Algebra - I

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1. Lecture 25 [Problem Solving II]

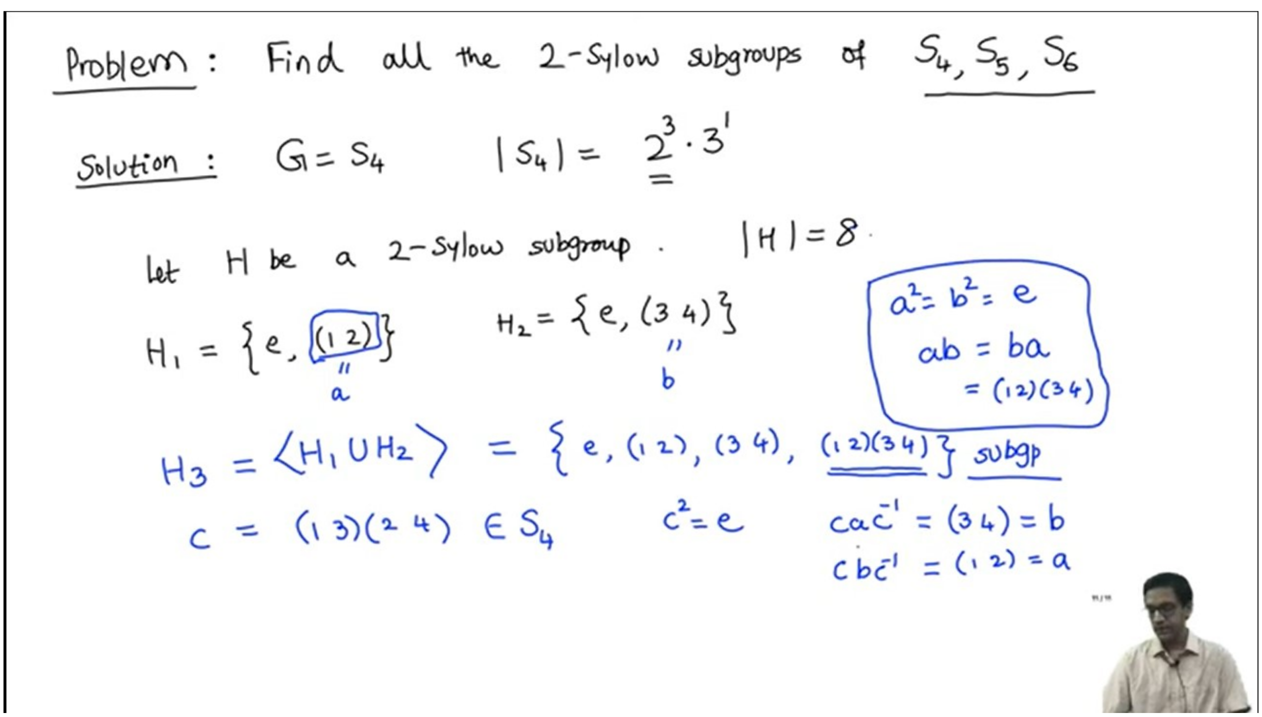
Let us do another problem. So, here is a problem again on Sylow theorems .
Problem:- Find all the 2-Sylow subgroups of the symmetric groups S_4, S_5 and S_6.
So, these are the the ; so, in each case, I want you to try and find the what are all the

2-Sylow subgroups, what they look like a explicitly in terms of permutations, ok.

So, again please do try this on your own . Let me work out the solution for you. So, let us do S_4 first. So, let us take the group S_4 . Its cardinality is 24, cardinality of $|S_4| = 24$ which is I am sorry $4 \cdot 6$ so, that is $8 \cdot 3 = 2^3 \cdot 3^1$. So, the subgroup that we are looking for the 2-Sylow subgroup is subgroup of cardinality 8 in this case ok.

So, I am trying to find the let H be a 2-Sylow subgroup; H be a 2-Sylow subgroup . The cardinality of $|H| = 8$. Let us try and find the subgroups whose cardinality is 8 this case . Now, there are some obvious things we can start with .

So, let us first do so, I am looking at all permutations of the numbers $1, 2, 3$ *and* 4. So, for instance here is the here is 1 possibility, I take the identity let me call the identity as



$H_1 = \{e, (12)\}$ now and the trans position (12). So, this is a subgroup, it is got cardinality 2 rather than 2 cubed ok because (12) this is a transposition, square is identity .

Let us look at another $H_2 = \{e, (34)\}$ again a subgroup of cardinality 2 . So, let me give these these elements names $a = (12)$ and $b = (34)$, these transpositions (12) and (34) . So, observe they are both elements of order 2 and further since they are sort of disjoint, they commute with each other $ab = ba$ are actually equal to each other and their product is nothing, but this (12)(34) again an element of order 2 ok . So, what we can do next is to; so, I am; I am sort of trying to get to this cardinality 8 by looking at other possible powers of 2 as starting points . So, I have constructed subgroups of cardinality two now .

Using these two, we can construct a subgroup whose cardinality is 4 . How is that? Well, actually in this case, we can just look at the subgroup generated by by their union if you wish . So, let us put a, b together I mean we want a subgroup which contains H_1 and H_2 so, that is a subgroup generated by them. And, because of these; these relations here, all I need to do is just look at the subgroups whose elements are $H_3 = \{e, a, b, ab\}$ I do not need to include any further elements. This is the smaller subgroups which contains both H_1 and H_2 ok .

This is the subgroup, this is like the Klein 4-groups if you wish copy of cyclic group of cardinality 2 with itself . So, this guy is again a subgroup , but its cardinality is only 4 subgroup of cardinality 4 and what we really want to do is to go one further level up . We want to double this further in some way ok .

Now, in order to get order 8 , here is my here is the element . So, that will do a trick for us. So, let me tell you what it is first and then, we will see why it does the job . So, consider the element (13)(24) ok. It is not the one I wrote here that is, this guy is (12)(34). I am looking at the product $c = (13)(24) \in S_4$.

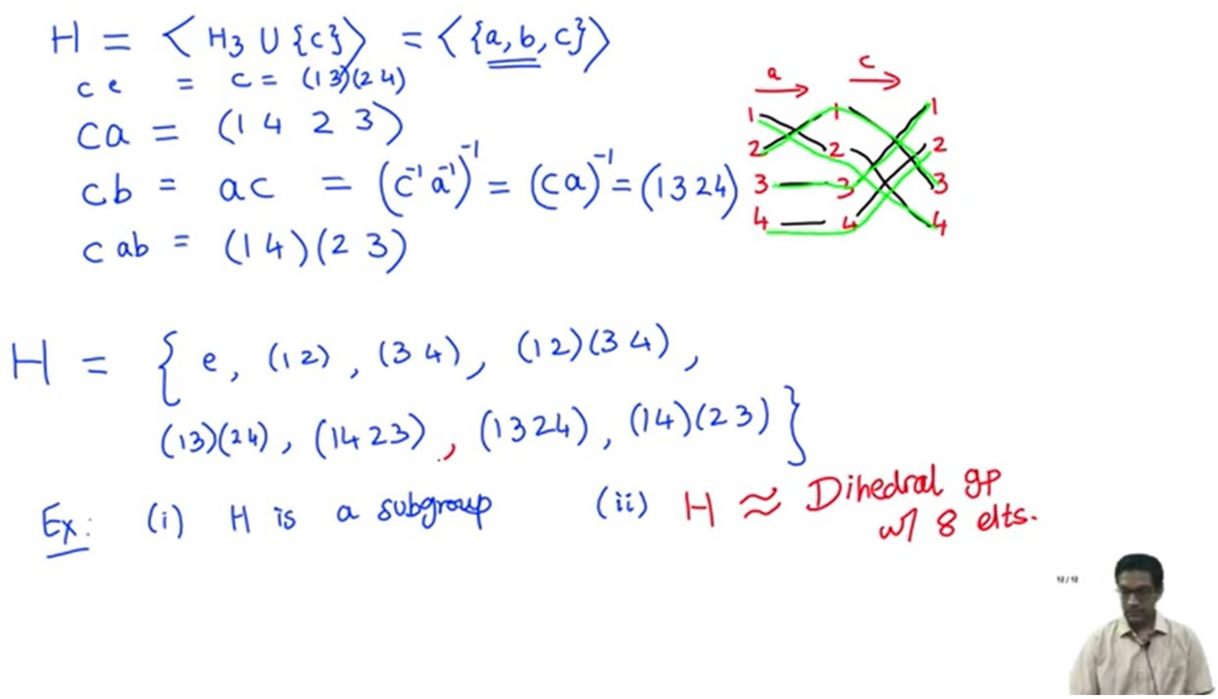
So, let us; let us define this element . Observe that what are the properties of c. Firstly, it is ; it is an order two element that is ok . Unlike this; this; this other element (12)(34) this element c does not commute with a and b ok.

So, in fact, if you; if you sort of conjugate so, if you look at what is cac^{-1} which is if you remember how conjugation works in a symmetric groups , all you have to do is to replace so, take $a = (12)$ and you just replace 1 and 2 by the numbers that they map to under the transformation under the permutation c ok.

So, in this case 1 maps to 3 , 2 maps to 4 . So, cac inverse will just be 1 maps to 3 , 2 maps to 4. So, this is going to become the permutation $(34) = b$ which is exactly $cac^{-1} = b$ ok. By the similar token cac^{-1} will just be 3 maps to 1, 4 maps to 2 this will just become $(12) = a$ ok . So, that is so, in particular tells you that is c does not commute with a and b because $cac^{-1} \neq a$ it is in fact, b in this case ok.

So, we have; we have sort of thrown in this additional element . Now, let us see what we have to , what we get if we also include this element . So, let me now take the subgroup I generated already to that I will add in this additional elements c and now take the subgroup generated by this this collection. So, in other words, I am trying to look for if you wish the subgroup which is really generated by the set $a, b\ and\ c$. So, I have three elements whose order is two and if I just take the first two guys, I know it gives me what I called H_3 . Now, I also want a throw in this additional element c and ask what that gives me ok.

So, now if you; if you look at , so what we need to do ? Once I throw in an additional element, I am must try and multiply it with what I already have and see if what new elements



arise . So, let me take c and I will multiply it with a , I will multiply c with b and also multiply c with ab and let us compute in each case what one gets .

So, let me write out the answers for you this is $ca = (1423)$ the four cycle that is what you get . If you compute ca ok . So, let me just verify that , then I will leave the rest for you to check. So, what is ca mean ? How does one compute products of permutations ? So, first I this is supposed to be the composition of the two maps ac ok . So, let me write out this a , here c which is again $1, 2, 3, 4$. So, what does the the map a do?

So, ca is suppose to be thought of as the compositions of c with a and composition of bijections. So, permutation is a bijection here . So, recall what was a ? A maps $1 \to 2$ and $2 \to 1$ ok. It maps these two to themselves that was what a does , that is what a did. Now, what is c? c maps $1 \to 3$, $3 \to 1$, it maps $2 \to 4$ and $4 \to 2$ right . Let us go back and see what c does ? c was (13)(24) so, it maps 1 *and* 3 its swaps them, it swaps 2 *and* 4 ok. So, that is was this is $1 \to 3$, $3 \to 1$, 2 *and* 4 go to each other .

So, now I have to compose these two maps and ask what the full composition looks like ok. So, let us see , let us chase the arrows. So, what does 1 map 2 under this? 1 goes to 4 ok so, that is what you see here 1 goes to 4. Now, where does 4 go to ? 4 goes to 2 so, that is 4 goes to 2 . Now, so, I have; I have figured this out, I have figured this out let us see what 2 goes to? 2 maps to 1 which maps to 3 ok . So, 2 goes to 3 and then finally, 3 goes to 3 goes to 1 so, 3 goes to 1. So, that is this , that is the cycle here ok. So, what I have done here is just written out the permutation in cycle notation. So, that is (1423). Now, let us compute the other ones cb for example, it turns out to be well I mean you could do the same thing write it out like this, but we could also use the various relations that we have already know, cb is same as ac because $cbc^{-1} = a$.

And what is ac ? Well, I know how to do ca . If I know how to do ca , then I know also how to do ac, ok why is that because $ac = (c^{-1}a^{-1})^{-1} = (ca)^{-1}$ So, I have to swap the order , then take inverses .

But observe that a and c where their squares were identity which means a and c are their own inverses . So, this is just $(ca)^{-1}$ whole inverse ok. Now, I know what (ca) looks like and the inverse of a four cycle is just obtained by reading the cycle in the in the other order ok instead of clock-wise you read it counter-clock wise. So, in this case , it is just I start with 1 , but then I read in the other order I sort of go first to 3 and then to $2, 4$. So, that is (1324) ok.

Now, similarly you have to compute cab and let me just tell you the answer it turns out to be (14)(23) ok. So, what I have it done? I have; I have manufactured when I multiplied everything by c and of course, I should multiple c by the identity which just gives me c itself ok c was (13)(24) ok. So, I get these elements and you can check that if I just put all these 8 elements together, it is in fact, a subgroup ok, it is closed under multiplication .

So, what is my subgroup H in this case ? H turns out to be the elements I have manufactured in this manner:

$$H = \{e, (12), (34), (12)(34), (13)(24), (1423), (1324), (14)(23)\}$$

So, I have manufactured these these 8 elements and like I said you have to check, exercise well two exercises in fact, check that H is a subgroup it is got cardinality 8 and if you stare really hard you will notice something interesting that this subgroup has well it is got this element whose whose order is 4 ok so, I have another element of order 4 .

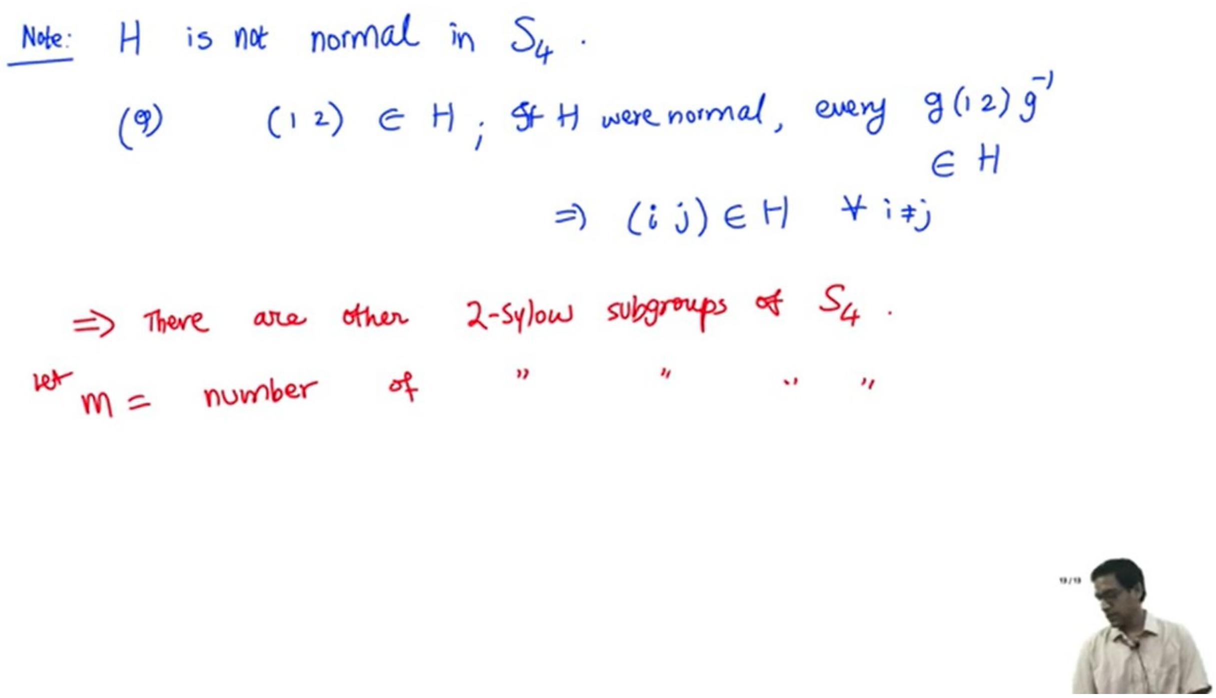
In fact, if you notice that if I take this, let us look at the cyclic subgroup generated by (1423) that is just these four elements $\{e, (1423), (1234), (1324)\}$ ok . So, I have got this subgroup cyclic subgroup of cardinality 4 and then, all the other elements that remain are all a elements you know whose square is identity, their all elements of order 2 ok and this should remined you of something that you know this looks like very much like the dihedral group in some sense ok.

So, that is the second exercise show that H is actually a isomorphic to the dihedral group with the 8 elements that is what we called D_4 ok dihedral group with 8 elements ok and you can you can pretty much see where its coming from ok. So, interesting thing we have seen is that the the Sylow 2-Sylow subgroup of S_4 is actually the the the you know it is got 8 elements and its isomorphic to the dihedral group on on 8 elements ok.

Now, observe something else this subgroup H is not normal ok. So, now, you know I so, far I have constructed one, 2-Sylow subgroup for you, but where are all the others ok? So, observe for a start that this subgroup H is not normal inside S_4 . So, observe note : H is not normal . Why not? Well, because if what is normal mean for example, I take the transportation $(12) \in H$.

If H were normal, it will mean that every conjugate of (12) is also in H right . So, if H were to be normal , if H were normal , that would mean that every conjugate , every element of the form $g(12)g^{-1} \in H$. But then, we know what conjugation does right. What does conjugation do ? It will just map it to every other transposition of this form , in a the other guys of the form $(ij) \in H$ for all $i \neq j$ok, i *and* j running between 1 and 4 in this case.

So, every other possible transposition would also live in H but observe from our description of H that in fact, but H only has two of these transposition. It is got to (12) and it is got (34) ok. In fact, there are how many in all?



Well, there are actually 4 choose 2 transposition which is 6 transpositions inside the the group S_4 and H only contains two of them ok. It does not contain the the other transposition. So, H is not normal and in fact, how do you get? Therefore, what it implies is that there are more Sylow subgroups so, which means that there should be more their exist more 2-Sylow subgroups ok.

How many are there ? How many 2-Sylow subgroups are there? So, this means that there are more . There are other 2-Sylow subgroups as well ok and the question is how many are there ok? So, let us as always so, let us give that a name. So, m denote the number of let this the be the number of 2-Sylow subgroups of S_4 .

Now, a recall from last times problem session, we sort of know how to compute this number by the orbit stabilizer theorem . This number which is just the cardinality of the orbit under the conjugation action. This is the cardinality of the group divided by the cardinality of the stabilizer of any one 2-Sylow subgroups ok.

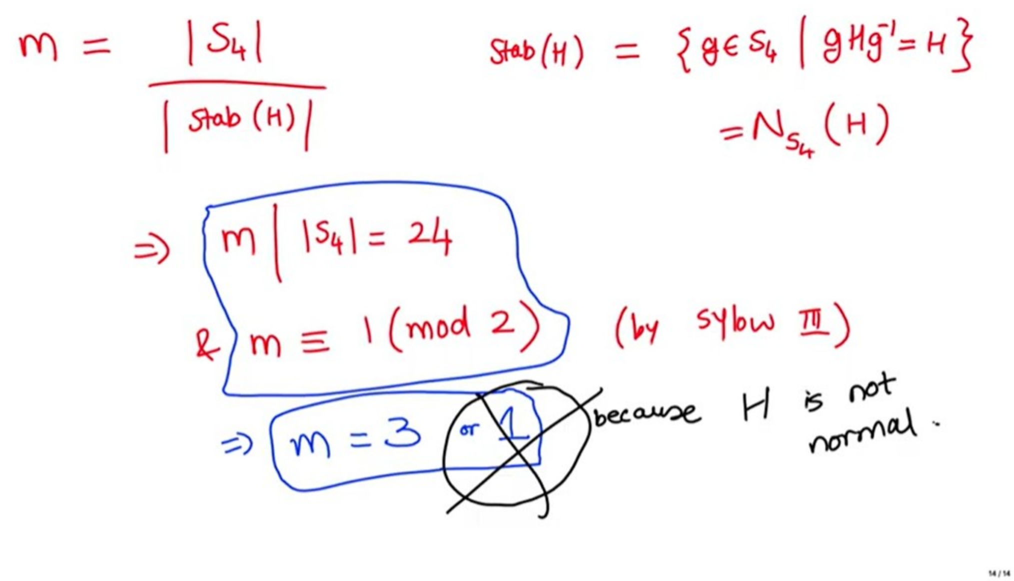
Here, I have picked one which is H and I ask how many elements are there in the stabilizer and recall the stabilizer of H under the conjugation action is exactly what we call the normalizer.

$$stab(H) = \{g \in S_4 | gHg^{-1} = H\} = N_{S_4}(H)$$

. The group is S_4 here ok.

And so, this means that in particular this tells us that this number m must divide the cardinality of S_4 because it is S_4 divided by something. So, m has to divide the cardinality of S_4 other words, m must divide 24 ok and we also know by Sylow theorem number 3 that $m \cong 1(mod\ 2)$ ok. This guy is by Sylow III.

Now, we have two facts about m that m is a divisor of 24 and m is an odd number ok. Well, what is that mean ? This 3 *or* 1 right what are the the odd divisors of 24 , they can



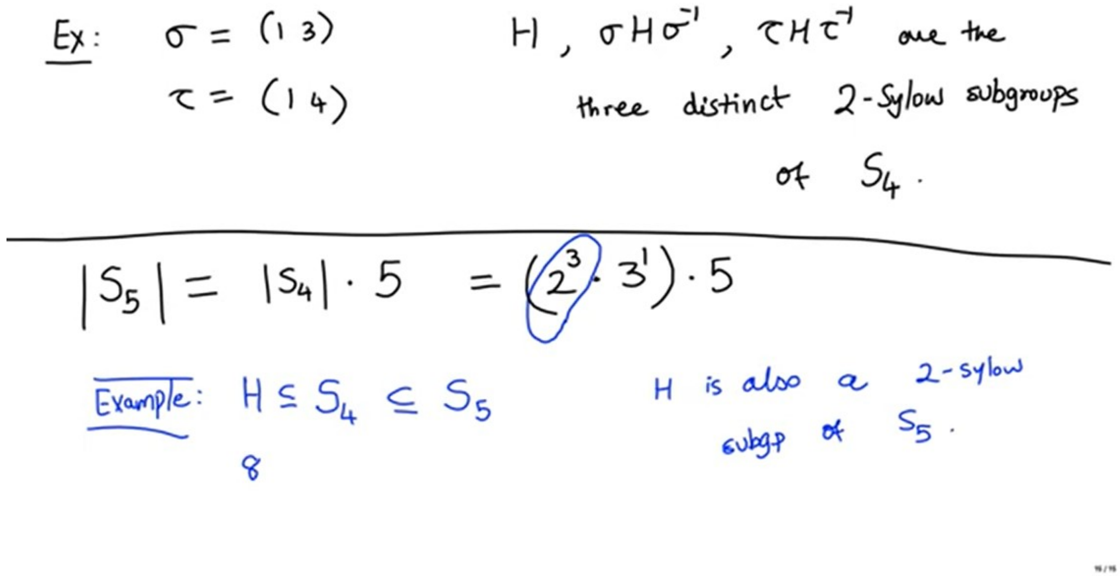
only be 1 or 3, but as we sort of did last time as well, m cannot be 1 because if $m = 1$, that will tell you that H is normal ok.

So, this cannot happen, this is not possible because H is known to be not normal right. We just argued that H is not normal because it contains two transpositions, but not the other four ok. So, that tells you that m is actually 3 that there are exactly three 2-Sylow subgroups ok. They are all of course, conjugates of each other, they are all isomorphic each other as well as a consequence. So, their all copies of the dihedral group.

And here is the to finish off; finish off let me tell you how to get other ones. Let us take two of the other transposition which are missing from H which are $\sigma = (13)$ and $\tau = (14)$. Then, look at the conjugation. So, H itself is $\sigma H \sigma^{-1}$ and $\tau H \tau^{-1}$. These are the 3-Sylow subgroups. These are the three; three distinct 3-Sylow subgroups sorry 2-Sylow subgroups of S_4 ok.

So, we are succeeded in determining what the Sylow subgroups 2-Sylow subgroups of S_4 look like, ok. But the problem also asked us do same thing or S_5 and S_6. So, let us go on S_5 cardinality of S_5 is well what said into 5!. So, it is in fact, the cardinality of S_4 multiplied with an additional 5 right. So, in terms of S_4 remember was $(2^3 3^1)51$ this was a cardinality of S_4 . All you are doing is sort of appending and additional prime to it which is the prime 5 in this case ok.

So, what is that mean? Well, the 2-Sylow subgroup of S_5 still has cardinality 8 ok the highest power of 2 in this case is still 2^3 ok. And let me find another for you and leave the problem finding all of them for you to work out. So, here is an example how does one find a subgroup, I need to find a subgroup of cardinality 8. So, observe that in some sense, we have already solved the problem right. So, observe S_4 is actually a subgroup of S_5 ok. So, because of S_4 is a subgroup of S_5, any subgroup of S_4 is automatically a subgroup of S_5. So,



inside S_4 , I had determined I already had you know my my 3-Sylow subgroup of H and this guy has cardinality 8, this is this is the one we just did you know.

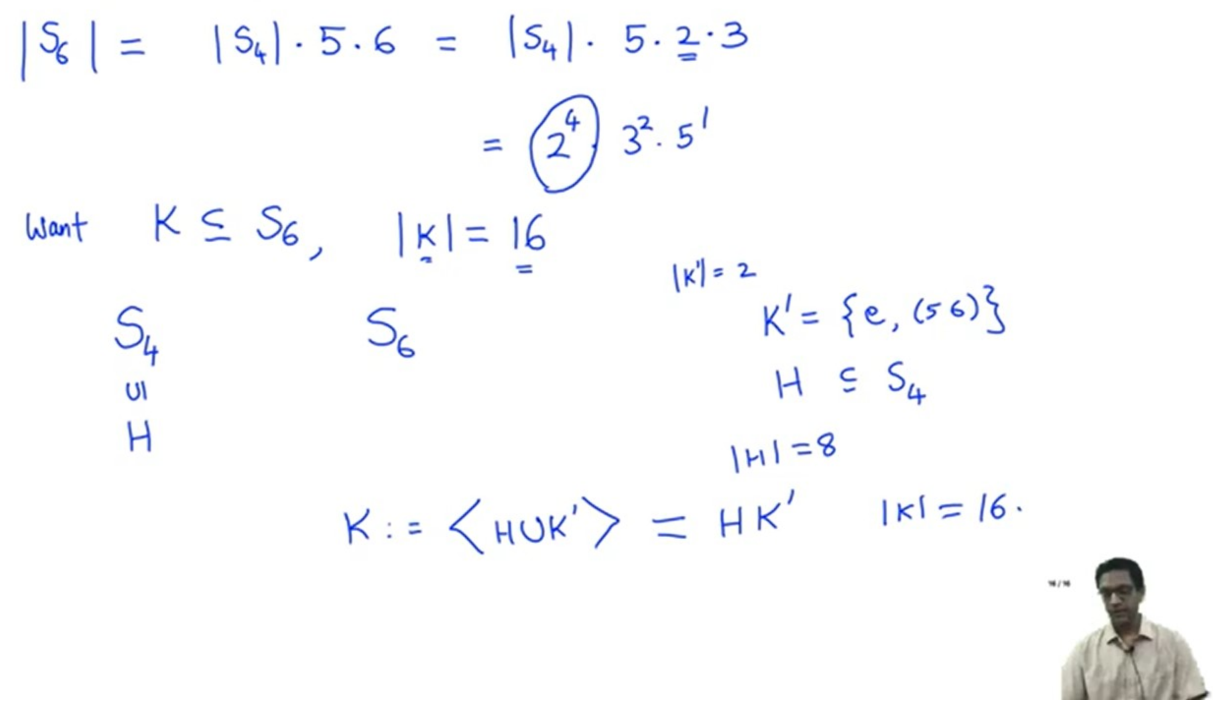
So, this 8 elements subgroup is a Sylow subgroup of S_4 , but it is also a Sylow subgroup of S_5 ok because the the power of 2 you are looking for still the same. So, observe the same answer works H is also a 2-Sylow subgroup of S_5 . So, I given you one example. Of course, like a said, exercise find all of them, how many are there and what are they look like ok.

Let me move on to S_6 now. Let us look at the group S_6 . How does this behave? Well, what we do I have the cardinality now is cardinality of S_4 which is $4! \cdot 5 \cdot 6$. So, in other words, the $6 = 2 \cdot 3$ ok. So, I have cardinality of S_4 , but then I have now increased my power of 2 by 1 ok. So, if I write this out fully, this $2^4 \cdot 3^2 \cdot 5$ ok. So, I am know looking for subgroup whose cardinality is 16 rather than 8, ok.

So, how do I find the subgroup K of cardinality of 16 inside a S_6? That is now my my question right. So, I want K let us call it subgroup of S_6 whose cardinality is now 16 ok. Again, a moments thought, we have solve the problem for a S_4 right. So, inside S_4 , we have determined subgroup whose cardinality is 8. Now, what is S_4 ? It is all permutation of $1, 2, 3$ *and* 4. S_6 is all permutation of $1, 2, 3, 4$ *and* $5, 6$ ok. So, I have two more elements.

Now, how do I construct K is suppose to have twice the number of elements as has H ok. So, here is the; here is the idea. So, just look at these two additional number that we have . You know we have the number of 5 and 6 now when you go from S_4 to S_6. So, let us also look at the the subgroup which is just generated by the transposition (56) thing which just permutes to over them. So, let us; let us give this name let us call this $K' = \{e, (56)\}$. This is a subgroup whose cardinality is 2 ok.

Now, in addition, I already had the subgroup of H which I constructed in the first instance when I was looking for a S_4 . I have already had H in my hand. H has 8 elements. K' has



2 elements. Now, let me sort of smash them together and see what I get. So, let me define K to just be the subgroup generated by H and K' ok.

And, again I leave this as an exercise because this in some sense, K' every element of K' is commutes with the every elements of K ok why because H only affects the first four numbers right, this all elements of H are all permutation of $1, 2, 3$ *and* 4 whereas elements of K' or permutation of the other numbers 5 and 6. So, in some sense, they do not interact with each other. They act not disjoint subsets of numbers. So, every element of H and every elements of K' actually commute with each other ok.

So, if you; if you take their union and look at the subgroup generated, it is just going to be all you know all products, all possible products of elements from H and K' ok. So, this is just going to be everything in H multiplied with the elements of K' and so, this will have turn out that this is exactly cardinality 16 ok. So, cardinality of K turn out to be exactly 16 ok.

And, you can if you wish write it out all the element so, what are the elements of this K look like let me go back the elements of H . Here are the elements of H. So, K is nothing, but the the same elements of H union, the elements of H multiplied by the transposition $5, 6$ ok. So, I can write all these out, then I can also write out the the once with $5, 6$ so, let us do that. So, this is my H. So, lets copy H here and let us go here let me look at .

So, what is K now? K is all these very same elements union these elements multiplied with $5, 6$. So, $1, 2, 56$ and so on. So, you will get 16 elements when you do this. So, let me just write them all out ok. And I am again going to leave the question of how many there are, how many 2-Sylow subgroups are there? So, I just constructed one for you is a 2-Sylow subgroup of S_6, how many 2-Sylow subgroups are there and how do you determine all of them ok.

$$K = \{e, (12), (34), (12)(34), (13)(24), (1423), (1324), (14)(23)\}$$

$$\cup \{(56), (12)(56), (34)(56), (12)(34)(56), (13)(24)(56), (1423)(56), (1324)(56), (14)(23)(56)\}$$

$$\subseteq S_6$$

So, I would encourage to sort of explore problems of this kind you know try doing it for 3-Sylow subgroups or 5-Sylow subgroups and so on for the various symmetric groups ok. So, you all you know it is a; it is a very interesting exercise to see we what sort of permutation should take and so on ok .