Algebra - I
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

ALGEBRA I

1. Lecture 43:Fraction fields

Ok let us talk about Fraction Fields. The first example of a ring that we encounter in life is probably the ring of integers $\mathbb{Z}$ and the first example of a field that we encounter is the field of rational numbers. You cannot think of rational numbers as just pairs of integers you can have the same fraction given by two different pairs of integers . So, you have this relation that $\frac{p}{q} = \frac{r}{s}$, if and only if $ps = qr$. The passage from integers to rational numbers works in a much more general setting namely, when you replace the integers by any integral domain.

So, let us start with R any commutative integral domain and look at pairs in R . So, we will define $\tilde{F}\{(p,q), p \in R, q \in R \setminus 0\}$. Define the equivalence relation $(p,q) \equiv (r,s)$ if $ps = qr$. Whenever you have an equivalence relation you can talk about equivalence classes. Define F to be the set of equivalence classes in $\tilde{F}$. This is going to be the field of fractions . So, on this we will define firstly, we will define addition and multiplication on F analogously to the rational numbers.

So, what we have is the set F, with two operations ; addition and multiplication. And, then you can check that with these definitions F is a ring. We will use the notation $\frac{p}{q}$ to denote the equivalence class of (p,q).

Let us just point out some features of this ring. The 0 element is 0 of the ring, and its equivalence class contains all pairs $(0,p)$. The multiplicative identity is $(1,1)$. So F is a field. What is more? R itself sits inside F as $r \to (r,1)$.

Theorem 1.1. *The field of fractions of a (unital) commutative integral domain contains R as a subring and is the smallest field that has this property.*

Let us look at some more examples . So, the first example that we look at is when we take for the integral domain $K[x]$. So, this as you recall is the ring of polynomials in one variable x with coefficients in K . This K itself is a field . And then, in the field of fractions is called is usually denoted $K(x)$ and it is called the field of rational functions . And, a typical element of $K(x)$ is $\frac{p}{q}$ and where p,q are polynomials and $q \neq 0$.

So, if S is the set of roots of q it is the set of x in K such that q x is equal to 0 . Then, let us call this F of x then you can define a function from $K \setminus \{x\} \times K \setminus S \to K$. You can think of this rational functions function from $K \setminus S$ to K which takes an element x to $p(x)/q(x)$.

Let us look at a 2nd example in this case we will take R to be the ring of Gaussian integers. Remember, these are just integers of the not integers of the form complex numbers of the form a plus b i , where a and b are integers and we had seen that this is in fact, a Euclidean domain . So, now, what is a typical element of the field of fractions? So, so, if F is the field of fractions, then a typical element is of the form we can write it as $\frac{a+bi}{c+di}$, where we assume that either c or d is nonzero ok. But, there is another way of describing this field of fractions I will define $Q[i]$ as the set of complex numbers whose real and imaginary parts are rational.