Algebra - I
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ALGEBRA I

1. Lecture 40: Examples of ideals in commutative rings

The first way to construct new ideals from old is addition of ideals . So, if I_1, I_2 are ideals , then you define $I_1+I_2 = \{x_1 plus x_2 | x_1 \in I_1, x_2 \in I_2\}$. Then there is intersection of ideals .

Example 1.1. *Fix a field F and a set X and let $F[X]$ be the set of all functions $f : X \to F$. Now, this is a ring under point wise addition and multiplication. So, if we have f_1 and f_2 in $F[X]$, then we define $f_1 + f_2(x)$ to be $f_1(x) + f_2(x)$ and multiplication similarly.*

Now, let me give you some examples of ideals in this ring. So, if M is a subset of X , define $I(M)$ to be the set of all f from X to F such that $f(x) = 0$ for all x in M . So, this is the set of all functions whose restriction to M is identically 0. This is an ideal in $F[X]$.
Every ideal in $F[X]$ is of the form $I(M)$ for some subset M.

Example 1.2. *consider any field F and the ring of polynomials $F[t]$. The ring $F[t]$ is a Euclidean domain and therefore, it is a principal ideal domain. So, every ideal of $F[t]$ is generated by a single element f . The ideal generated by f is equal to the ideal generated by g if and only if f is u times g for some unit u of $F[t]$. The units of $F[t]$ are non-zero constant polynomials or non-zero constants.*

Now, recall, a polynomial is said to be monic, if its top degree coefficient is equal to 1. Now, given any polynomial you can make it monic by just dividing it by its top degree coefficient and therefore, every ideal is generated by a monic polynomial . Moreover, if two monic polynomials generate the same ideal, then these two monic polynomials would differ by a unit, but that unit can only be 1 because both of them have the same leading term. So, what we conclude is that ideals of $F[t]$ are in bijection with monic polynomials in F t . So, every ideal is generated by a monic polynomial and that monic polynomial is unique.

There is another ideal that we have which is the 0 ideal.

Now, you can ask given two ideals I_1, I_2, what are their sums and intersections. So, what is f plus g and what is f intersect g ? So, for this well, recall that f plus g well. So, what is f plus g? It is you can see that it is the same as the ideal generated by f and g and this is the principal ideal. So, this is the ideal generated by the gcd of f and g which I explained to you how to compute in the section on Euclidean domains and the ideal generated by f intersect the ideal generated by g is the ideal generated by the lcm of f and g where you can define the lcm of f and g as just f, g divided by the gcd of f, h . I leave you to think about these two and figure them out. Now, let us take a special case of this example where F is an algebraically closed field . An algebraically closed field is a field where every polynomial can be written as a product of linear factors . So, for example, you could take F to be the complex numbers that is an algebraically closed field ok. So, in this case, if F is algebraically closed , then every polynomial; every monic polynomial has decomposition $f(x) = (x-\lambda_1)^{m_1} \dots (x-\lambda_r)^{m_r}$ where $\lambda_1, \lambda_2, \lambda_r$ are distinct elements of F and $m_1, m_2, \dots, m_r$ are positive integers. The set of roots of this polynomial are the numbers $\lambda_1, \lambda_2, \lambda_r$ and m_i is the multiplicity of the λ_i. A multi-set is a set of the form well is an object of the form which I will write like this $\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_r^{m_r}$ to denote m_i repetitions of λ_i for each i.

So, now, what we have is ideals in $F[t]$ are in bijection with multisets in F and corresponding to the multiset, $\{\lambda_1^{m_1}, \dots, \lambda_r^{m_r}\}$ you get the ideal generated by $(t-\lambda_1)^{m_1} \dots (t-\lambda_r)^{m_r}$.