Algebra - I
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ALGEBRA I

1. Lecture 39: Factorisation and the Noetherian condition

Let R be commutative integral domain. One way to generalize a prime number is the notion of an irreducible element. An element x belongs to R is said to be irreducible, if whenever $x = ab$, either a is a unit or b is a unit. Primes are irreducible in the ring of integers.

An ascending chain of chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ is said to satisfy the Noetherian condition if there exists an integer N such that $I_n = I_{n+1}$ for all $n \geq N$. The condition is named after Emmy Noether. She was one of the pioneers of ring theory. Whenever we have an increasing chain of ideals it must stabilize at some point. If R is a commutative integral domain, that is also Noetherian, then every element x admits a decomposition $x = p_1, p_2, \dots, p_k$ into irreducible elements $p_1, \dots, p_k$.

A commutative integral domain R is called a principal ideal domain if every ideal I of R is of the form (p) for some element $p \in R$. All Euclidean domains are principal ideal domains. Now, one very nice property of principal ideal domains is that they are Noetherian.

Theorem 1.1. *Every principal ideal domain is Noetherian.*

Proof. Given a chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ in R, let $I = \cup_j I_j$. Then I claim that I itself is an ideal because, if $a \in I$, then $a \in I_k$ for some k. And therefore, $ar \in I_k$ for all $r \in R$ which implies that $ar \in I$ for all $r \in R$. So, $I = (a_0)$. Thus $a_0 \in I_k$ for some $k \geq 1$. Which means that $I_k = I$ because, $a_0 \in I_k$ and a_0 generates all of I. □

Every element in a principal ideal domain is a product of irreducible elements. This factorisation is unique upto a unit. The proof is similar to that of the integers. Now, if you take this theorem here and go back to the proof of the fundamental theorem of arithmetic, that I did a few lectures ago. You will be able to work out all the steps exactly as it was there and complete the proof. This is something you should all try to do.