Algebra - I

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1. Lecture 32 [Generators and Relations of Symmetric Group - I]

Let us talk about Generators and relations for symmetric groups. We already done an instance of this in the previous problem session, when we looked at the symmetric group S_3. And, recall we said that there exist a surjective homomorphism from the free group on to generators to S_3 ok. We constructed such a map.

Now, how did we define this map? Let us call this map π , it was given by the following, you map the generator a to the transposition (12) and b to the transposition (23) ; mapped it to the two simple transpositions.

And, if you specify the values on a and b that is enough to give you a group homomorphism from the corresponding free group that is by the universal property of the free group ok ah. We said a few things about π . So, π is an onto map, its surjective. Why is that? Because, (12) and (23) together generate the group S_3 that is one.

Now, what we want to do really when we say generators and relations? So, the word relations is really something to do with the kernel of π . So, what we want to do is to

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$\underline{S_3}$

$$\mathcal{F} = \mathcal{F}(\{a, b\}) \xrightarrow{\pi} S_3$$

$$a \longrightarrow \underline{(1\ 2)}$$

$$b \longrightarrow (2\ 3)$$

- π is surjective
- $\underline{\ker \pi = ?}$

a) Normal subgroup of $\mathcal{F}$

b) $[aa] \in \mathcal{F}$

$$\pi([a\,a]) = \pi([a])\,\pi([a]) = (1\,2)(1\,2) = id \in S_3$$

c) $\pi([b\,b]) = id.$

$[aa]$, $[bb]$, $[ababab]$ $\in \ker \pi$

$\pi([\underline{ab}]) = \pi([a])\, \pi([b]) = (1\,2)(2\,3) = (1\,2\,3)$

$(1\,2\,3)^3 = id \implies \pi([ab]^3) = \pi([ababab]) = id$

Prop^n: $\ker \pi$ = normal subgroup of F generated by $\{[aa], [bb], [ababab]\}$.

understand what the kernel of this homomorphism looks like ok. Now, there are some obvious elements in the kernel which we can immediately write down. So, let me give you a few elements of the kernel, before that recall that the kernel well what is it?

It is certainly a normal subgroup of this is a normal subgroup as we know of the free group. So, let me just abbreviate the free group to F now ah. So, this is property 1, the kernel and here are some elements in the kernel. So, observe that the element (12) , the transposition has a square identity ok. So, if I similarly look at the square of this word a, in other words let me look at the word a occurring twice and recall that elements of the free group are all equivalence classes of words.

So, I look at the word aa and if I compute what its image under π is then here is what I will observe $\pi([aa]) = \pi([a])\pi([a])$ because π is a homomorphism . And, this is just $(12)(12) = id$ the square of this and that is just the identity. So, let me write id for the identity element of the group S_3 . Now, similarly more or less by a similar calculation, I also take the word bb and observe that its image under π is also going to be the identity ok.

So, what is this mean it means that I know some elements of the kernel the word $[aa]$ is in the kernel, the word $[bb]$ is in the kernel . These are both elements of the kernel of π . In fact, there is another yet another word which now involves both a and b which we can write down; so, I leave space for that. So, what is that? Well, let us look at the product of the two. So, suppose I take a word ab which is the product of the words a and b and I compute the image of that under π .

It is π $([a])$ π $([b]) = (12)(23) = (123)$ in S_3 and we know that 3 cycles have order 3, if you raise it to the power 3 then it just gives you the identity. So, what this means is; so, observe (123) being a 3 cycle, if I cube it I get the identity . So, in particular it means if I compute the image of π under the cube of this word. So, I take ab whole cubed $.\pi([ab]^3) = \pi([ababab]) = id$ so, I just repeat ab thrice .

Def: Given a subset $T \subseteq G$, the normal subgp generated by T is $\displaystyle\bigcap_{\substack{N' \text{ normal subgp of } G \\ N' \supseteq T}} N'$

$a^2 = [aa]$ $\quad b^2 = [bb]$ $\quad (ab)^3 = [ababab]$

let N = normal subgp of $\mathcal{F}$ generated by $\{a^2, b^2, (ab)^3\}$

Claim: $N = \ker \pi$

So, if I compute this then that is going to be the identity again ok. So, what that means, is that I can write out another word. So, this is just $ababab$ and these 3 words certainly belong to the kernel of this homomorphism π ok. Now, what we are going to claim is that in some sense these are these are the 3 words that you need, these are all the the words that you need ok. So, here is the the proposition .

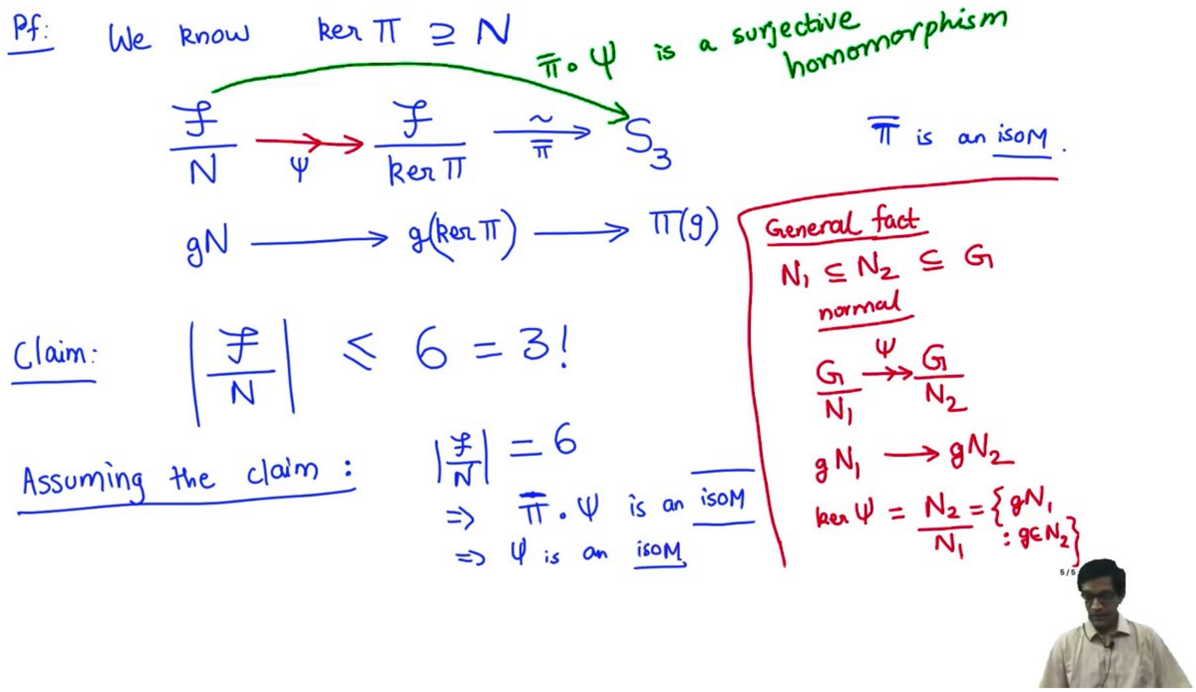
Proposition: The kernel of π is just the normal subgroup generated by $\{[aa], [bb], [ababab]\}$ these 3 words.

It is the normal subgroup of the free group generated by these 3 words ok, which is aa ; so, generated by the set containing these 3 words ok. So, what is the the terminology? Normal subgroup generated by something mean, that just means you take the smallest normal subgroup of the ambient group which contains the given subset.

So, this is just a sort of the obvious definition. So, definition given a subset given a subset of a group ; given a subset T of a group G, the normal subgroup generated by T is just the intersection of all normal subgroups. So, let us call it N $'$ ah. What is N $'$? N $'$ is a normal subgroup of G which contains the given set T ok.

And, it is an easy exercise to check that when you intersect an arbitrary collection of normal subgroups, the answer is still a normal subgroup ok. Arbitrary intersections of normal subgroups is a normal subgroup again. And so, the intersection of all normal subgroups which contain a given set is necessarily a normal subgroup and that is called the normal subgroup generated by the given set T ok.

So, in our case; so, let us come back to our situation. So, we said take the set comprising aa ab and a ab whole cubed. So, it is sort of cumbersome to keep writing this again and again. So, I will just call this word a square b square. So, just mild abuse of notation here



and I will call this ab whole cubed is just the word ababab . So, I am dropping the the sort of the square brackets for the equivalence classes ok.

So, I claim that. So, let N be the the normal sub group generated by these 3 elements by $\{a^2, b^2, (ab)^3\}$ and I claim ah. So, we need to prove really which is the proposition, to prove the proposition.

Claim: N $=$ Ker(π).

ok. So, recall that is the proposition, the kernel is exactly the normal subgroup generated by these 3 elements.

So, let me give the normal subgroup generated by these 3 elements a name, I am calling it N and I am going to show that N is the same as kernel of π . Now, what we know; so, let us prove the claim what we have just said. So, we know the following that the kernel contains N .

We know the following we know that the $N \subseteq ker(\pi)$ right because the 3 elements with generate N are all in the kernel and the kernel is a normal subgroup. So, N is the smallest normal sub group containing those 3 guys, it is the intersection of all the normals.

And so, in particular N has to be contained inside the ker(π) ok. So, that is the first thing we know. There is another thing we know, we know that the free group modulo the kernel , the quotient group by the first isomorphism theorem for groups or the fundamental theorem of group homomorphisms says that this is in fact, homomorphic or rather isomorphic to the image of this map ok; to the image of the map from $F \to S_3$.

So, I have an isomorphism and recall what is this, what is this isomorphism do? If I take an element $g \in F$, and I look at the coset $gker(\pi)$ right, the coset of g is just mapped to $\pi(g)$; that is what this this map does ok, that is the isomorphism that comes from the the

$$\Rightarrow \ker\psi = \frac{\ker\pi}{N} = \{id\} \Rightarrow \ker\pi = N \qquad \blacksquare$$

$$\left|\frac{F}{N}\right| \le 6 \qquad \{[w]N : [w] \in F\} = \frac{F}{N}$$

(1) $[a]N = [a']N$ $\qquad [a']^{-1}[a] = [a][a]$

$$= [aa] \in N$$

(2) $[b]N = [b']N$

given map π . So, maybe I will call this map $\bar{\pi}$ which is just induced by π ok. So, I know that this is an isomorphism. So,$\bar{\pi}$ is an isomorphism of groups ok, is an isomorphism.

Now, let us look at N now. So, let us look at $\frac{F}{N}$, let us look at the quotient $\frac{F}{N}$; I claim that since $N \subseteq ker(\pi)$, there is actually a map from $\frac{F}{N} \to \frac{F}{ker(\pi)}$ ok. And this is in fact, a surjective map. So, we usually indicated by this this double arrow here ah. What is this map? So, this is again a general fact. So, maybe we will write this down here separately, here is a general facts which fact which works for all groups and normal subgroups and so on.

So, suppose I have 2 normal subgroups, suppose I have $N1 \subseteq N2$ ok. They are both normal subgroups of a given group G. So, let us say $N1$ and $N2$ are both normal then I can talk about the the 2 quotient groups. So, there is this group $\frac{G}{N1} \to \frac{G}{N2}$, And what I have is a map there always exists a map from $\frac{G}{N1} \to \frac{G}{N2}$ surjective map ok. So, we will call this ψ maybe.

And what is this map do? Well, the most obvious thing takes the coset of $gN1$, the left coset to the left coset $gN2$ ok. So, this is the obvious thing to do and here is simple exercise check that ψ is in fact, the homomorphism of groups that it is onto. And in fact, its kernel, the kernel of this map is exactly well what is it? Its everything which maps to the identity coset; so, that is going to be all elements the cosets of elements from $N2$.

So, this is usually what we call $\frac{N2}{N1}$ ok. So, $\frac{N2}{N1}$ just means I take all cosets $\{gN1 | g \in N2\}$ ok. So, it is sort of a short aside on homomorphisms between such quotient groups. Now, our situation is similar we have in fact, $\frac{F}{N}$ and $\frac{F}{ker(\pi)}$, N is contained in the kernel. And so, actually I have a map from $\frac{F}{N}$ that is a coset of N maps to the coset $gker(\pi)$ which in turn maps isomorphically to S_3 under the map $\bar{\pi}$ ok.

General fact: $N \subseteq G$ normal. If $xN = yN$, then $g_1 x g_2 N = g_1 y g_2 N$ $\forall g_1, g_2 \in G$

Pf: Given: $x^{-1} y \in N$

To prove: $(g_1 x g_2)^{-1} g_1 y g_2 \in N$

$$\underbrace{(g_1 x g_2)^{-1} g_1 y g_2}_{\parallel} \qquad g_2^{-1} x^{-1} y g_2 = g_2^{-1}(x^{-1}y) g_2 \in N.$$

So, $gker(\pi) \to \pi(g)$ ok great. So, I have through this I have managed to get a sequence I have managed to get a homomorphism from $\frac{F}{N} \to S_3$, let us call this ψ. So, in effect let us which is give the whole thing a name. So, this map now which goes from $\frac{F}{N} \to S_3$; so, this is just $\bar{\pi}\psi$ is an is a surjective homomorphism ok.

Now, what we are trying to prove really is that N and the kernel of π are the same. And if that were true then of course, you can see that this map psi would just be sort of like the identity map between them. And, this this composition here would just be the same as the original map $\bar{\pi}$ ok. Now, so what I claim is the following. So, let us let us prove this as follows .

We claim that this group $\frac{F}{N}$ that we do not know much about at this point is in fact, a finite group and has at most 3 factorial elements or at most 6 elements. So, 6 is 3 factorial in this case which is the number of elements in S 3. I claim $\frac{F}{N}$ as at most 6 elements ah, we will prove this in just a second. But, observe that if we assume that the claim is proved then it automatically shows that N and kernel π are equal ok.

So, for the moment assume the claim assuming the claim , let us see what what we can deduce. So, observe because $\frac{F}{N}$ surjects on to S_3 $\frac{F}{N}$ must have at least 6 elements right, because I have an onto map which sort of goes down to a set of 6 elements. Then the the the other set on the other side should have at least those many elements right, just a property on to maps. And, on the other hand the claim says that this has at most 6 elements.

So, assuming the claim observe we would conclude that $\frac{F}{N}$; in fact, has exactly 6 elements ok, because the ontoness gives you the other inequality. And, what does this mean? This in turn would imply that this map that we have constructed $\bar{\pi}$ composition sorry $\bar{\pi}$ composition psi is in fact, an isomorphism because it is both 1 to 1 and onto now . It is an isomorphism of groups.

COR: $[\omega_1][a][\omega_2]N = [\omega_1][a'][\omega_2]N \quad \forall [\omega_1],[\omega_2] \in \mathcal{F}$

$$\underbrace{\cdots\cdots}\; a \;\underbrace{\cdots\cdots}\; N = \underbrace{\cdots\cdots}\; a' \;\underbrace{\cdots\cdots}\; N$$

$$\frac{\mathcal{F}}{N} = \{ [\omega] N \} = \{ [\omega] N : \omega \text{ word in } a, b \}$$

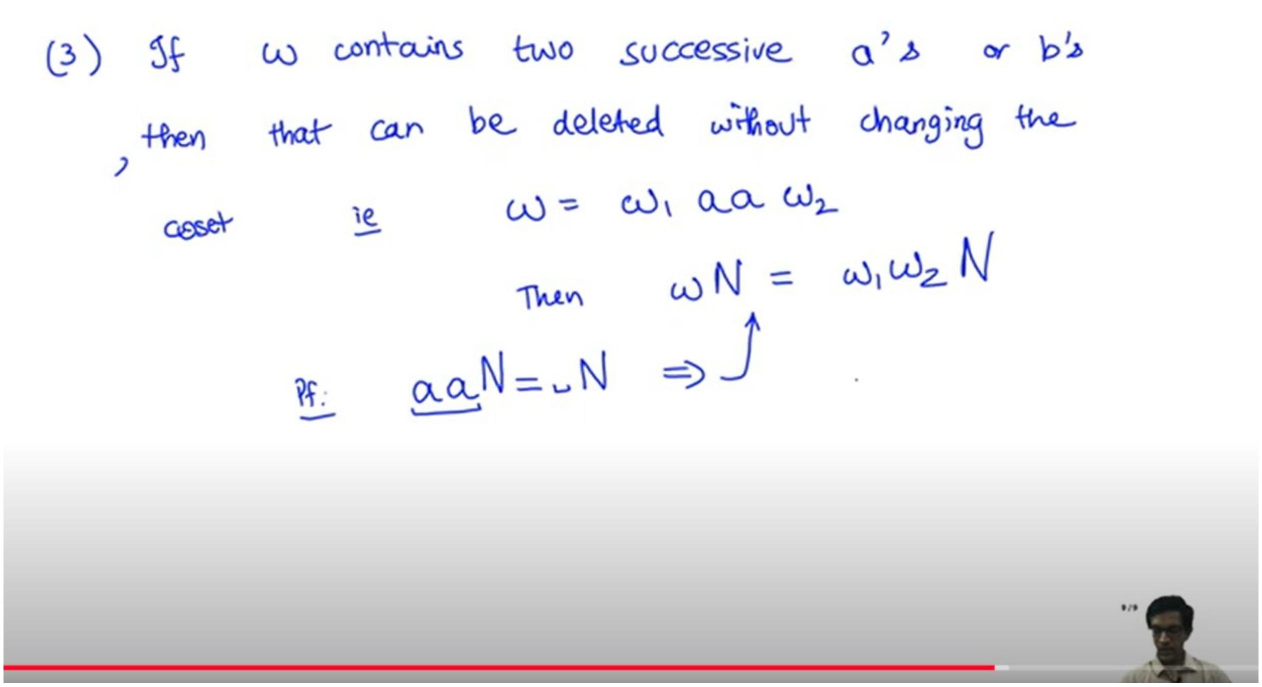
Now, if that is an isomorphism of groups then the only option I mean $\bar{\pi}$ is already an isomorphism; the only option is that psi is also an isomorphism. So, psi is also an isomorphism ok. And what would that imply; that means, in particular that psi is is a 1 to 1 map, psi was already onto claim a size 1 to 1. Now, observe we we sort of know what the kernel of psi looks like, kernel of psi is just the quotient $\frac{N2}{N1}$. In this case it is just $\frac{ket(\pi)}{N}$ ok.

So, in our case what would we get? We would get therefore, that kernel of π which is known to be I am sorry kernel of psi which is $\frac{N2}{N1}$ which in this case is kernel π by N is just the trivial group ok. The kernel π just has only the identity that is exactly what we want to prove ok and so, we are done. So, the proof is done. So, assuming the claim we are we are done.

So, all we need to do really is to prove this claim that this group $\frac{F}{N}$ has at most 6 elements ok and that is really where the the heart of the proof is . And, for a start we should observe that this is a very surprising fact on the face of it. Because what are the elements of F itself? Well, the elements of F look like this, they are all words, the equivalence classes of words.

And now, we are saying let us look at the cosets of you take all possible words in F and look at their cossets left cosets with N. And, this collection is finite has at most 6 elements ok. But, observe the number of words themselves is infinite right . Recall, if I write words I mean what are the elements of the free group words in $aa'bb'$? There is an infinite number of words I can write. But, somehow when you take cosets with N then this reduces to just a finite number ok. So, that is somehow the point.

So, let us try this let us see how this this finiteness would come about. Well, it it comes about because many different cosets should end up giving you the same answer . So, I am looking at all . So, this is my $\frac{F}{N}$. I claim that many cosets give me the same you know many w's give me the same coset ok. For example so, let us see an instance of this. So, let us look



at the word a and let us look at it is sort of the the $^{-1}$ word which is $a\,'$. I claim a and $a\,'$ are actually the in the same coset model of N ok.

Why is this? Well, let us prove this uh how does one prove two things are in the same coset, you just look at $a\,'\,^{-1}$. So, you push this $a\,'$ over to the other side, $a\,'\,^{-1}$ times a. So, you compute what this is, well $a\,'\,^{-1}$ recall $a\,'$ was you know that word was the a^{-1} in fact. So, this is a again . So, this is just $[a']^{-1}[a] = [a][a]$. So, that is just the word $[a]^2$ or $[aa]$ and recall because of the definition of N, $[aa] \in N$. So, this means exactly that these two cosets are equal, that $a\,'\,^{-1}\ a \in$ N.

Similarly, the same thing holds for b , same proof $[b]N = [b']N$. So, these are two examples of different words which give you the same coset. And in fact, here is the here is the general fact again , general fact about cosets of normal subgroups in group. So, suppose I have N which is a normal subgroup of G and suppose I have two different elements which have the same coset.

And, suppose if x and y are 2 elements of the group which have the same coset modulo N. Then in fact, I can do the following, I can hit x on the left and on the right by some 2 group elements , that will have the same coset as$g1\ x\ g2$ N $= g1\ y\ g2$ N , for all $g1$ and $g2$ in my group G ok ah.

Again proof is very simple, let us prove this very quickly. Proof: what is given is the following that x and y are in the same coset which means that $x\ ^{-1}\ y \in$ N , this what is given. And what is it that we want to prove, that if I take $(g1\ x\ g2)\ ^{-1}\ g1\ y\ g2 \in N$, I need to prove that this is also an element of N ok. So, we just compute.

So, let us see what would you get this equals $g2\ ^{-1}\ x\ ^{-1}$, $y\ g2$ of course, the $g1\ ^{-1}$ and g cancel each other off. So, I can get rid of them . Now, what is left is just $g2\ ^{-1}\ x\ ^{-1} = y\ g2$ $= g2^{-1}x\ ^{-1}\ yg2$ was already in , that is the given data. And, this is just a conjugate of that ok because N is a normal subgroup, when you conjugate an element from N, it is again an N. So, this is obviously, in N. So, we are done ok

List all cosets $[\underline{w}]N$

(1) $\_\, N$ (2) aN (3) bN (4) abN

(5) baN (6) $abaN$ (7) $babN$ ....

Claim: $abaN = babN$

Pf: $(bab)^{-1} abaN = b^{-1}a^{-1}b^{-1}abaN$

$= b'a'b'abaN$

$= bababaN$

So, this little general fact is very useful, it says that once I know you know when when 2 elements are representatives of the same coset. Then hitting stuff on the left and the right will again produce 2 representatives of the same coset ok. So, now let us look at how that is going to be used in our case. So, I have a and a' have the same coset. So, what that means, is if I hit a on the left by any word and on the right by some word, that is in the same coset as the corresponding operation to a' ok.

So, here is our corollary to our instance to our situation. So, I take a word a another word, its coset is the same as word a' word N and this is true for all words $[w1]$, $[w2]$ ok. So, maybe I should really put the equivalence classes here . But what does this mean? This just says that if I take any word .

So so, what this means is suppose I have a word here which looks like this is a bunch of letters, there is an a in it and there is another bunch of letters. Then the coset of this word is going to be the same as, well I take these letters keep these letters the same as they are here . Replace, just the a with an a' ok and then keep these letters the same .

So, you you make only one change which is a changes to a', then these two are in the same coset. These two words are going to be in the same coset ok. So, again I am going to drop those square brackets on the 2 sides ok. So, you can replace an a with an a' in a word and that does not change the you know that that still gives you two things with the same coset ok.

Similarly, same thing for b and b' ok; if I change b with with b' then you know it just gives me another representative of the same coset ok. So, that is the the first observation that you can change a's and b's, a's and b's to a's and b's without changing the cosets. So, it is sort of enough therefore, to you know since I am trying to understand this, I just want to know what are all the different; how many different cosets can I get, when I take different words w here? Ok.

BUT $(ba)^3 \in N \qquad ((ba)^3)^{-1}N = (bababa)^{-1}N$

$$= a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}N = ababab\,N$$

$$= N$$

$$\Rightarrow ((ba)^3)^{-1} \in N \Rightarrow (ba)^3 \in N.$$

Claim: If ω is a word in a, b without two succ. a's or b's, then ωN equals one of (1)–(6).

Pf: Induction on $\text{len}(\omega)$

$\text{len}\,\omega = 1, 2, 3$ ✓

$\text{len}\,\omega \geq 4$: $\quad \omega = \ldots baba$

$$\omega N = \ldots \underline{baba\,N} = \ldots bb\underline{ab}\,N$$

$$= \underbrace{\boxed{\ldots ab}}_{\text{len} = \text{len}(\omega) - 2} N \Rightarrow \text{done by induction} \blacksquare$$

Well, the the thing is it is enough to restrict to words w which contains only a's and b's ok. Let us throw out the a 's and b 's because of this this fact here ok. So, it is enough to say, let us look at cosets of words which only involve a and b ok. So, such cosets must be the same as the collection of all cosets ok.

Now, how many such cosets are there? So, we need a few more reductions now. So, here is reduction number 3 which says if this word; so, I am now remember w is a word only with a's and b's. If w contains two successive a's or two successive b's, two successive a's or b's then I can delete those two. Then that can be deleted without changing the coset of w, can be deleted ok.

In other words more formally, if $w = w1aaw2$ then the coset of w is the same as the coset of $w1$ $w2$ as I have deleted the a's from it . Hey, why is this? Well, it is again an instance of the same general principle that I talked about. Proof: recall a square which is aa is the same as so in fact, aaN is the same as just the empty word N ok, because $aa \in N$.

So, aa and the empty word both have the same coset which means I can hit stuff on the left and the right and I will still get things in the same coset ok, which implies well exactly this fact that $w1aaw2$ and $w1$ empty word $w2$ have the same coset ok. Same thing with b's instead of a's. So, this is the next reduction principle, it says you can throw out two consecutive occurrences of a's or b's without changing the coset of your word ok. So, now, that is that is the important reduction.

So, now let us try and list out all the different possible cosets. So, I am going to try and list all cosets of the form $[w]N$ ok. And where I will use my various reduction rules, I will not list cosets which are you know which can be obtained using different representatives. So, here are my different cosets. So,

(1) $\_\_N$.
(2) aN.
(3) bN.
(4) abN.
(5) baN.
(6) $abaN$.
(7) $babN$.

So, I just I am I am taking care not to have two successive a's or two successive b's, that is it ok. So, here is the here is a list, it still looks like an infinite list, but here again comes the next reduction step. We claim that these two are actually equal to each other, these two cosets are in fact, equal ok.

So, that is the next observation; claim: the coset $abaN = babN$ ok. Again proof; what we need to do? We need to take one of them other side as inverse So, look at (bab) $^{-1}$ $abaN$ and we need to simplify this. So, remember $(bab)^{-1}$ is what? b $^{-1}$ a $^{-1}$ b $^{-1}$ $abaN$ inverses remember are the same as $'$s b $'$ a $'$ b $'$ $abaN$. Now, we have already proved the following that whenever you see $'$s in a word, you can replace the $'$s by the un$'$d alphabet and that still gives you the same coset. So, this element here has ok. So, this is now $bababaN$ ok. Now, look at this this word here $bababa$ that is well, what is $bababa$? That is just the word $(ba)^3$ ok.

So, I claim that this word $(ba)^3 \in N$, but observe that the word $(ba)^3$ is actually in N ok. How do we see this? So, recall N had $(ba)^3$ and not $(ab)^3$, but that is not too hard to change this to the other. So, let us look at ba cubed inverse ok. So, what is ba cubed $^{-1}$? It is now going to be instead of $(bababa)^{-1}$; so, its ends with an a. It will now be ah; so, what is this $(bababa)^{-1}$? .

So, I just have to write it out in the other order. So, that is going to be a $^{-1}$ b $^{-1}$ a $^{-1}$ b $^{-1}$ a $^{-1}$ b $^{-1}$, but then recall I can change all the $'$s to the uninversed alphabets without changing the N. So, this so, I have to put N's everywhere. So, look at ba cubed N , its a

same as this, its a same as this which is the same as this, but then ab whole cubed is already in N. So, this is just the same as N ok.

So, what this means is that ba whole cubed is I mean ba whole cubed N. So, so I have shown ba whole cubed $^{-1}$ N is N. So, of course; that means, that ba whole cubed $^{-1}$ is in N . But, then if an element is in N, N is a sub group of course, if the $^{-1}$ is in N, the original element is also in N ok. So, what this means is; so, this this proves what we set out to prove that the the word 7 or the coset 7 is actually the same as the coset 6 ok. So, you do not actually get the 7th guy in this list and well what we now show is that you will not get the 8th guy or the 9th guy or the 10th guy. Every other word that we would have potentially written down will turn out to coincide with one of the first 6 words ok or with one of the first 6 cosets.

So, that is going to be our next observation fact or proposition; claim: if if w is a word in ab, w is a word in a and b without two successive a's or b's, without two successive a's or b's then the coset wN . So, let us forget the brackets now that wN coincides or equals one of the cosets 1 through 6 . In other words this list here empty, a N, b N, abN, baN or $abaN$ ok.

So, we have already proved this for words whose length is at most 3 right, because those are exactly the first 7 guys that I have written here and 7 turns out to give me the same answer as 6 ok. But, what if my words were longer? So, in some sense we just have to prove this by by induction. So, to get a sense of the proof you should probably just work out the next few cases. But, this proof is really by induction on the length of this word induction on the length of this word w.

So, observe that ah; so, I proved it already for lengths length equals 1, 2, 3 we are done . Now, if length is at least 4; suppose the length of w looks if w has length at least 4 then what does w look like? Well, remember it is a word with a's and b's, no successive a's or b two successive a's or b s. So, let for example, suppose the last letter was an a ok, we we can repeat the same argument with b as well. But, if the last letter is an a then the letter preceding it must be a b again a and remember length of w is at least 4 which means I surely have at least one more b. After that I do not know what happens, there could be more letters, there may not be more letters; I do not I do not know. So, I will just put dots there ok. It it may or may not exist ok.

So, w looks like this. So, let me look at the coset of w now , w N is what is this? This is dot dot dot baba N ok, but remember aba N that is the coset number 6 is the same as coset number 7 which was bab N ok. So, by this general principle that we talked about earlier this can therefore, be written as words until b and I will just replace the last 3 guys aba with bab ok.

This only replaces the last 3 with an N ok. Now, let us stare at this. So, we already observed some simplification which is that the 2 b's occur next to each other. So, that is going to cancel or give me the identity. So, this becomes sorry I delete the the b's and let us see what I get, I get the same 3 dots whatever was there; the b's are gone and now I have ab N ok. Now, look at this, this, this new expression that I have gotten.

What is this? Well, this is now a word of length length, the original length minus 2 right; because the 2 of the b's disappeared from this list. So, this fellow has length equal to length of w minus 2 which is strictly smaller than the length of w ok. So now, since I am doing induction on the length of w, now I am done because this is again of the form some word which has a's and b's. Well, you may worry a little bit about the fact that you know I have assumed there are not two successive a's or b's. But, you will notice that if these dots

$\therefore$ we have proved $\left|\frac{\mathcal{F}}{N}\right| \leq 6$.

& hence $\ker \pi = N$

$$\frac{\mathcal{F}}{\langle a^2, b^2, (ab)^3 \rangle} \approx S_3$$

S_3 is the gp on generators a, b

& relations : $a^2, b^2, (ab)^3$

actually existed, if if say this next dot was actually a letter that letter would be an a ok; which would further cancel this a. Or, if if the one before it was a b again that would further cancel the b and what is left is still again of the same form ok.

So, there is just a tiny little bit that you will have to do to convince yourself that the argument works without a problem, but it does. And so, what we have done is by induction the proof is done ok. So, which implies done by induction ok. So, what does that mean? We have shown that any coset of any word therefore, is has to coincide with one of these 6 cosets, 1 through 6 which means that our original claim was proved.

Therefore, we have managed to prove we have proved that $\frac{F}{N}$ has at most 6 elements ok. And therefore, we have proved the the original thing we set out to prove which is that ah. And, hence we have shown that the kernel of this map, the the original map was exactly the normal subgroup generated by just those 3 elements, those 3 relations ok.

So, this this fact that we have shown that F modulo the you know the the the normal subgroup generated by $\{a^2, b^2, (ab)^3\}$. So, I will just use this this sort of angled bracket here to say normal sub group generated by them . So, we have shown this is S_3, this is sometimes expressed as follows; we say S_3 is the group on three generators . So, we say S_3 has 2 generators, a and b and relations.

So, relations is the new word here which says what is in the kernel $\{a^2, b^2, (ab)^3\}$ ok. So, there are 2 generators and 3 relations, between them they give you a nice presentation is is the other word ah. This is suppose, this is sometimes called a presentation of S_3 on 2 generators and 3 relations ok.