Algebra - I
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

ALGEBRA I

1. Lecture 49: Problem solving

This problem session is going to be based on a very special ring, the ring of quaternions. The ring of quaternions was discovered by Sir William Rowan Hamilton an Irish mathematician in the year 1843. So, the time of Hamilton we already knew about the complex numbers. So, this gave a way of multiplying pairs of real numbers. We could think of a complex number as a pair (a, b), a pair of real numbers denoted $a + bi$ where $i^2 = -1$.

What Hamilton was trying to do is he was trying to find a way to multiply triples of real numbers. So, he would try to write a triple of real numbers as a + bi + cj and then try to see if he could find a rule for multiplying them that would result in a ring and Hamilton was not able to do that. But, one day he suddenly realized that if instead of triples you take quadruples of real numbers, then you can find a ring. So, let us see how Hamilton did this. We are already familiar with the group of quaternion's which we talked about in the 1st week of this course. So, in order to have the group of quaternion's we had four matrices one which is just the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the matrix which i called i which is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; the matrix j which I called $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$and the matrix k which is $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

And, we have seen that these matrices satisfy the relations $i^2 = j^2 = k^2 = -id$ and that $ij = k, ji = -k, jk = i$. The ring of quaternion's is defined as H is the real span of 1, i, j, k . We endow it with two binary operations addition and multiplication inherited from matrices.

Example 1.1. *Show that H is a ring.*

We need to show that $(H, +)$ is an abelian group; we need to show that $(H, \cdot)$ is a monoid; we need to prove associativity and we need to prove unitality. So, let us start with the 1st which is trivial. It clearly has a unit and is closed and associative by inheriting these properties from matrices.

So, we can conclude that H is indeed a unital ring or what we just call a ring ok.

Example 1.2. *Is H commutative?*
Clearly not since $ij = -ji$.

Let us take a closer look at the multiplication. So, let us write $v = v_1 i + v_2 j + v_3 k$ and $w = w_1 i + w_2 j + w_3 k$. So, we are going to think of vectors in 3-dimensional Euclidean space as quaternions. The exercise is to write down the product of v and w .

So, $v \times w$ is a vector which is lies in the plane in the line perpendicular to the plane spanned by v and w and its direction is given by the right hand rule and the magnitude of this vector is magnitude of v times magnitude of w times sin theta where theta is the angle between from v to w . Sorry so, this is v . In the plane spanned by v and w ok. So, that was the definition of cross products just to remind you ok. Let us move on part b of this problem . So, say $v^2 I$ will use the same notation as before is not = 0, then show that . So, this p remember is the quaternion given by ok and so, let p be the quaternion given by s times identity s times 1 + v . Show that p inverse is = s - v divided by $s^2 + |v^2|$ and this is very easy you just look at s + v times s - v and then you multiply we already know something about the product of two imaginary quaternion's . So, we will use that. So, this becomes s times v - s times v, these cross terms um s commutes with v because the identity matrix commutes with everything. And, then we have s^2 of course, and then we have the term v dot v multiplied by, but so, this will be $-v \times -v^2 - v \times v$, but that is 0 because the angle from b to itself is 0. So, that sin term will kill it and so, what we get is that this is just $s^2 + 0$. So, this is just a multiple of 1 and so, when I say this you can think of this as scalar multiple and so, from this the identity follows ok .

So, let us just pause to make this observation here that every non-zero quaternion is invertible. This is very rare in ring and we have we know some examples of rings where every non-zero element is invertible and those are fields like the real numbers, the complex numbers, the rational numbers and then finite fields , but in fields multiplication is commutative and in the quaternion's every non-zero element is invertible, but multiplication is not commutative . So, this quaternion's an example of what is called a division ring . So, quaternion's form a division ring . Every non-zero element of H is invertible this is what we call a division ring a ring not necessarily commutative where every non-zero element is invertible .

Example 1.3 (Quarternion rotation identity)**.** *We now come to the most interesting problem of this session which is the which is called the quaternion rotation identity. Suppose I take u to be a unit vector. So,*

$u_1^2 + u_2^2 + u_3^2 = 1$ *and I can think of u this as an imaginary quaternion where* $u = u_1 i + u_2 j + u_3 k$. *And, I will think of vector v in* $\mathbb{R}^3$ *as a quaternion* $v = v_1 i + v_2 j + v_3 k$. *Let q=cos of alpha by 2 times 1 of course, . So, this is the quaternion + u times sin of alpha by 2 . So, this is an element of H ok. So, the problem is show that the function v goes to* qvq^{-1}. *This function rotates the vector v about the axis u. So, let us see how we can solve this problem . Let us just introduce some notation. Let us take* $r_{u,\alpha}(v)$ *to denote the rotation of v by angle* α *about u. So, to start with observe that both v goes to qvq inverse and v goes to* $r_{u,\alpha}(v)$ *are linear transformations from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$. *But, a linear transformation is completely determined by its value on any spanning set. So, it suffices to show that they are equal on some spanning set of* $\mathbb{R}^3$. *So, to start with we will take a case 1, where v is u itself ok the unit vector u. So, in this case of course it is u itself. So, we need to just calculate q u q inverse . So, that is* $ucos(\alpha/2) + usin(\alpha/2)(ucos(\alpha/2) - usin(\alpha/2)$ *. That is u . So, we see that the rotation and the qvq inverse they agree upon the vector u itself ok .*

And, now let us take case 2, where v is perpendicular to v if we can show that it is true for $v = u$ *and v perpendicular to u then we have shown that these two linear transformations are equal for every vector which is an exercise.*

This quaternion rotation identity gives a very simple way to compute the rotation of vectors about any axis by any angle and this is used all the time in computer graphics . So, when you have your video game and you are running around and turning every time you turn while you have an axis by which you turn an angle by which you turn and those computations before rendering are done using the quaternion rotation identity . So, Hamilton's quaternion's discovered in 1843 are still used today to give a fast rendering of 3D graphics in video games, now, is not that cool ?