Algebra - I
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

ALGEBRA I

1. Lecture 47: Ring homomorphisms

When we studied groups, we also studied group homomorphisms. They were functions between groups that preserved the group operation similar to that there is the notion of a Ring Homomorphism . So, suppose you have two rings R and S then a ring homomorphism is a function is a function $\phi : R \to S$ which basically preserves all the structures that are involved in the definition of a ring namely it preserves addition and multiplication. Note that ϕ is a group homomorphism and we take axiomatically that $\phi(1_R) = 1_S$.

For example, if you take R and S to be any rings and you take $\phi(x) = 0$ for all x in R, then this will satisfy 1 and 2, but it will not satisfy 3 ok and we can talk about isomorphism an isomorphism of rings is a bijective homomorphism and if ϕ from R to S is an isomorphism then ϕ^{-1} from S to R is also an isomorphism ok let us look at some examples.

While studying groups we had seen that the homomorphism which takes x to x mod n the residue class of x mod n is a group homomorphism, but this is also a ring homomorphism because residue classes are respected under addition and multiplication.

Let us look at the slightly more interesting example take the ring of polynomials in some field x. So, this is the set of all polynomials in the variable x with coefficients in the field F and you can define a a function from f of x to f by just taking the polynomial f of x to f of a for some fixed a. So, let us say fix a in F for some fixed a in F .

Take the ring of polynomials in a variable x and then you take the ring of polynomials in another variable y whose coefficients are polynomials in x ok. So, an element here looks like $p_0(x)+p_1(x)y+\ldots+p_n(x)y^n$ for polynomials $p_i \in F[X]$. You can think of a polynomial in y whose coefficients are polynomials in x as a polynomial in x and y and so, what we are saying is that the ring is isomorphic to polynomials in two variables $f(x, y)$. The isomorphism is defined by just expanding out these coefficients and getting polynomials in two variables. So, this is an example of a ring isomorphism.

Another example of a ring isomorphism would be $\phi : \mathbb{Z}[x] \to \mathbb{Z}_n[x]$ by reducing the coefficients to their residue classes.