Algebra - I
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

ALGEBRA I

1. Lecture 48: Quotient rings

Let us talk about Quotients of Rings by ideals . The easier case is that of a commutative ring , let us begin with that . Say, suppose R is a commutative ring , ok. Remember when R is a ring , then R comma + is an abelian group , ok. And what you can do is, you can take the quotient of R by an ideal . So, let I be an ideal of R . So, what is an ideal ? It is a normal subgroup of R under addition; but also it has a property that, if you take an element of I and multiplied by any element of the ring, the product is again an element of I. The quotient ring is defined as quotient group R/I with product $(x + I)(y + I) = xy + I$. Check that this is a well-defined product and associativity of the product, that there is a unit, and that it distributes over addition. The unit of the ring R/I is just the coset of the unit in R. The only issue is when I is actually equal to R; when I is equal to R, then R/I is trivial and then this ring cannot have a unit.

Recall the definition of a prime ideal: an ideal I of R is said to be a prime ideal if $xy \in I \implies x \in I$ or $y \in I$. Here is a result about quotients by prime ideals:

Theorem 1.1. *R/I is an integral domain , if and only if I is a prime ideal.*

Proof. Suppose I is a prime ideal. Let us take $(x + I)(y + I)$: if this is equal to 0 in R/I, that means it is equal to just I itself . Then, that means that , in fact this is an if and only if this is happens if and only if $xy \in I$. This implies that x belongs to I or y belongs to I, which implies that x + I is 0 in R/I or y + I is 0 in R/I , ok. So, if I is a prime ideal, then R/I is an integral domain.

Conversely, if R/I is an integral domain , we just reverse this argument , suppose we have $xy \in I$; this this implies that $(x+I)(y+I) = 0$. Since, R/I is in integral domain, this implies that $x + I = 0$ in R/I or $y + I = 0$ in R/I ; but that is the same as saying that x belongs to I or y belongs to I . □

An ideal I in R is called a maximal ideal if whenever J is an ideal which contains I then either J is R or J is I.

Theorem 1.2. *I is a maximal ideal if and only if R/I is a field.*

Proof. Suppose, I is a maximal ideal. Take an element $x + I$ in R/I . So, $x + I$ is not equal to 0 in R/I means that x does not belong to I , right. So, this is an if and only if, so if x is not in I . So, now consider the ideal J equals all elements of the form $rx + a$, where r belongs to R and a belongs to I ; it is not difficult to see that J is again an ideal and that J contains I and it is contained in R . But notice that J also contains the element x, but x is not in I; so that means J properly contains I . So, this implies that J is equal to R. In particular , the unit of R belongs to J. So, this means that R/I is a field ; every non-zero element of R/I has a multiplicative inverse . Conversely, suppose R/I is a field . I want to show that , every ideal of R that contains J is either I or R . So, suppose J is an ideal that properly contains I and is contained in R ; I want to show that J is actually equal to R . But, then what you do is, you look at the image of J in R/I. So, what I am saying is, you look at $J + I$. So, this is the set of all elements of the form $j + I$, where j belongs to J ; this is a subset of R/I. And in fact, it turns out that $J + I$ is an ideal in R/I . And in fact, it is a non-zero ideal . Why ? Because J properly contains I. So, there is an element of J which is not in I. So, there is at least one non-zero element in $J + I$, a non-zero element of R/I in J +I . But in a field, the only non-zero ideal is the field itself, this is something you can check ; this means that J + I is equal to R/I , which implies that J is equal to R . □

Corollary 1.3. *Every maximal ideal is a prime ideal.*

The reason for this is every field is an integral domain . Let us look at some examples of ideals which are prime, but not maximal . Now, if you look at the integers ; the only ideal that is prime, but not maximal is (0). For prime number p , the ideal generated by p is a prime ideal and a maximal ideal . The only ideal larger than p is Z itself , and this corresponds to the fact that Z mod p Z is a field , ok. So, it turns out that in the integers, most ideals are prime ideals .

But let us look at polynomial ring in two variables . So, let us take say $R = \mathbb{C}[x, y]$; then here is an example of an ideal let I be the ideal generated by x and y . This is a maximal ideal . It consists of polynomials with in two variables with trivial constant term. So, there is no constant term and that is the only condition . And, what you can show is that , R/I is isomorphic to the complex numbers . How do you construct this isomorphism? Given a polynomial f , you map it to $f(0, 0)$ that is a complex number . And, every polynomial that is in this ideal will get mapped to 0 and you can show that this is an isomorphism. So, this is a maximal ideal ; because the quotient is the field of complex numbers , ok. But in $\mathbb{C}[x, y]$ you just take the ideal generated by x , this is a prime ideal; but this is not a field. So, it is not a maximal ideal . In fact, this ideal x is contained in the ideal (x, y) , which is contained in $\mathbb{C}[x, y]$. When we take quotients in non-commutative rings, we need to be a little more careful. So, let us try doing the quotient process which we did for commutative rings in the case of a non-commutative ring and see what goes wrong . So, now suppose R is possibly non-commutative ring and let us say I is a left ideal in R . So, that would mean that, I is a sub group of R + , and for every x in I and r in R , r x again belongs to I . Now, I want to define a ring structure on the quotient , additive quotient group R/I . So, let us try to define a ring structure on R/I . So, we do this x + I into y + I is equal to x y + I . We need to check if this is well defined, ok. So, is this well-defined ? So, as before we just look at x + a + I times x + b + I and try to see if this is equal to x y + I ? But as we saw this is x y + a y + x b. I is a left ideal, so that means that, since b is in I, $ab \in I$; since b is in I, x b is in I , but we do not know that a y is in I. So, if a y is not in I, then this can go wrong .

So, what we need is two sided ideals. So, let us take I to be a two sided ideal in R . Then this will also be in I , and so we will get that this is equal to x y + I as needed . So, multiplication is going to be well defined on R/I, provided I is a two sided ideal .

Theorem 1.4. *If R is a ring and I is a two sided ideal of R ; then R/I has a ring structure with multiplication given by $(x+I)(y+I) = xy+I$.*

The main point is that this product is well defined only when I is a two sided ideal and that is what you need to take quotients of rings . Let us look at an example of a quotient ring construction in the case of a non-commutative ring . So, let Q be a quiver. So, it is a quadruple V, E, s, t , where V is the vertex set of the quiver, E is the edge set and s and t are functions from E to b , which tell you where each edgE^*ts and terminates . And then, we have E^* is the set of all paths in Q and the path algebra K Q is is an algebra is a ring; it is as a as a set, it is the vector space, it is a vector space with base is given by paths in Q and a multiplication is defined by concatenation of paths , this is the path algebra of Q . Now, what I had mentioned earlier and left for you to check was that, if you define N to be the span of p in E^* , such that the length of p is positive, then N is a two sided ideal . In fact, in Q there are two kinds of paths , so, the trivial paths, so we have the paths e_i , $i \in V$. So, this is the trivial path at I; it is a path of length 0 does nothing, just stays at the vertex i and then there are paths of length , positive length . So, $K[Q]$ has a vector space over K , this is spanned by paths of positive length. Ttherefore, we can define isomorphism from $K[Q]/N$ just a linear map to $K[\{e_i\}]$ using this decomposition which is a projection map.

What this map does is; if you have any linear combination of paths, you just take all the paths of length greater than 1 and send them to 0 and just keep the paths, the trivial paths. This linear map has kernel exactly equal to N . So, what we get is $K[Q]/N$ is isomorphic to i belongs to V $K[\{e_i\}]$. And this in fact turns out to be a ring isomorphism.