Algebra - I
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ALGEBRA I

1. Lecture 44: Path algebra of a quiver

Today, I am going to introduce you to a factory for making rings, ok. And, it makes very interesting non commutative rings. So, the rings that this factory produces are called Path Algebras of Quivers. So, what is a quiver? Let me begin with an example. So, here is an example for quiver. You have a bunch of nodes 1, 2, 3, and then you have a bunch of arrows, so the arrows a, b, c, you can even have arrows going in both directions, d, ok. So, formally what is a quiver? Formally a quiver consists of 4 pieces of data, V which is a vertex set, then E which is an edge set, then there are two functions called the source function s from E to V. So, this is called the source function. And then finally, t from E to V called the target function. In this example, they are as follows. V is just the set 1, 2, 3. E is the set a, b, c, d and let me tabulate the source and target functions for a, b, c, and d, so you have a, b, c, d and we can ask for their source and target. So, source of a is 1, because the arrow starts from a and the target of a is two because the arrow ends at a. Source of b is 2, target of b is 2. Source of c is 3, target of c is 2 source of d is 2 and target of d is 3, ok. Now, given two vertices, let us say i and j of the quiver a path, so a path from i to j is a sequence of edges. And I will write this sequence from, right to left a 1, a 2, a n, such that the source of a 1 is i, the target of a 1 is the source of a 2, and so on. The target of a n minus 1 is the source of a n, and the target of a n is j, ok. So, in this example I can write down a path from 1 to 2 as; so, I have to start with one I have to follow arrows, and each time I follow an arrow I write down that latter, and I write them from right to left. So, for example, I start with 1. Let us use the arrow the only way to get out of 1 is to use the arrow a, ok. Now, I am at 2. So, what do I do? I could either do b or I could do d, let us do b and I am back at 2, I will do b again. So, I will write it as b b, but I write it as b square that is the usual convention. Then, I can go to 3 using d and then I can come back to 2 using c and then I will do b again and end up at c. So, this is a path from 1 to 2. A special kind of path is the trivial path. So, for every i in V, we denote by epsilon i, the what is called the trivial path from i to i. So, what we are thinking of this path as doing is you are at the vertex i, and you do not use any arrow at all. So, this is a path of length 0, ok. So, so this path means do nothing starting at i, ok. And so, E star is the set of all paths in Q, ok. And if p equals a n, a n minus 1 dot dot a 1 is in E star, we write s of p the source of p to be the source of a 1 and the target of p to be the target of a n, ok. So, then this p is a path from source of p to target of p. And the empty path corresponding to i is a path from i to i. So, this will be the convention, ok. So, so much for the definitions of quivers and paths in quivers. Now, we are ready to define the path algebra of a quiver. For this we need to fix a field F, any field, your favourite field, you can even take a ring, ok. And then I define the path algebra of Q of a quiver Q over F is denoted by F square brackets Q, ok. So, this is going to be a ring. And is the F vector space with basis, 1 sub p where p is a path in Q. So, this E star recall denotes a set of all paths in E. So, typical element of F Q is of the form, summation p in E star a p 1 sub p, ok. And we can talk about, now we have I am going to turn this thing into a ring. So, I have to define multiplication. In order to define multiplication, we consider concatenation of paths . So, suppose you have two paths, p equals a 1, a 2, dot dot dot a n and Q equals b 1, b 2, dot dot dot b m. Then, if target of a n is the source of b 1, then you can define q p to be b m dot dot dot b 1, a n dot dot dot a 1 and this is a path and it is called the concatenation of q and p, ok. Now, I define multiplication in the algebra F Q by defining the multiplication of two basis elements. So, it is defined on basis elements as follows 1 q multiplied by 1 p is defined to be 1 sub q p if the target of p is equal to the source of q and 0 otherwise. And using bilinearity this, so you want to say that the multiplication is linear in each of the arguments. So, this can be extended to the path algebra. So, you define summation a p 1 p, p belong to E star multiplied by summation q belong to E star, oops the star goes up, b p 1 p is defined as summation p q in E star a p b q 1, oops this is q 1 p 1 q. So, this is the definition of multiplication in the path algebra of a quiver. So, it is a not difficult to check that the path algebra of a quiver is a ring with this definition of multiplication. Addition is just you know the vector space addition . I will not do the check of that is a ring; I will; if you want you can try it yourself, instead we look at some examples. So, here is the simplest example maybe I will call it example 0, where q is the quiver where E is just a singleton set 1, and let us say V is the empty set. So, I do not have to specify what s and t are because they are functions from the empty set. So, in this case we will just represent it by a single point which is 1 and there are no arrows at all. Then what is F Q? So, Q is 1 empty set and whatever s and t the functions from the empty set to the set

1. So, F Q then has bases. So, what are the paths? There is only one path which is epsilon 1. So, the empty path from 1 to 1, and if you multiply epsilon 1 by itself you get epsilon 1. This is just isomorphic to the ring c, oh oops this should be F. This is isomorphic to the field F which we think of as a ring, ok. So, that is a trivial example. Let us look at a slightly more interesting example. And let us take E to be 1, but now I will define I will have V to be just one path which I will denote by t, and s of t equals t of oops, maybe I should not call it t, let me call it u s of u equals 1 and t of u equals 1. So, if you want to draw this quiver it looks like this. You have this 1 node 1, and then you have u is 1 arrow going from 1 to 1. So, then E star consists of all the paths start ended at 1. There is the empty path, then there is u, then there is u squared, there is u cubed, and so on So, F of E star is just the vector space spanned by epsilon 1 u, u squared, u cubed and so on. And what we have is 1 sub u multiplied by 1 sub u to the power k multiplied by 1 sub u to the l is 1 sub u to the k plus l. So, I claim that k E is isomorphic to F E sorry, the polynomial algebra in one variable and this isomorphism is given by taking 1 sub u to the k to the monomial u to the power k, ok. This is easy to check. Let us look at one more interesting example. Let us take I will just draw the quiver now. So, you have two nodes 1 and 2, and we have one arrow which we call a. So, what are the paths of Q? So, there are 3 paths that I can find, there is epsilon 1, there is a, and there is epsilon 2. So, epsilon 1 goes from 1 to 2, a goes from 1 to 2 and epsilon 2 goes from 2 to 2. And the products are given by a epsilon 1 that means, first you go from 1 to 1 by the empty path and then you go from 1 to 2. So, this is a and then we have epsilon 1 a is 0 because the target of a is 2, but the source of epsilon 1 is 1. Then, we have a epsilon 2 is 0 and epsilon 2 a is equal to a. So, this algebra has 4 basis vectors and this is how they multiply. Now, this algebra actually has another familiar [FL]. And consider the following algebra a to b the set of all matrices of the form a, b, 0, d, where a, b, and d are arbitrary elements of F. Now, if you add or multiply two upper triangular matrices you still get an upper triangular matrix. So, this is a ring, right. And then you take epsilon 1 goes to the matrix 0 0 0 1. Epsilon 2 goes to the matrix 1 0 0 1 0 sorry and a goes to the matrix sorry; ye, maybe I should not have called this a let us call this c b 0 d and a goes to the matrix 0 1 0 0. Now, if you do the matrix multiplication, so if you take the matrix corresponding to a that is 0 1 0 0. So, let us call this map phi. So, this by this is defined on the basis element, so this gives rise to a map phi from F Q to a. And if you look at this, so this is, this is phi of a and this is phi of epsilon 1 that is 0 0 0 1. If you multiply them you will get

0 in the first column, in the second column you will get 1 0 which is phi a. So, phi of a times phi of epsilon is phi of a epsilon 1. And similarly, you can do all the other checks and you can see that phi is actually defines a ring homomorphism. And in fact, phi is an isomorphism of rings.