Algebra - I
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ALGEBRA I

1. Lecture 46: Products of rings

In this lecture , I am going to talk about Product of Rings . Suppose, you have two rings R and S , then you can form a new ring by putting a ring structure on the Cartesian product and this structure is just a component wise addition and multiplication . So, if you have $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2)$. It is not difficult to check that this operation will inherit all the ring axioms from the corresponding axioms for r and s .

Just to make some things clear , let me give a few examples . If we have two integers m and n, then $\mathbb{Z}_m \times \mathbb{Z}_n$ is a group , but it is also a ring under component wise addition and multiplication . Suppose, R and S are two rings not necessarily commutative , we can form the ring $R \times S$ and now I ask what are the left ideals in $R \times S$ and here is one easy way of constructing a left ideal in $R \times S$. If I is a left ideal in R and J is a left ideal in S , then $I \times J$ is a left ideal in $R \times S$.

Now, you can replace this left here by also by right or by 2-sided and it works exactly the same way and here it will be a right ideal or a 2-sided ideal . So, if I and J are right ideals in R and S respectively , then $I \times J$ is a right ideal in $R \times S$ and if I and J are 2-sided ideals in $R \times S$ respectively ; R and S respectively , then $I \times J$ is a 2-sided ideal in $R \times S$. It turns out that every left ideal of $R \times S$ is of the form $I \times J$ where I is a left ideal of R and J is a left ideal of S . So, how do we prove this ? Well, so, starting with a left ideal in $R \times S$, we will try to identify what I and J would be . So, let I just be the set of all $a \in R$ such that $(a, 0)$ belongs to K and let J be the set of all $b \in S$ such that $(0, b)$ belongs to K . It is very easy to show that $I \times J$ is contained K because if $(a, b) \in I \times J$, then $(a, 0) \in K$ and $(0, b) \in K$ which implies that a , b belongs to K which shows that $I \times J$ is contained in K .

The converse is easy to prove. Similarly, you can show that b is in J which implies that a , b is in $I \times J$ and hence, we have that K is contained in $I \times J$. So, K is equal to $I \times J$. So, what we have seen is that if you take a product of two rings , then all the left ideals of that product are products of left ideals of the two rings themselves. The same holds for right ideals or for 2-sided ideals .