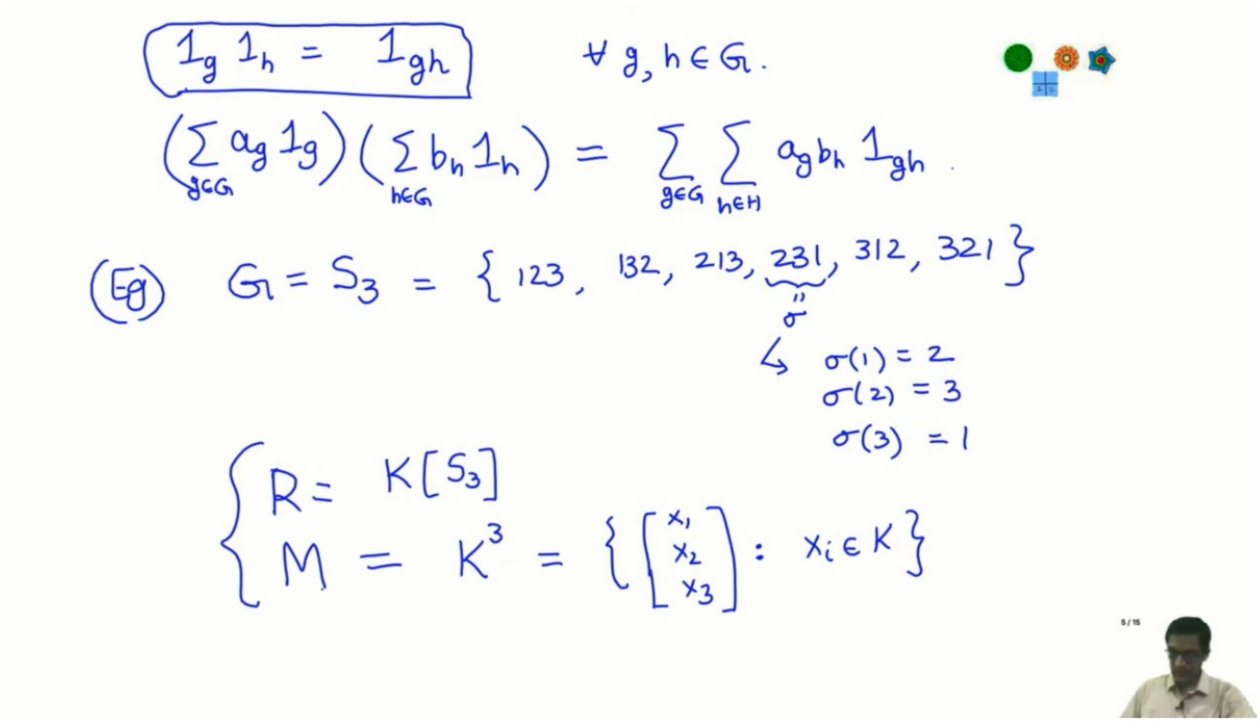


typical element firstly? A typical element of R looks like summation $\sum_{g \in G} a_g 1_g$ right; it is a linear combination of the basis. So, this is a typical element of R. So, let us call it g in G and to this if we add another element $\sum_{g \in G} b_g 1_g$ in G . Because, the answer is just because these form a basis, I can just add them like this $1_g, g \in G$ ok . So, that is the operation of addition in this group ring and recall multiplication can be defined as follows . Ah it is enough to sort of define it on the basis elements. If I take the the product of the element 1_g with the element 1_h just gives me the element ah 1_{gh} corresponding to the product gh .

So, this was the definition of the multiplication of course, to define it in full you will also have to say what how you multiply 2 linear combinations . So, I take ah linear combination like this, and another linear combination like this, $h \in G$, $g \in G$. Now, I I I sort of just use this rule above and and compute it by linearly in other words I just expand it out in full sum over all g in G, sum over all $\sum_{g \in G} \sum_{h \in G} = a_g b_h 1_{gh}$. So, this is so, I am just recalling the definition from the the previous lectures on rings . So, this is how the group algebra or the group ring over K, it is defined ok. And as was checked there it is it is a ring under this these operations . So, let us do one particular example here. Let us take a simple example of a group ring. Let us take the group S_3 , which is the symmetric group on 3 letters . So, in ah 1 line notation recall the elements look like ah this is $\{123, 132, 213, 231, 312, 321\}$ ok, these are just the elements written in 1 line notation .

So, example if I pick ah say if I call this as my element $\sigma = 231$. So, let σ denote the 1 line notation permutation 231, this means that it is a function . So, permutations are functions from the set 1 2 3 to the set 1 2 3 bijective functions, σ maps $1 \to 2$, it is a first entry σ maps $2 \to 3$ and $3 \to 1$ so, 231 . So, this is what σ does? ok. So, σ is a function from the set 1 2 3 to itself ok .

So, that was our definition ah of the symmetric group and and ah permutations. Now, let us take the ring R to be the ring the group ring of S_3 over this field K ok. So, $K[S_3]$ is our



ring R . So, recall what does this mean, it means I have a basis uh as a vector space, this is a 6 dimensional vector space it is got basis indexed by you know I put $1_{123}, 1_{132}, 1_{213}$ etcetera ok .

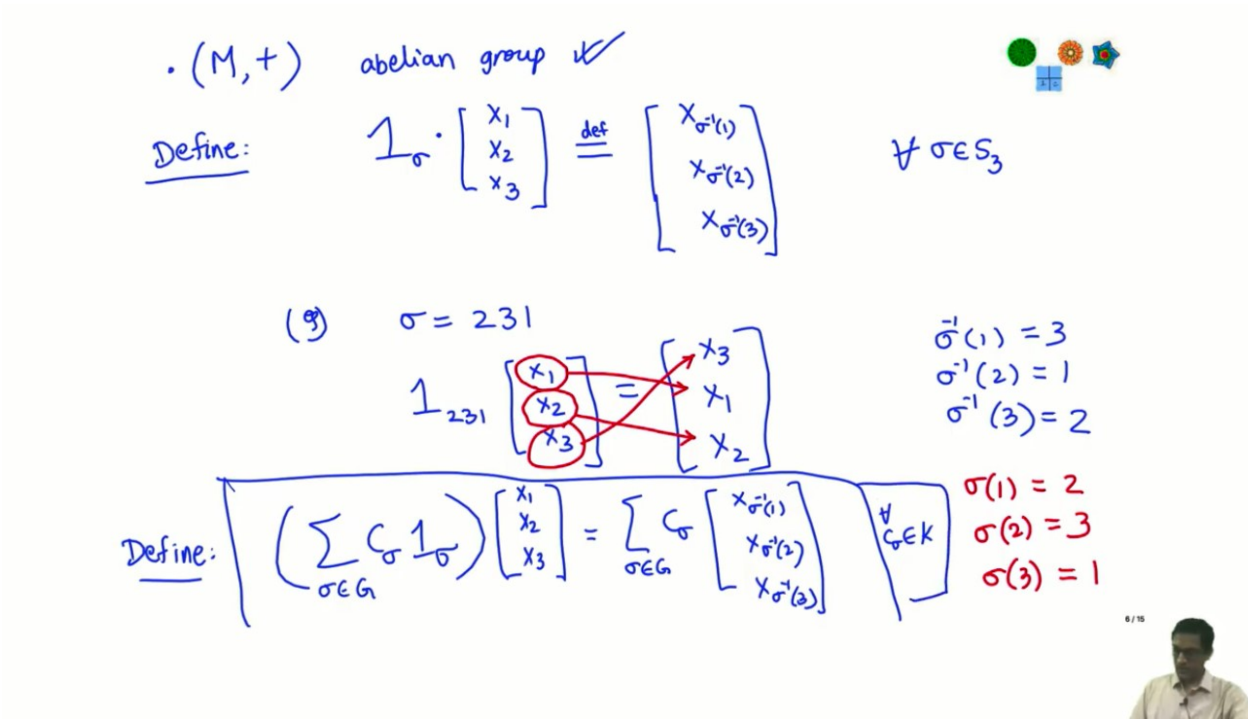
Each each element of S_3 to each element that corresponds a basis of this space, and the multiplication is the key thing here it is 1_g into 1_h multiplies in this particular way ok. So, let us construct a left module for for this ring R ok. So, I am going to construct the following left module. So, let me tell you what this space M is going to be, M is just a space K^3 ok; by which I mean, ah the the 3 dimensional vector space over K, which I will think of as column vectors of size 3 .

So, let us think of K^3 to be the following set, consisting of all column vectors $X_1X_2X_3$, where xi s are all elements of K ok. So, this is a 3 dimensional vector space if you wish over over K, over the field K. Now, I claim that this set M can be made into an R module ok . So, I have to define the ah the R the scalar multiplication .

Observe M already has an addition right; this is just all column vectors it is in fact, a vector space over K, I certainly has an addition ok under that addition it is already an abelian group . So, that part is ok it is only the scalar multiplication that we have to define. So, let us define it in this in the following manner . So, we say that so, we now define . So, I if I so, I have to tell you given an element $X_1X_2X_3$ of my ah module M of my space M, I have to tell you how to scalar multiply it with every element of my ring R ok .

Now, ah let me tell you for a start how to scalar multiply it with a particular basis element of my ring R. This this is special element 1_g for all g . So, let me take some σ ok. So, let me take σ to be some element of my group S_3. For this element σ I will now tell you how to scalar multiply the vector $X_1X_2X_3$ by the ring element 1_σ ok .

So, what is this? . So, here is the definition this is defined as follows it is x. So, it it permutes the Xs just like whatever σ would do ok. So, this is $X_{\sigma^{-1}(1)}$ and ah we will just



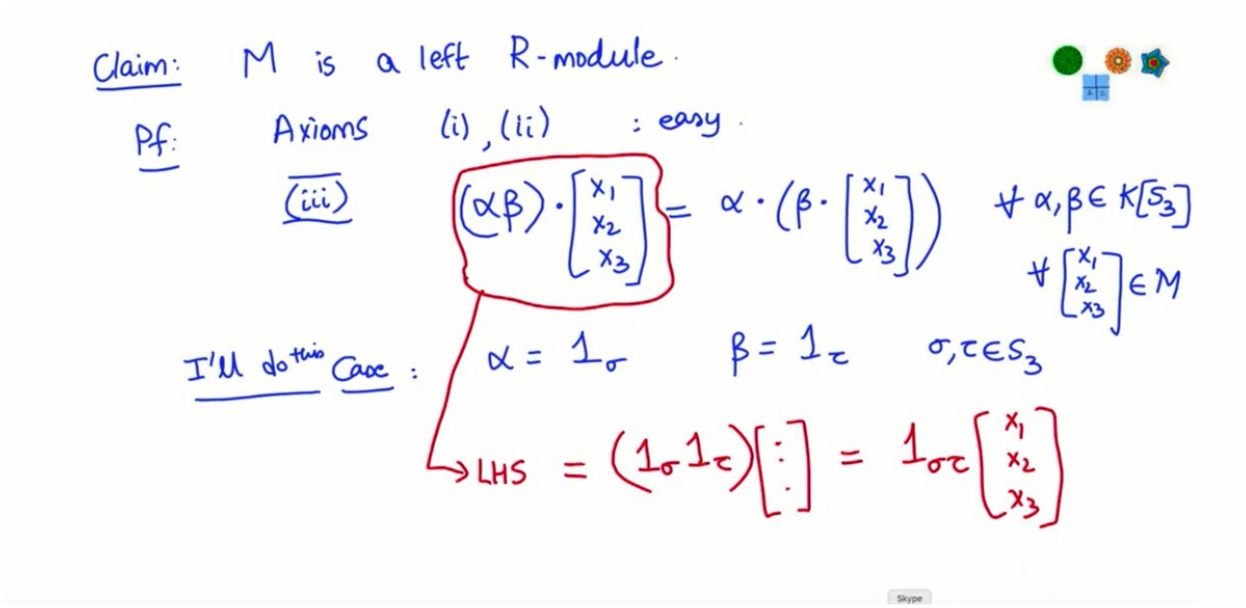
see why the inverses are required in a second. So, I put $X_{\sigma^{-1}(2)}$ $X_{\sigma^{-1}(3)}$ ok. So, this is my ah definition each vector $X_1X_2X_3$ column vector is mapped to the vector $X_{\sigma^{-1}(1)}$, $X_{\sigma^{-1}(2)}$, $X_{\sigma^{-1}(3)}$.

So, let us let us do an example, ah this is for any σ . So, maybe I should yeah so, this is for all $\sigma \in S_3$. Let us take that particular σ that we we looked at in the ah the previous one take this example ok. So, if I take this particular $\sigma = 231$ example, if $\sigma = 231$ what will 1_{231}, do when it acts on $X_1X_2X_3$. Well the answer is it will map it according to this formula. So, recall what was σ doing σ was mapping ah $123 \to 231$ ok .

So, in particular; that means, if I take the inverse of σ , the inverse map, then σ^{-1} will map $1 \to 3$ ok . So, let us write that down what does σ^{-1} do? σ^{-1} maps $1 \to 3$ ok. So, let us go back up again $\sigma^{-1}(2)$ is upstairs here is 1 $\sigma^{-1}(2) = 1$ and σ^{-1} of what is left $\sigma^{-1}(3) = 2$ ok. So, that is what it does? . So, let us just see what; that means. So, this is X_3 on top X_1 X_2 ok. So, the the element 1_{231} acting on $X_1X_2X_3$ gives me $X_3X_1X_2$ ok . So, you see what it has done, it has taken X_1 the element X_1 here, which is which occurs in the first place and it has moved that X_1 from the first place to the second place ok . So, remember that is that is how σ acts right.

The action of σ let us write σ here σ maps the number $1 \to 2$, maps the number $2 \to 3$, and $3 \to 1$ right . So, now, this this action here you should remember it as as follows it maps it takes the element in the first position and moves it to the second position ok. Now, $\sigma(2) = 3$. So, the X_2 which is in the second position is now moved to the third position. And, 3 which is in the third position X_3 gets moved to the first position ok .

So, this is what ah this this particular element 1_{231} does more generally if I take 1_σ acting on $X_1X_2X_3$ it will act in the same way here. So, I need to put the inverses here for a reason we we will see soon, it is to make it a left module rather than a right module. But, the



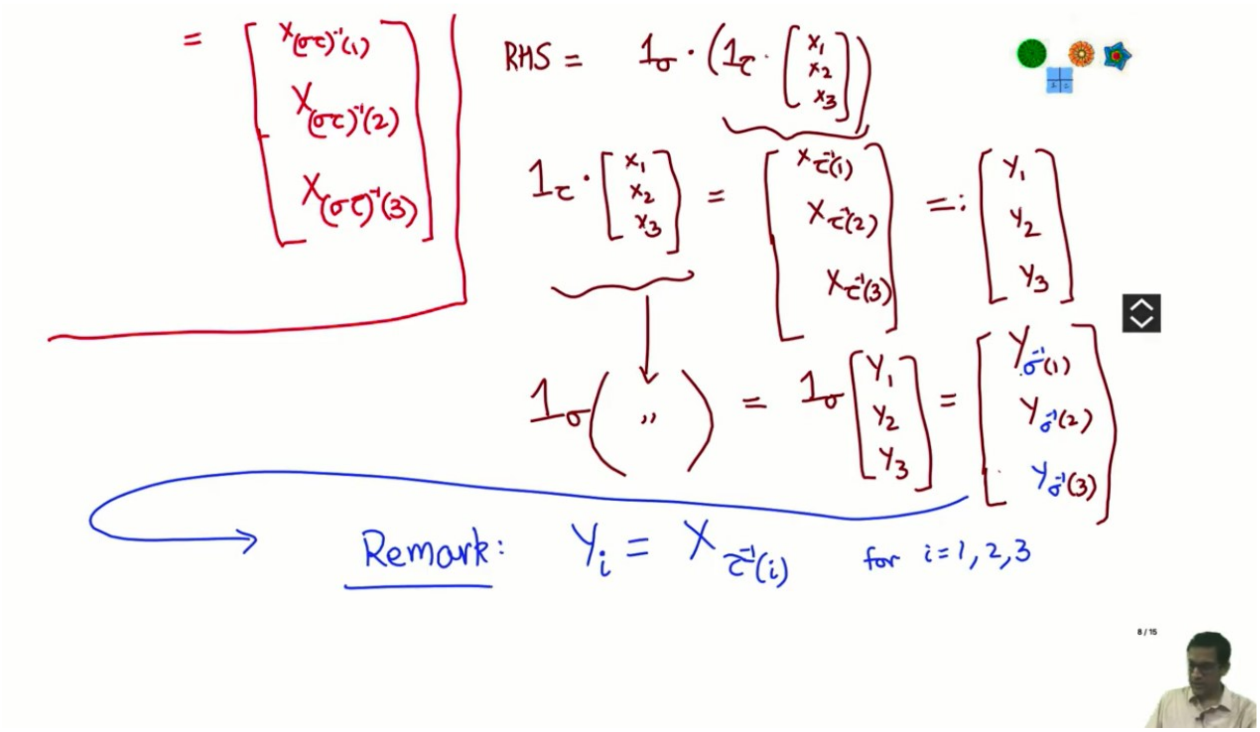
basic idea is that it permutes these these 3 components $X_1X_2X_3$ sort of the same way that σ permutes the numbers 1 2 1 3 ok .

So, I have defined this this action of course, I have only defined it on the basis elements ah it is sort of clear what I need to do in general. So, more generally I need to complete the definition by saying, if I have a linear combination of these guys $\sum_{\sigma\in G} C_\sigma 1_\sigma$ if we wish, σ running over G, then that acts on a vector $X_1X_2X_3$ I just expand this summation here $X_1X_2X_3$. The answer is just summation σ , C_σ and 1_σ acting on this vector which I know is $\sigma^{-1}(1)$, $\sigma^{-1}(2)$, $\sigma^{-1}(3)$ ok. So, I I complete the definition in this way, this is the full full fledged definition .

So, this is for C_σ's . So, what are the C_σ's, here C_σ's are just ah elements of K. So, they are just for all constants for all elements $C_\sigma \in K$ ok and for all $X_1X_2X_3$ also in K. So, this is also for all $X_1X_2X_3$ running over K ok. So, this is the full full definition of the the scalar multiplication . So, now, we need to check that this is a this is in fact, a module ok, which means I need to check all the axioms again. So, let us check that claim this makes M into a left R module ok. And, let us prove this this just involves checking all the axioms ah and axioms i and ii as usual are rather easy. So, I will leave that for you to check just the distributivity properties .

So, let me check axioms 3 and 4. So, axiom 3 is important, that is the one which ensures it is a left module, not a right module. So, let us compute. So, suppose i i so, what do we need to check for axiom iii, I need to check that if I take say α acting on sorry the product α β acting on a vector should give me the same answer, as ah successive actions α β acting on an element $X_1X_2X_3$.

Should give me the same answer as α acting on β acting on the element $X_1X_2X_3$ and this should be true for all α β coming from a ring. The ring here remember is the group ring of S_3 and for all elements $X_1X_2X_3$ coming from my module M ok. Now, in this case of course, I should in general pick α β to be any elements of ks 3 meaning any linear combinations. So, I should pick for example, α to be something of this kind β again to be some such linear



combination, but ah it is sort of enough to check it on on just the basis elements for a start and then you know.

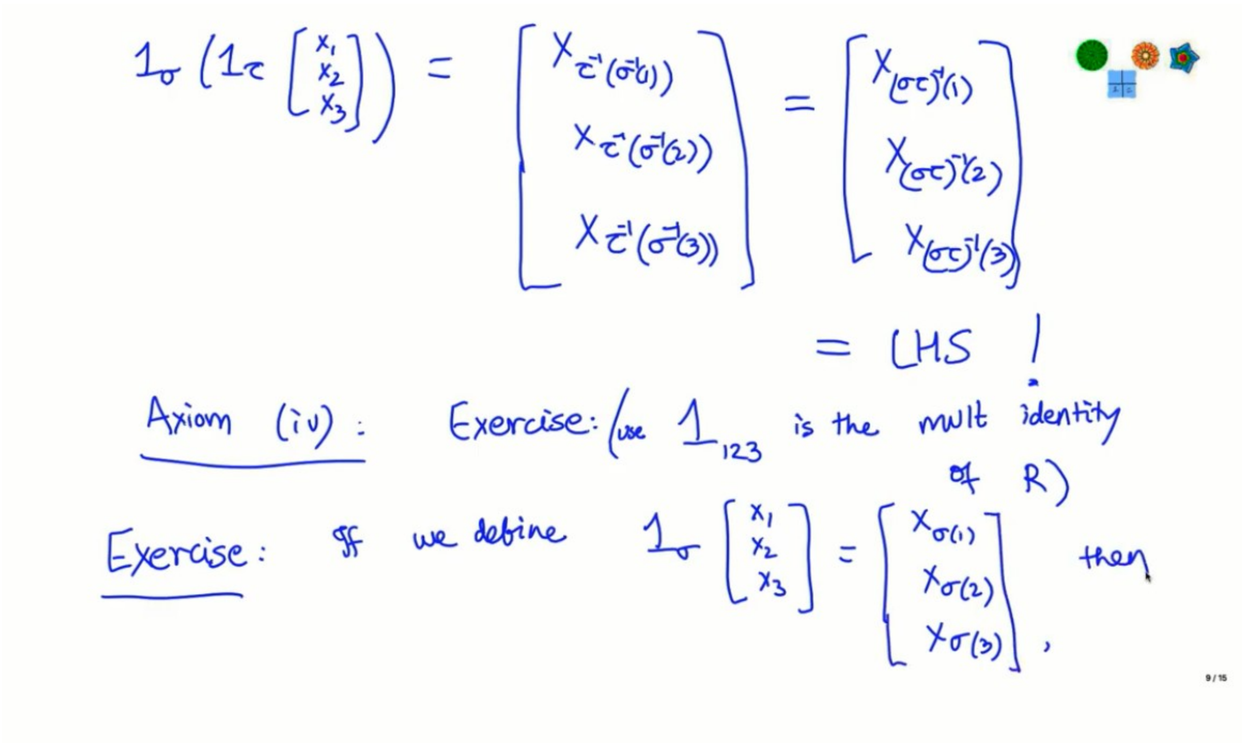
If it works nicely there then taking sums will usually work out nicely. So, let us just so, let me just check it on the basis elements and allow you to check the the entire ah linear combination. So, let me just take this simple case. So, I will just do it in this simple case, that $\alpha = 1_\sigma$ ok. It is just a single basis element $\beta = 1_\tau$ ok, where what are σ and τ there are some particular elements of S_3 ok.

So, let me check ah so, I will just do this case I will do this case and leave the full summation to you. So, let us compute the left hand side ok. So, what is the left hand side here? It is this this guy so, let us compute this the left hand side, it turns out to be $\alpha\ \beta$ times this and what is $\alpha\ \beta$ here it is $1_\sigma\ 1_\tau$ that is the product $\alpha\ \beta$ acting on this vector $X_1X_2X_3$ ok .

But, $1_\sigma\ 1_\tau$ by definition is $1_{\sigma\tau}$ acting on this vector $X_1X_2X_3$. And, that by our definition is just you have to change the Xs like this $X_{\sigma\tau^{-1}}(1)$, $X_{\sigma\tau^{-1}}(2)$, the last is X ah ok maybe I should just write it (Refer Time: 23:46) . So, let me just write it on the next page. So, this is the same as the vector $X_{\sigma\tau^{-1}}(1)$, $X_{\sigma\tau^{-1}}(2)$, $X_{\sigma\tau^{-1}}(3)$. So, that is my column vector. Now, let us compute so that is 1 1 answer. So, we have found 1 answer. So, let us do the right hand side as well. So, what is the right hand side in this case?

So, recall the right hand side is 1_σ followed by 1_τ acting on $X_1X_2X_3$ ok. So, we have to do this little more carefully. So, let me let me do it slowly if I take 1_τ first. So, let me compute what is on the inside I take 1_τ and I act on $X_1X_2X_3$.

Now, when I do that by definition this is the vector $X_{\tau^{-1}}(1)$, $X_{\tau^{-1}}(2)$, $X_{\tau^{-1}}(3)$ ok. So, that gives me some some new element. So, let me call this element something let me give it a name, let me call it $y_1y_2y_3$. So, the 3 numbers the 3 components are $y_1y_2y_3$. So, this is



just definition calling like this ok. Now, let us act 1_σ further on this, now I take 1_σ and I try to act on this this element here ok .

So, what is that? That is just 1_σ acting on so, it is since I have called it $y_1 y_2 y_3$ for short let me just use that notation 1 σ acting on $y_1 y_2 y_3$, which by definition is it moves whatever in the first component . So, it is just y τ inverse 1, y τ inverse 2, y τ inverse 3 . So, that is my ah 1 σ acting on 1 τ acting on $X_1 X_2 X_3$ ok.

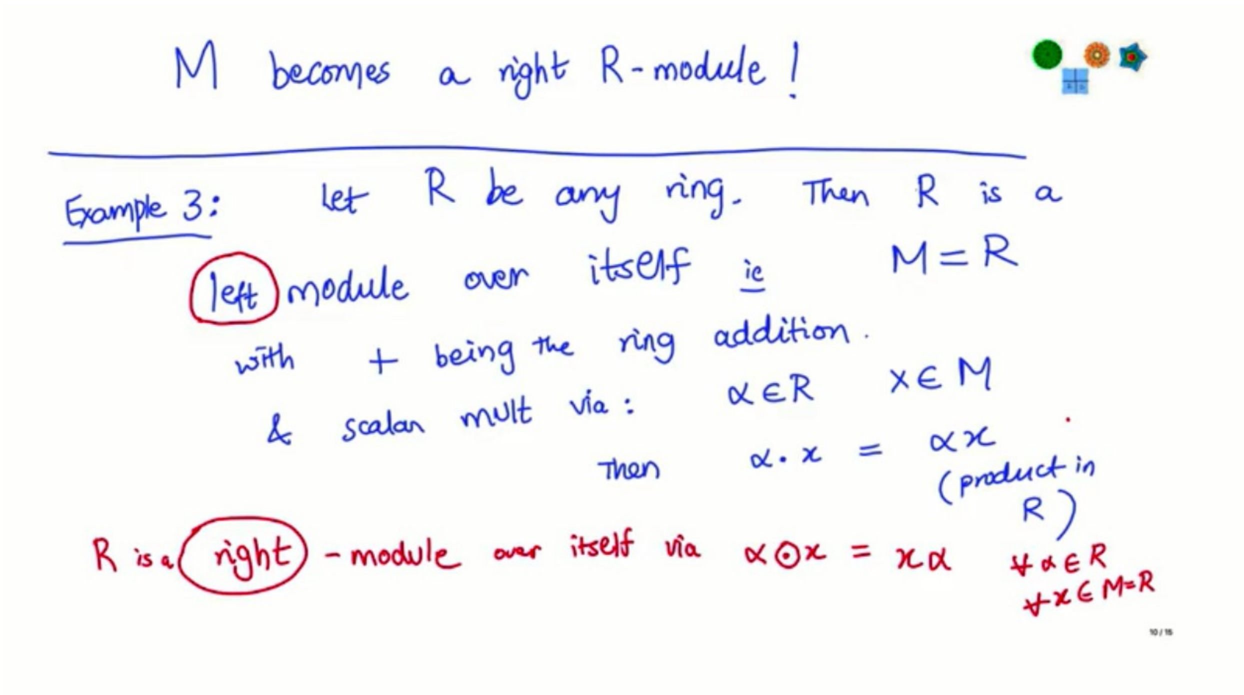
Now, let us unravel the last step let us look at what we got as the answer . So, let us look at this answer here and see what that becomes. Now, remember y τ inverse 1 . So, what is y of anything ah y i is same as X_τ inverse of i ok. So, when I say y of τ inverse 1 ah.

So, oh sorry I I made these τs instead of σ s . So, sorry recall I am I am hitting it with a 1_σ . So, sorry these should all be σ s instead of τs . So, let us make that change. So, this should be actually $y_{\sigma^{-1}(1)}$, $y_{\sigma^{-1}(2)}$, $y_{\sigma^{-1}(3)}$. Now, ah what is y of of any any index is just from here I observe it is y uh it is X_τ inverse of that index ok. So, so let us let me just ah remark, let us observe that, if I want to compute y_i , I just have to take $X_{\tau^{-1}(i)}$.

So, this is the bit that 1 has to be little careful about X_τ inverse i X_τ inverse i ok. So, let us this. So, this is for i equals 1 2 and 3 ok. So, let us plug that in into the next ah ah page. So, I conclude that if I take 1_σ and then act it on 1_τ on $X_1 X_2 X_3$. The answer is ah X, let us go back up, it is $y_{\sigma^{-1}(1)}$, but $y_{\sigma^{-1}(1)}$ is $X_{\tau^{-1}\sigma^{-1}(1)}$ ok .

So, it is $X_{\tau^{-1}}$ acting on $\sigma^{-1}(1)$, same thing $X_{\tau^{-1}}$ acting on $\sigma^{-1}(2)$, and this becomes $X_{\tau^{-1}}$ acting on $\sigma^{-1}(3)$ ok. So, what does that mean finally, I conclude that, this is the same as X .

So, $\tau^{-1}\sigma^{-1}$ remember it is just $\sigma\tau^{-1}$ the whole inverse, $X_{\sigma\tau^{-1}}$ the whole inverse and $X_\sigma \tau$ the whole inverse 3 ok. So, that is the same as what we got on the left hand side ok . So, this proves that axiom 3 holds in other words it is a it is a left module ah axiom 4 is is again



easy. So, let me leave axiom ah iv as an exercise . So, you just go to check what the identity element is recall the identity element of ah the this ring R is just the element which is called 1_I the identity ok. So, exercise is use the fact that this is the multiplicative identity of this ring ok .

So, the key point is that this this inverse is the is the little twist here that, ah that is required to make it work out right. And, little exercise again, if you did not put an inverse there exercise, if we define the action as follows 1_σ acting on $X_1X_2X_3$ to be the same formula without inverses $X_{\sigma(1)}$, $X_{\sigma(2)}$, and $X_{\sigma(3)}$. If you did this then what happens well then M becomes a right module ok; so, that is the that is the interesting bit here ok, excellent and my final example for now is ah sort of a general one. So, this is if uh let R be any ring, let R be any ring, then R actually is a is a left module over itself in other words i i e .

So, what do I mean I I can take my module to be my ring itself to be my underlying set of the ring ah with what addition do I take? Well I take the same addition of the ring itself, with ah + being the ring addition . So, I I just think of the ring under addition as an abelian group ok. So, that is my underlying abelian group of M. And, I have to now tell you what scalar multiplication is and scalar multiplication is defined as follows . Well I just do left multiplication and scalar multiplication is defined via the following, if I take an element $\alpha \in R$ and if I take an element $X \in M$. So, remember M is actually R again then I define the scalar multiplication $\alpha \cdot X$ to be just the product of these two elements. So, this is now the product in the ring R just the usual multiplication, observe both αX are actually elements of R .

So, I am just using two different notations for R to say you know I think of one as being the sort of the scalars and the other as being the ah the module itself . So, that is the first ah first thing one can do we can think of a ring as left module over itself. And, again just like we did in the two previous examples you can change left to right in the following way.

So, this is a left module, but I can as well define an alternate right module structure as follows . So, I can change left to right, if I do the following, I define this operation $\alpha \odot x$ differently. I now define $\alpha \odot x$ ah it becomes a right module. So, R is a right module over itself via well the same addition, but multiplication is now multiplication on the right. So, this is for all for all αx in R which is same as M ok. So, under the right multiplication operation, if I take the scalar α and I multiply it on the right of x. Then, that gives R the structure of a right module ok. And, if I multiply the scalar α on the left, it gives me the structure of a left module .

So, R is in fact, a module both the left and the right module of itself ok. So, we have seen in some sense 3 examples of modules which are well 3 examples of left and 3 examples of right modules. A remark on notation so, we keep talking about left and right modules. Now, ah recall when we say that something is a right module right module over a ring R just means that ah I have elements α for all $\alpha \in R$. So, M is a right R module just means that ah I have a notion of so, there is a scalar multiplication there is a notion of scalar multiplication, of scalar multiplication. And, we denote this scalar multiplication by; so our notation for the scalar multiplication is like this right.

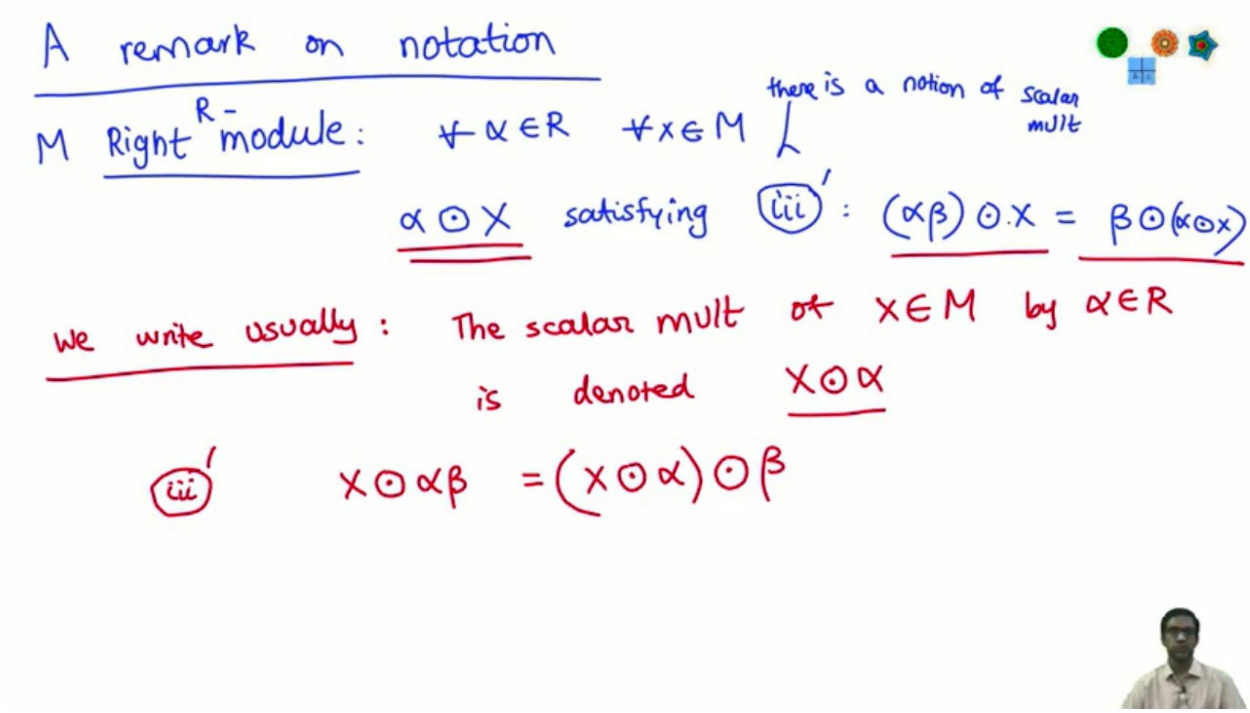
So, we all we always say, let us say $X \odot \alpha$ is ah the scalar multiplication and it satisfies the the well the other axioms are the same this axiom iii dash that we talked about, that is the key here, which is if I multiply α and β and then I scalar multiply it with X. Then the answer is β, scalar multiplying α, scalar multiplying X right that is the order .

Now, ah looking especially at the examples that we have done this time the the 3 examples, especially the last one, where R acts on itself on the right by right multiplication in some sense. Ah. An alternate notation so, here is a more conventional and alternate notation for this scalar multiplication. So, instead of so, if you have a right module then we usually write the scalar on the right of the vector ok .

So, we write more conventionally usually in the following order. So, this is only for right modules. So, if ah M is a right R module, then the scalar multiplication, the scalar multiplication by α ok, scalar multiplication of $X \odot \alpha$ if you wish, scalar multiplication of the element $X \in M$ by the scalar $\alpha \in R$ is denoted as follows. We say X scalar multiplication α well α is thought of as being a scalar that multiplies on the right ok . Now, ah why is this it is it is slightly strange, if you, if you have used to see in scalars you know thought of as being multiplied on the left.

Now, ah why is this alternate notation somewhat better, because it captures this axiom iii prime notationally in a more pleasing way, which is that $X \odot (\alpha\beta)$ is well what does this say . It says, if you want to scalar multiply $X \odot (\alpha\beta)$ you must first multiply by the scalar α, then by the scalar β.

So, this is in in our new notation it just becomes this X. First you scalar multiplied by α, then by β . So, in some sense it flows in the same order ok . So, you will usually see right modules written in this way, where the scalars are thought of as sort of acting from the right ok . But, of course, I mean it is it is it is just a question of notation there is nothing fundamentally ah new happening here. You could if you choose just choose to do it the way we did it that the scalars act on the left , but the composition rule is is sort of switched that when α β, tries to act on X.



You must first act α then β ok, or you could choose to just use this usual notational convention, where the scalars are thought of as somehow coming from the right. And, in which case it is easier to remember the the order in which it flows ok .