Algebra - I
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ALGEBRA I

1. Lecture 42: The ring of formal power series

The ring of formal power series over a field F is denoted $F[[t]]$ and comprises of infinite sums $\sum_{i=0}^{\infty} a_i t^i$, for elements $a_i \in F$. Sum and product are defined as for polynomials.

Example 1.1. *If you take $a_0 = 1$, $a_1 = -1$, $a_n = 0$ for n greater than 1, then you get $1 - t$. If you take $b_i = 1$ for all i, then you get $\sum_i t^i$. Multiplying these yields $(1 - t)(\sum_i t^i) = 1$.*

This is the formula for the geometric series, but we have proved a version of it that does not involve any convergence. So, this is an identity that holds true in the ring of formal power series. And it can be proved in the ring of formal power series without any reference to the notion of convergence.

Example 1.2. *Another example of an element in the ring of formal power series is $\sum_{n=0}^{\infty} n! t^n$. This is a perfectly valid example of an element of the ring of power series even though if we were working over real numbers this would not converge for any non-zero value of t.*

Let us find out what the units in the ring of formal power series are. Gven a unit $g = \sum a_n t^n$ there must exist an element $f = b_n t^n$ such that $fg = 1$. Solving for the coefficient of each power of t we have the system

$$\begin{aligned} 1 &= a_0 b_0, \\ 0 &= a_1 b_0 + b_1 a_0, \\ 0 &= a_1 b_1 + a_2 b_0 + a_0 b_2, \\ &\dots \end{aligned}$$

We can solve this system recursively for the b_i, provided a_0 is nonzero. Thus the units are all power series with nonzero constant coefficient. What are the ideals in the ring of formal power series? To understand this, it is useful to have the notion of the valuation of an element in F. So, for any f in $F[[t]]$, define the valuation of f denoted $v(f)$ as the minimum value of m greater than or equal to 0 such that $a_m \neq 0$. I claim that the ideal generated by f is the set of all g in $F[[t]]$ such that valuation of g is greater than or equal to valuation of f. What we have is that the ideal generated by f is the same as the ideal generated by $t^{v(f)}$ and every ideal is of this form.

So, what we have concluded is I is equal to the ideal generated by t^m for some integer $m \geq 0$. And there is one more ideal namely the ideal 0. And one interesting point about these ideals is that they form a chain, and $(t^i) \cap (t^j) = \max(i, j)$ and $(t^i) \cup (t^j) = \min(i, j)$