Algebra - I
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Let us solve some problems together.

Problem 1.

Consider the matrices $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}, i=\begin{pmatrix}0&-1\\1&0\end{pmatrix}, j=\begin{pmatrix}0&i\\i&0\end{pmatrix}, k=\begin{pmatrix}-i&0\\0&i\end{pmatrix}$.

Show that $Q=\{\pm e,\pm i,\pm j,\pm k\}$ is a group under matrix multiplication.

I would urge you all to think about this problem. Firstly, for a group we need to know that we have a binary operation. So, we need to check that multiplication of matrices is a binary operation from $Q\times Q\rightarrow Q$. So, we need to check that if you take any two matrices in Q and multiply them you once again get a matrix in Q.

So, firstly, note that it is enough to check this for any two of e, i, j, and k.

$row\times column$	e	i	j	k
e	e	i	j	k
i	i	-e	k	-j
j	j	-k	-e	i
k	k	j	-i	-e

From this table it follows that Q is **closed** under multiplication.
It is **associative** since multiplication of matrices is associative.
The **identity** element of Q is e.
We need to check the axiom of the **inverse**. From this table we see that the inverse of i is -i, the inverse of j and k likewise are -j and -k respectively.

These relations, which may be familiar to you, are the relations for quaternions. So, this group Q is called the quaternion group.
Let us move onto problem 2.

Problem 2.
What are the orders of the elements of Q? Let us construct a table whose entries are the powers of each of the elements e,i,j,k until these powers yield the identity.

	1	2	3	4
e	e			
i	i	-e	-i	e
j	j	-e	-j	e
k	k	-e	-k	e

Observe that the orders of -i,-j and -k are the same as the orders of i,j and k respectively. Also note that the order of -e is 2 since $(-e)^2=e$.

Problem 3.
What are the conjugacy classes in Q?
Let us take the element e: clearly $ueu^{-1}=e,$ for any $u\in Q$. So **{e}** is a conjugacy class.
By the same token **{-e}** is a conjugacy class as well.
Now, let us conjugate i by the other elements of the group. Conjugation by the $\pm e$ gives us back i. Consider $j^{-1}ij$: we know that $j^{-1}=-j$, and $-j(ij)=-jk=-i.$ Similarly $k^{-1}ik=i.$
Conjugating by -j: $(-j)^{-1}i(-j)=-i,$ and $i^{-1}ii=i.$ Clearly conjugating i by any element in Q results in $\pm i.$ So **{i,-i}** is a conjugacy class.

Similarly it is easily verified that **{-j,j}** and **{-k,k}** are conjugacy classes. Q has 5 conjugacy classes, 3 of size 2 and 2 of size 1.

Problem 4.

What are the subgroups of Q?

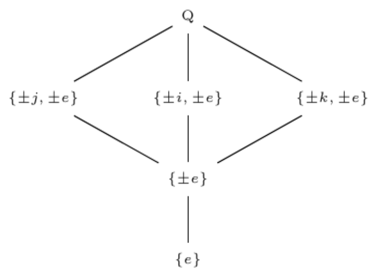
In any group there is always 1 subgroup which is **{e}**.

Next consider **{e,-e}**. This is closed under multiplication and contains the inverses of all its elements and is thus a subgroup.

Consider a subgroup that contains i. Then it must also contain all powers of i. In particular it must contain $i^2=-e, i^3=-i$, and of course the identity. This gives us the subgroup **{e, i, -i, -e}**.

Similarly we have the subgroups **{e,j,-j,-e}** and **{e,k,-k,-e}**.

Note that a subgroup containing any one of i, j,k must also contain the negative of that element, and also contain -e. Suppose a subgroup contains at least two of i,j,k: By multiplication it also contains the third element. It also thus contains the negatives of i,j,k as well as -e. That is, if a subgroup contains any two of i,j,k then it is the whole group. This gives us the final subgroup which is **Q** itself.



This is a graph with the subgroups of Q as the vertices and edges between two subgroups if one of them is contained completely in the other. It is called the *lattice* of subgroups.