Algebra - I
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

ALGEBRA I

1. Lecture 45: Ideals in non-commutative rings

We have studied Ideals in Commutative Rings extensively. We need to be a little more careful while studying ideals in non-commutative rings. The slight complication arises, because multiplication does not commute. And there are two different flavors of ideals; there are left ideals, and right ideals. So, a left ideal is just additive subgroup of R, so it is a subset of R that is closed under addition contains 0 and has additive inverses . This part is the same as in the definition of the ideal, but the additional condition is that $rx \in I$ for all x in I and r in R. Now, this condition looks exactly the same as the definition of an ideal in a commutative ring, but there is a different variant of this where we will put the R on the right. So, right ideal again is an additive subgroup of R such that $xr \in I$ for all x in I and r in R . And then finally, we have the notion of a two sided ideal; a two sided ideal is just an ideal which is a left ideal and a right ideal.

In a commutative ring, these three notions all coincide; left ideals are the same as right ideals, and they are all two sided ideals . Let R be the ring of n by n matrices, I will just call it $M_n(K)$, all n by n matrices with entries in a field K. For a matrix A in R; I will use the following notation, A_{ij} denotes the element in the ith row the entry in the ith row and jth column of R; the ijth entry of of A . And I will use A_{i*} to denote the ith row of A, so you think of it as a row vector; and A_{*j} will be the jth column of A column vector . And I will use the notation $\mathcal{R}(A)$ to denote the row space of A, so that is the span of A_{i*}. And similarly $\mathcal{C}(A)$ will be used to denote the column space of A. Let me give you a construction of some left ideals and right ideals. So, let V be any subspace of K^n, and then we can define two subsets of R, we will define $R(V)$ to be those matrices a such that row space of A is contained in V . And we will define $C(V)$ to be those matrices a in R such that column space of A is contained in V, ok. Now, recall an important fact from linear algebra that the row space of AB is contained in the row space of B, and the column space of A B is contained in the column space of A . So, what this means is that you take any matrix whose row space is contained in V, and you multiply it on the left by another matrix, then the product matrix will still have row space contained in V; so that means that this is a left ideal . And here if you have a matrix whose column space is contained in V and you multiply it on the right by another matrix, then the column space of the product is still contained in V; so that means that this is a right ideal . It turns out that these are all the left and right ideals of R . How do we see that? So, to see that we need to given an left ideal, let us let us do it for left ideals. So, given I in R a left ideal, we need to identify the subspace V for which this ideal I is of the form $R(V)$. So, how would we; how would we get that subspace V? Well, an obvious choice if you think about it a bit is to let V be the span of the row spaces of all the matrices in I . So, you take all A in I and you take all I between 1 and n ok. So, this is of course huge infinite set usually, but it is going to be a subspace of K to the n its going to be a finite dimensional vector space and its going to be generate spanned by a finite set of such rows, ok.

And now, what I want to show is that I is actually equal to $R(V)$. So, firstly if you have a vector if you have an a matrix A in I, then of course the row space of A is going to be contained in V because that is one of the matrices whose rows span V. So, if A belongs to I, then row space of A is contained in V which implies that A belongs to R(V) . So, what this says is that $I \subseteq R(V)$.

Suppose A belongs to R(V), then each row of A; so in particular row space of A is contained in V, and which means that each row of A lies in V ok.

So, what we proved is that R(V) is contained in I and so R(V) is equal to I. So, this shows that every left ideal in R is of the form R(V) for some vector space, subspace V of k n. And a similar reasoning using columns instead of rows, will show that every right ideal is of the form $C(V)$ for some subspace V of K^n.

What about 2-sided ideals in R? So, suppose I is a 2-sided ideal in R ok; then I is of course, going to be of the form C of V for some subspace V in K n, ok. Now, consider two cases firstly; if V is the 0 dimensional subspace, so if V is just the subspace consisting of the single vector 0; then of course, I is 0 itself consists only of the 0 matrix just by the definition of C V. Now, if V is not 0, then take some non-zero vector v in V ok. And now let us look at the matrix $A(j)$, the matrix whose jth row is V not capital V, is the is the vector v and all other rows are 0 . So, this matrix $A(j)$ looks like something like this in the jth row, we have the vector v, and everywhere else we have 0. So, now I ask you what is the row space of $A(j)$? The row space of $A(j)$ is the subspace spanned by the rows of A; at least one of these coordinates in v is non-zero and so the row space of $A(j)$ is going to be the jth coordinate line . So, this means that ok, so firstly $A(j)$ belongs to I for all I, for all j between 1 and n just because the column space of $A(j)$ is contained in the subspace capital V. But on the other hand the span of $A(j)_{i*}$ is all of K^n. What we have seen here is that there are only two 2-sided ideals in R, namely the ideal consisting only of the 0 matrix, and the other one is the ideal consisting of all matrices in R. So, let Q be a quiver . So, it is of the form V, E, s, t; where V is a set of vertices, E is a set of edges, S is the source function, tells you where each edge starts and t is the target function, tells you where each edge ends . And the path algebra if you remember this is as a vector space, it is spanned by all the paths in Q . Now, I will give you some examples of left ideals, right ideals, and 2-sided ideals in K Q. So, fix a subset S of V, and you define I of S to be the span of all paths with source in S . So, this is span of paths p, in E^* such that source of P belongs to S . And similarly, I define J of S to be span of all paths that end in S, this is span of p in E^* such that t of p belongs to S . I claim that this is a left ideal, I S is the left ideal in K Q, and this is a right ideal in K Q . And if you want an example of a 2-sided ideal in K Q, you just define N of S to be span of all paths of length greater than 0, all paths of positive length . Remember the path algebra the the set of paths also consists of paths of length 0, one for each vertex of the quiver. Now, if you look at the span of paths of positive length, you can show that they form a 2-sided ideal. You should try to prove these three assertions; about left ideals, right ideals, and 2-sided ideals in the path algebra of a quiver.