Algebra - I
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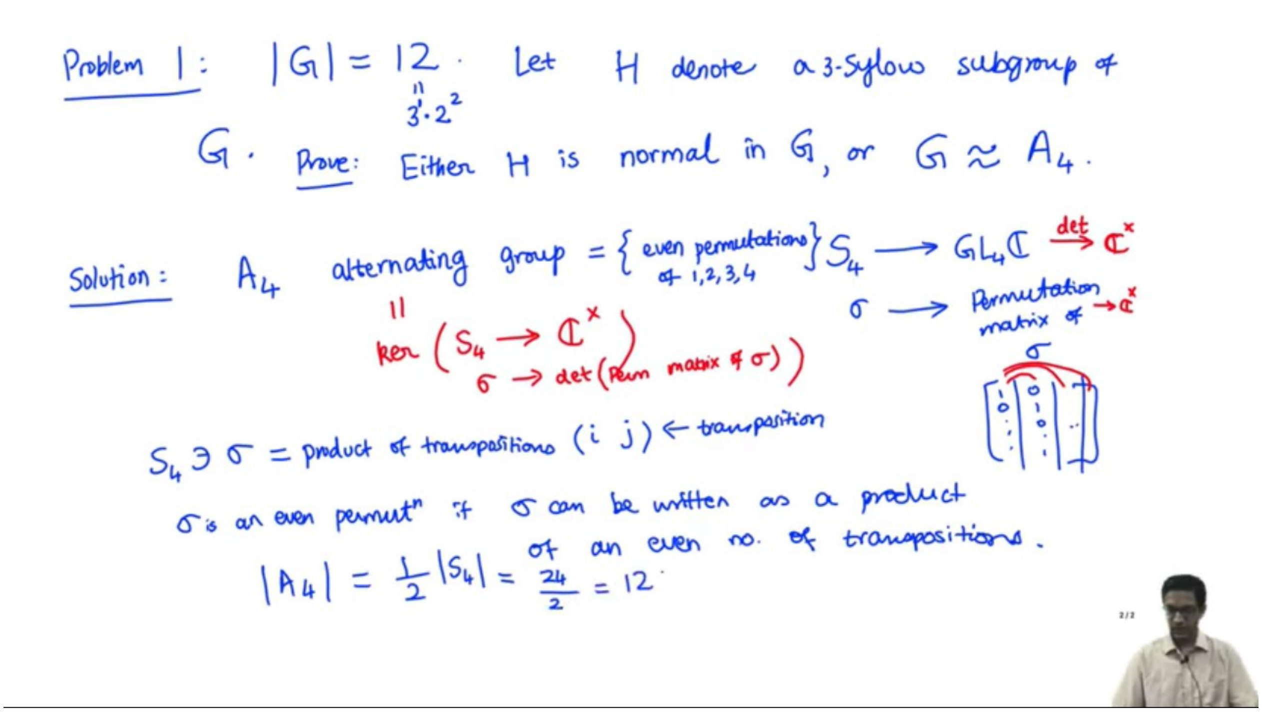
1. Lecture 24 [Problem Solving I]

Let us do some problems on applications of Sylow theorem of all three Sylow's theorems, applications of the Sylow theorems ok. So, yes the first problem suppose G is a finite group whose cardinality is 12 ok, then prove that prove that .

So, maybe we should give it Sylow's subgroup name as well. So, let H denote Sylow 3 or 3 Sylow subgroup of G . So, 12 of course, 3×4, so that is 3×2^2 square that is the prime factorization and so by Sylow's theorem, we know that there is 3 Sylow subgroup which is basically a subgroup whose cardinality is 3^1 . So, pick any such subgroup and call it H.

So, then the what we need to prove is the following then prove the following either H is normal either H is normal in G . So, it is a normal subgroup or the group G is isomorphic to A_4 ok. So, this is what we need to prove. Either the 3 Sylow subgroup is normal in this group or the group itself is isomorphic to the group A_4 ok.

So, let us work out the solution to this problem. First, let us recall what the group A_4 was. So, what is A_4 ? This is the what we call the alternating group. So, you encountered this once before, the alternating group on four letters. How was it defined? So, if you look



back on the lecture on normal subgroups you will recall that this was defined as follows. Ah So, in general I take S_n or in this case let us do it for this case, I take the symmetric group S_4 of all permutations and I can actually map this to the group of all.

So, let us call it $GL_4(\mathbb{C})$ may be complex 4×4 complex matrices which are invertible and how do I do it? I take each permutation sigma in S_4 and map it to the corresponding permutation matrix.

So, recall we have what is called the permutation matrix associated to each permutation and how do you get it? Well, you start with the identity matrix right. So, I have the identity matrix whose columns are $[1, 0, 0, ..., 0]^t, [0, 1, 0, 0, ..., 0]^t, ..., etc.$ So, that is my identity and then the permutation matrix of sigma does the following it takes the columns of this identity matrix and permutes it, permutes these columns according to the permutation sigma ok. So, I sort of have to just permute my columns according to what sigma does ok . So, this is just something for you to recall if you look back on the lecture on normal subgroups.

So, this is a homomorphism of groups mapping a permutation to the corresponding permutation matrix and I can further look at the determinant map. So, I can further compute the determinant of each permutation matrix turns out it is either plus 1 or minus 1 in general. So, the determinant actually maps to so, the determinant is a map which takes invertible matrix to a non zero complex number. So, which I will denote as $\mathbb{C}^\times$ with a cross on top.

So, this is the set of all non zero the multiplicative group of non zero complex numbers and the . So, if you sort of take this composition of these two maps to each σ you associate the determinant of its permutation matrix and so, you get a homomorphism and the kernel of that entire homomorphism. So, kernel so, how did we define the alternating group? Here is one way of defining it is nothing, but the kernel of the homomorphism from S_4 to the multiplicative group of complex numbers where this homomorphism is defined by saying each σ maps to the determinant of the permutation matrix of σ ok. So, this was how it was defined.

So, kernel of so in this case the multiplicative group of complex numbers, non zero complex numbers, it is it has identity being the number 1. So, what we are saying is its all permutation matrices whose determinant is $+1$ ok. So, of course, this one way of defining it another way of defining it is to say this somewhat more concrete, easier to understand. Ah, We say that this alternating group is nothing, but the set of what are called even permutations of $\{1, 2, 3, 4\}$. So, what is the set of even permutations ? So, recall that any permutation σ. So, if σ is an element of S_4. Any permutation can always be written as a product of transpositions ok.

So, what is a transposition? A transposition is just a permutation of the form (ij) which means that it leaves all the other numbers fixed and it sends $i \to j$ and $j \to i$ ok. So, that is your transposition. So, this sort of permutation is what we call a transposition and it is a fact that any any permutation can be written as a product of transpositions ok. This in general , we say that a permutation is even, if it can be written as a product of an even number of transpositions ok.

So, we say that so, this is the definition σ is said to be an even case an even permutation if σ can be written as a product of an even number of transpositions ok and turns out that this is another way of saying the same thing the determinant is $+1$ is the same as saying that σ can be written as a product of an even number of transpositions ok .

And the other important fact about A_4 is that the cardinality of A_4 is exactly half the cardinality of the entire symmetric group S_4 . So, S_4 is 4! cardinality which is 24 and so, this

H = 3-Sylow subgp of G $\Rightarrow$ $|H| = 3^1$

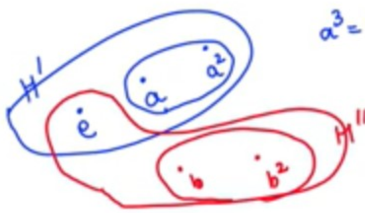
Suppose H is not normal, we need to prove that $G \approx A_4$.

Sylow III : Let m = no. of 3-Sylow subgroups of G

$\equiv 1 \pmod 3$ $\Rightarrow$ $m \in \{1, 4, 7, 10, \dots\}$

If H' & H'' are distinct 3-Sylow subgps $\Rightarrow$ $\underbrace{H' \cap H''}_{\text{proper subgp of } H'} = \{e\}$

$a^3 = 1 = b^3$



Claim: The no. of elements of G of order $3 = 2m$

Pf: Exercise!

is 24/2 which is exactly 12 ok. Now, let us look at what we need to prove, we are given the three Sylow subgroup is is called H and so, we know let H be what is given.

This was a 3 Sylow subgroup and since the cardinality of $|G| = 3 \cdot 2^2$, the cardinality of $|H| = 3$. The maximum power of 3 which divides the cardinality of G ok.

Now, what we need to prove let us look back on what we need to prove either H is normal or G is A_4 ok. So, let us let us do it as follows let us assume that H is not normal suppose, H is not normal ok we will we will show, we need to prove that the group then is isomorphic to A_4 ok. So, this is our our goal .

Let us try and use the Sylow theorems. So, we will use for a start, let us look at the 3rd Sylow theorem. So, we will first apply Sylow theorem number 3 . Ah What does this say? It talks about the number of 3 Sylow subgroups. So, let m denote the number of 3 Sylow subgroups . Ah, what the third Sylow theorem says is that this number $m \cong 1(mod\ 3)$.

What; that means, is that it can take one of the following values it can be 1, 4, 7, 10, etcetera right. It is one of these possible numbers numbers congruent to $1(mod\ 3)$ ok. Now, let us try and understand this 3 Sylow subgroup a little better. So, suppose if I have two 3 Sylow subgroups.

So, I have H', if H' and H'' are both 3 Sylow subgroups , two distinct 3 Sylow subgroups then what does that mean let us try and draw a picture. So, I have H', what does it have? Its got an element e. Let us call the other two elements as a and a^2 its a group of order 3. So, its got to be cyclic now I take H'' which is another such guy.

Now, the point is that these two subgroups if they are distinct then this automatically means that their intersection has to only be the identity . So, I can only get H'' looking like this that should be another b and b^2 ok.

Why is this; because well their intersection is after all another subgroup right. The intersection of two subgroups is a subgroup and this is a subgroup of both H' and H'' if you wish and its got to be a proper subgroup, because H' and H'' are two distinct sets or or they are not equal to each other. So, this intersection has to be a proper subgroup of H' ok, but $|H'| = 3$ and so, if you have a proper subgroup then that is only got to be cardinality 1 ok.

So, this tells us that any two of these 3 Sylow subgroups they must intersect in this manner. They have to have the identity common, but the other two elements are are separate meaning they have no intersections ok. So, what does that tell us in particular. So, if you look at what each 3 Sylow subgroup therefore, contributes, there is the identity and then there are two elements of order 3. So, observe this a, a^2, b, b^2 and so on, they are all elements of order 3 right, because the group has order 3 ok.

So, what does a subgroup a 3 Sylow subgroup contribute? There is one identity and two elements of order 3 . These are both elements of order 3. Similarly, if I take the other 3 Sylow subgroup here , its got these two elements whose order is 3 ok and so on.

So, in general I claim the little observation here is that the number of elements of order 3 , the number of elements of G of order 3 is exactly 2 times the number of 3 Sylow subgroups m ok. I claim that this must be true ok. So, a moments thought should convince you that this is in fact, right why is this, because if I well on the one hand we have just said that if I take a 3 Sylow subgroup , it contributes two elements of order 3 ok and these elements are all distinct. If I take two different 3 Sylow subgroups sort of intersecting each other I get two elements of order 3 from one guy, two other elements of order 3 from the other subgroup.

So, it is clear that there are at least these many elements of order 3 in my group ok, but if on the other hand if I have an element of order 3, then it generates a cyclic group of order 3 ok which is automatically a 3 Sylow subgroup ok. So, any element of order 3 necessarily belongs to some 3 Sylow subgroup ok. So, I sort of orally described the argument, but I would like you to try and formulate this into into a rigorous sequence of steps. So, proof exercise ok.

So, now we we sort of understand what these 3 Sylow subgroups are good for. They contribute elements of order 3. They contribute 2 elements of order 3 each and so, now, we we already said what are the possible values of m. They can be 1, 4, 7, 10 and so on, but we can already see that 7 10 and so on are all not possible values. So, I cannot have $m = 7$, because if $m = 7$ that will tell me that there must be 14 elements of order 3 in my group, but my group itself is only of order 12 ok. So, all these numbers 7 and higher will turn out to be not possible. They cannot be values of m, because they will tell us that there are too many elements of order 3 in my group ok.

So, observe that $m = 1 \; or \; 4$ since the group itself. So, since $2m$ the total number of elements of order 3 can at most be the cardinality of the group and the cardinality of the group is 12. So, this tells us that $m \leq 6$ for example, and among the possible values 1 and 4 are the only ones which satisfy this inequality ok.

So, there are two possibilities m is either 1 or 4 and let us see which of these is indeed possible. If $m = 1$ what does this mean? This means that there exists only one 3 Sylow subgroup. So, there exists a unique 3 Sylow subgroup ok, but what does this mean? So, recall all 3 Sylow subgroups are are mutually conjugate if you wish that is the second Sylow theorem. Ah, So in this case so H. So, this this H is the unique we already gave it a name, we picked one of them and called it H. H is the unique 3 Sylow subgroup and what does that mean?

If I conjugate H by any element of the group that is again a 3 Sylow subgroup, because its cardinality is 3, but there is only one 3 Sylow subgroup which is H itself. So, this must

$m = 1$ or 4 $\quad \because \; 2m \leq |G| = 12 \implies m \leq 6$

(i) $\underline{m=1}$ $\implies$ H is the unique 3-Sylow subgp

$$\underbrace{gHg^{-1}}_{\text{also 3-Sylow}} = H \quad \forall g \in G \implies H \text{ is normal}$$

$$\implies m \neq 1 \implies \boxed{m=4}$$

be true. The conjugate any conjugate of H must coincide with H ok, because any conjugate of H is also a 3 Sylow subgroup, this is also 3 Sylow subgroup ok.

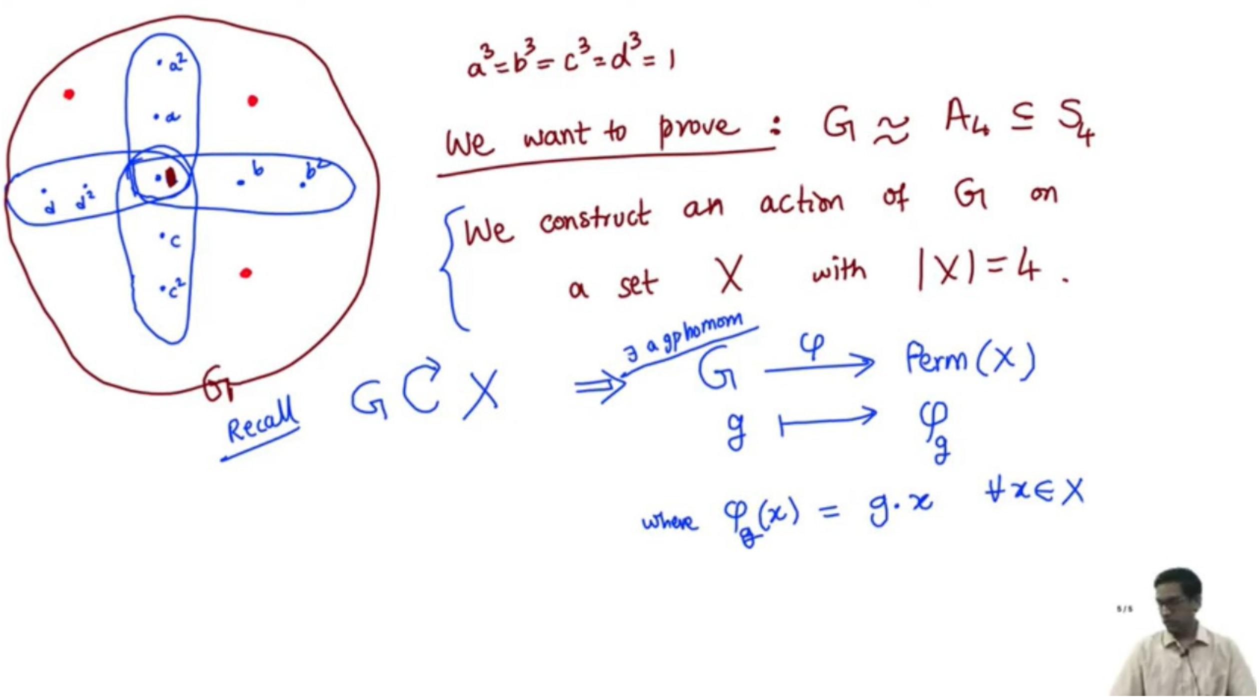
$$gHg^{-1} = H;\ \forall g \in G \implies H\ is\ Normal$$

Now, what does that imply that tells you? That H is normal ok. Every conjugate coincides with the group itself the subgroup itself ok, but we had assumed to start with that H is not normal that was our hypothesis ok. So, that implies that therefore, $m \neq 1$. So, this case cannot arise. So, we conclude therefore, that $m = 4$. So, the final conclusion here is that there are exactly four 3 Sylow subgroups in my group G ok.

So, let us see what that will give us. So, let us try and draw the the same sort of picture that we drew before on how many elements there are of of each of of order 3. So, let me draw my four 3 Sylow subgroups like this. So, here is my 3 Sylow subgroup identity $\{a, a^2, I\}$and then two more elements $\{b, b^2, I\}$ So, this is my identity element. So, here is a third of the subgroups $\{I, c, c^2\}$. So, I am just drawing my 3 Sylow subgroups c, c^2 and $\{I, d, d^2\}$ ok.

So, these are all my 3 Sylow subgroups. I have 4 of them and this accounts for $8 + 1 = 9$ elements of my group. So, I know there are three more elements left. So, what does my group look like? There are three more elements left in my group. So, what I have drawn here is a picture of my group G itself ok. So, think of this as my group G now sort of have some understanding about the orders of the various elements for example, ok . So, of course, I am saying a cubed b cubed c cubed and d cube are all equal to the identity. So, maybe let us call that 1 instead of e. So, they are all equal to 1 ok.

Now, what is it that we now need to prove we need to show that the group $G \cong A_4$ right that was our objective. Now, why should this group be isomorphic to A_4 ? In general when you are trying to prove statements like this that a group is isomorphic to say maybe some symmetric group even or a group is isomorphic to some subgroup of a symmetric group the



way one one tries to establish facts like that is by constructing an action of this group G on a set whose cardinality is 4 for example.

So, in our example; so what we want to prove? We want to prove that $G \cong A_4$. A_4 remember is some subgroup of the symmetric group S_4. When you want to do thing like this we will do it as follows; we construct an action ok we will construct an action of the group G on a set X with 4 elements ok.

Now, why does that help us? Why should we try and do it in this manner? Ah, How does that help; because recall from the the earlier set of problems that we did group actions can be thought of in an in an alternative way if a group G acts on a set X. So, recall , when I have this this means I can I obtain a group homomorphism from the group $G \to Perm(X)$ to the group of all permutations of X ok are all bijections from $X \to X$.

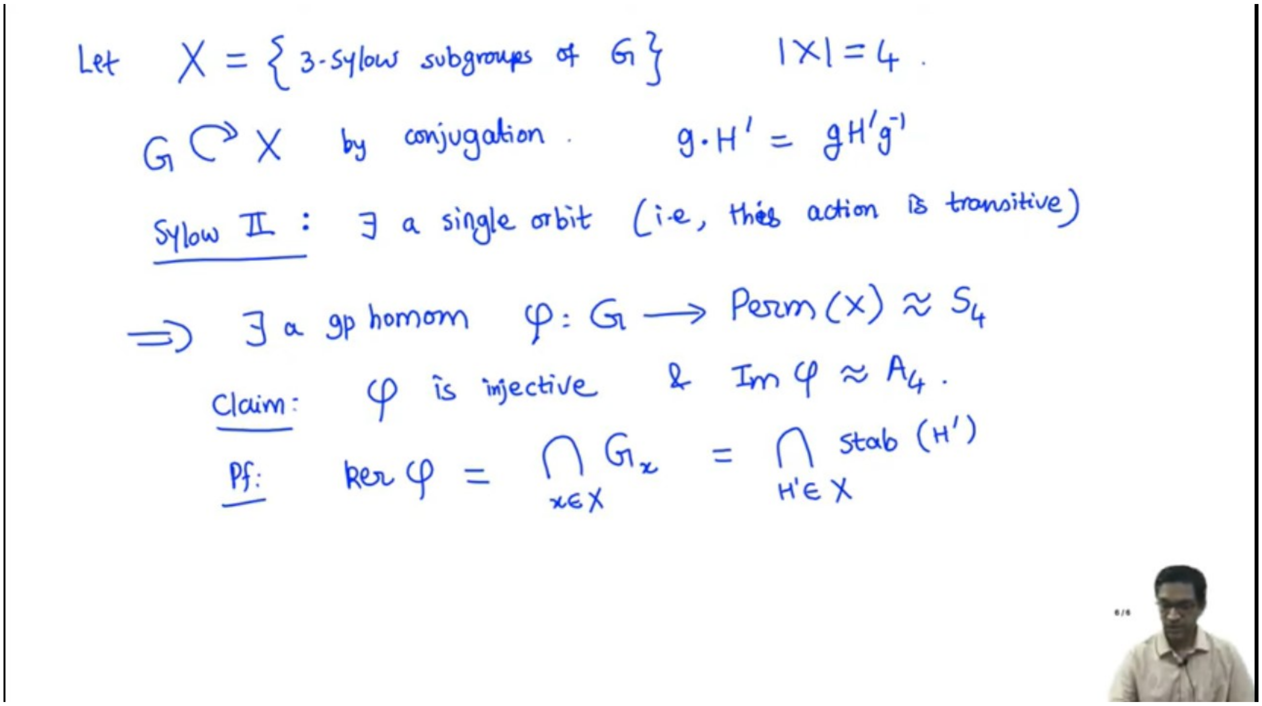
And if you recall how was this map defined each group element g was mapped to a certain permutation called ϕ_g where ϕ_g was defined as follows ϕ_g is a permutation of X which is just

$$\phi_g(x) = g \cdot x; \forall x \in X$$

ok. So, this is a group homomorphism. So, ϕ_g is a group homomorphism, recall given a group action there exists a group homomorphism ok. So, if I construct a group an action on a set of 4 elements then what I will get? At least is a homomorphism from my group G to the set of all permutations of a set of 4 elements ok and permutations of a set of 4 elements is is like S_4 right.

So, it is the the symmetric group. So, at least we will get a map a homomorphism from $G \to S_4$ and we then try and prove that this homomorphism is $1-1$ and its image is exactly A_4 ok that is the idea that is going to be the strategy ok.

Now, how do we how do we construct this this set X. So, so in in this case firstly, let us look for a set of four elements, it turns out the the correct object to take I mean there are many-many possibilities, but the correct object to take is what I have denoted in this diagram



here is the set of these four subgroups ok. There are four 3 Sylow subgroups and this four element set which comprises these four subgroups is what I need to take ok. So, let us let us make that precise. So, let us take X. So, I am going to define $X\{3 - Sylow\ subgroups\ of G\}$ as follows be the set of 3 Sylow subgroups of G. So, I know there are exactly 4 of them ok.

Now, let us construct an action of G on X that is easy too, that is its sort of natural. We just take the conjugation action . So, what do we mean by that? Ah, we just take g acting on any one of the Sylow subgroups will just give me $g \cdot H' = gH'g^{-1}$ the conjugate of that kind ok and the what Sylow theorem 2 tells us Sylow 2 tells us that all the Sylow subgroups are mutually conjugate to each other right.

So, what Sylow 2 tells us is that this action is transitive. In other words that that there is a unique orbit ok. Sylow 2 says that there exists, a single orbit in this case or to use a word which we introduced in the earlier problem session i e the this action is transitive this action is transitive ok.

So, the conjugation action on the set of 3 Sylow subgroups is is transitive according to the second Sylow theorem and that is a that is a set of four elements. Now, let us see if if that gives us what we want. So, now, by by our earlier principle, by the general principle I just talked about which implies I get a group homomorphism . Let us call it

$$\varphi : G \to Perm(X) \cong S_4$$

and the permutations of X as I just said is like the symmetric group S_4 because X has exactly four elements.

Now, let us try and establish the following facts:

Cliam φ is injective and $Img(\varphi) \cong S_4$

So, let us see how we would establish these two facts. So, proof of claim recall from the problem that we did last time, in the earlier problem session that we actually know how to where $Stab(H') = \underline{\{g \in G \mid gH'g^{-1} = H'\}} = \underset{\supseteq H'}{N_G(H')}$ "normalizer of H'"

$$4 = |orbit(H')| = \frac{|G|}{|Stab(H')|} = \frac{12}{|Stab\, H'|} \Rightarrow |Stab\, H'| = 3$$

$$3 = |Stab(H')| \geq |H'| = 3 \Rightarrow Stab(H') = H'$$

$$\therefore \ker \varphi = \bigcap_{H' \in X} Stab(H') = \bigcap_{H' \in X} H' = \{1\}$$

$\Rightarrow \varphi$ is injective.

compute the kernel of this homomorphism φ ok. So, the kernel of this this homomorphism when you have a group action and you consider the corresponding homomorphism, the kernel of that homomorphism we said is nothing, but the intersection of all the stabilizers ok.

$$Ker(\varphi) = \cap_{x \in X} G_x = \cap_{H' \in X} Stab(H')$$

So, I have to compute the the stabilizer of of all the various elements of X in this case. Now, what is that? So, this is just the intersection of the stabilizer of H', let us say H' is any one of the four 3 Sylow subgroups ok and what is stabilizer now means? Stabilizer by definition for the conjugation action where when I say the stabilizer so, I am just using this rotations tab for stabilizer of H' by definition is all those group elements which act on H' and leave it fixed ok.

So, $gH'g^{-1} = H'$ and this is what we usually call the normalizer. So, this has another name, this is sometimes called the normalizer of the subgroup H prime ok. In general the normalizer of a subgroup is nothing, but the stabilizer of that subgroup under the the conjugation action ok. So, we we are talking about the the normalizers, this case ok.

Now, let us see what what else can we can we conclude about the stabilizers. So, recall from the orbit you know the the counting theorem, not the one which counts orbits, but the cardinality of an orbit. So, recall how does one find the orbit of H' the orbit stabilizer theorem. So, the orbit is related to the stabilizer in this way right that is how we find the stabilizers if you I mean at least the cardinalities if you know the cardinality of the orbit that determines the cardinality of the stabilizer.

Now, observe in this case we know the the cardinality of the orbit right the orbit as we just said it is a transitive action. So, all 4 Sylow subgroups are in the same orbit. So, if I start with any H' and I apply the conjugation action I get all the other 3 Sylow subgroups. So, the orbit is equal to the entire set X, cardinality is 4 and cardinality $|G| = 12$.

So, from this I conclude that the $|stab(H')| = 3$ ok, but here is the the important observation from the definition its all those elements which when you conjugate with H', they give you back H', this certainly contains H'. At least H' certainly is inside this right, because if I take G to be an element of H' and I conjugate by that element G of course, the answer will lie again in H' right. If $G = gH'g^{-1}$ is also in H'.

So, the normalizer of of a subgroup certainly contains the subgroup itself. Now, the subgroup of course, has cardinality three as well ok. So, what is this say? We have just concluded that the stabilizer of $|stab(H')| = 3$, but on the other hand this stabilizer contains the original subgroup H', whose cardinality is also 3 ok. So, what is the conclusion, it just means this can only happen if the stabilizer is H' itself ok.

So, for any 3 Sylow subgroups we also conclude the following fact that its normalizer is itself ok the or the stabilizer for the conjugation action ok. So, what does that give us it determine it helps us determine the kernel. So, why did I get here; because I was trying to say I can try and determine the kernel of this map right.

So, I need to find this out that is the intersection of all the stabilizers now I know what the stabilizers are . So, therefore, conclusion the kernel of this map phi is nothing, but the intersection of all the stabilizers H' running over all the four subgroups, but the stabilizers are themselves H' ok and this as we know is just in fact, if you intersect any two of them it gives you the identity.

So, of course, if you intersect all four you get the identity itself right that is clear from our our picture. So, the intersection of these four subgroups is only the identity element ok. So, what does that mean it says that the kernel is just identity which means that ϕ is injective. So, this part is now done right. So, I I just concluded here that this is this of course, means that phi is injective ok. Already we are in good shape.

So, let us look at the second part of the proposition which is we need to show that the the image of phi is going to be isomorphic to A_4 ok. Now, this is again an interesting argument which is that so, let us look at the the map phi itself .

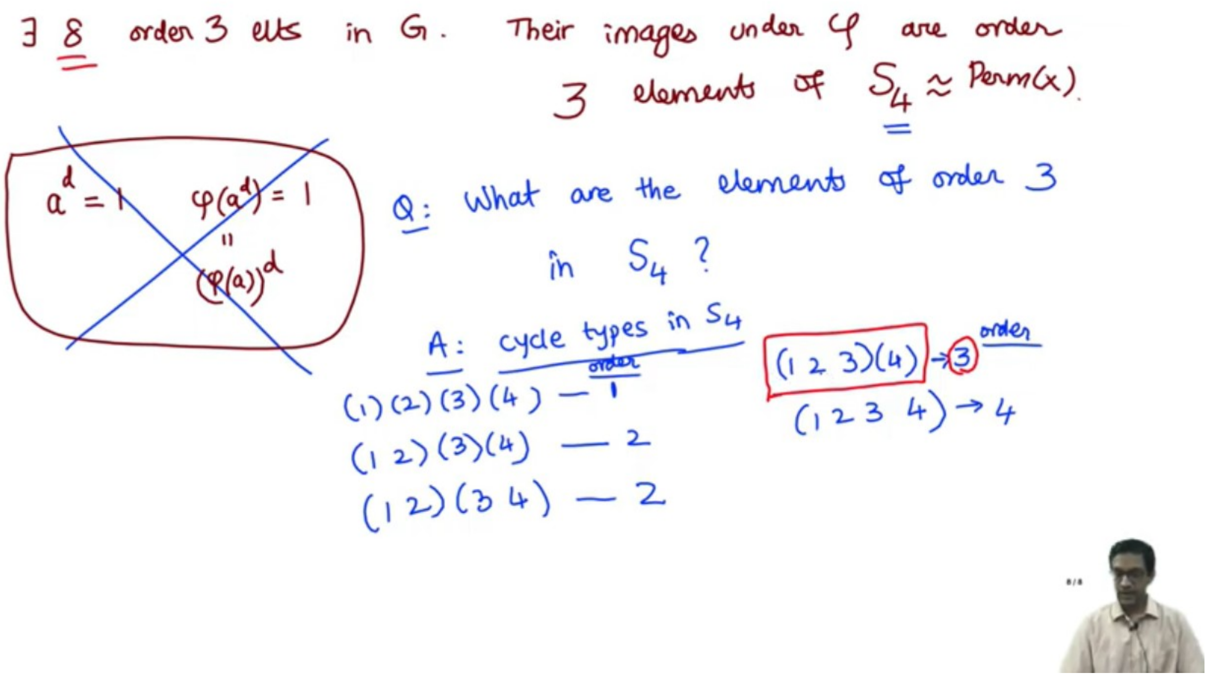
So, we look at the the order three elements of G ok. So, what I have here is a group homomorphism from G to the group S_4 or the group of permutations and I know now know that phi is an injective map.

So, let me look at the order three elements of G right. So, somehow I am basing all my intuition on on the factor I know what the order three elements are. So, there are these 8 elements, whose order is 3. Let me look at where they will map under this homomorphism phi ok.

So, let us look at the images of these these order 3 elements. So, there are so, recall there are 8 order 3 elements which are a, a^2, b, b^2 etcetera in G observe that since ϕ is injective their images under ϕ are also of order 3 their images under ϕ ok are also are well what are they they are now order three elements of if of S_4. Now, right of S_4 or S_4 which is the permutations of X ok.

So, at least I know that there are I am trying to understand what the image looks like I know at least this much that the image has these 8 order 3 elements ok and why is this because if an element has some order see if X we will call it something else maybe if I have an element a whose order is d in general if $a^d = 1$ and I apply a group homomorphism this tells me that $\phi(a^d)$ is also identity, but this is $(\phi(a))^d$ ok.

Now, what; that means, is that the order of $|\phi(a)|$ divides d in general, but the order of $|\phi(a)|$ since phi is injective it also means that you know d if d is the smallest number such



that $a^d = 1$ then d must also be the smallest number such that $\phi(a)^d = 1$. If not it will imply that ϕ you know ϕ is not injective ok.

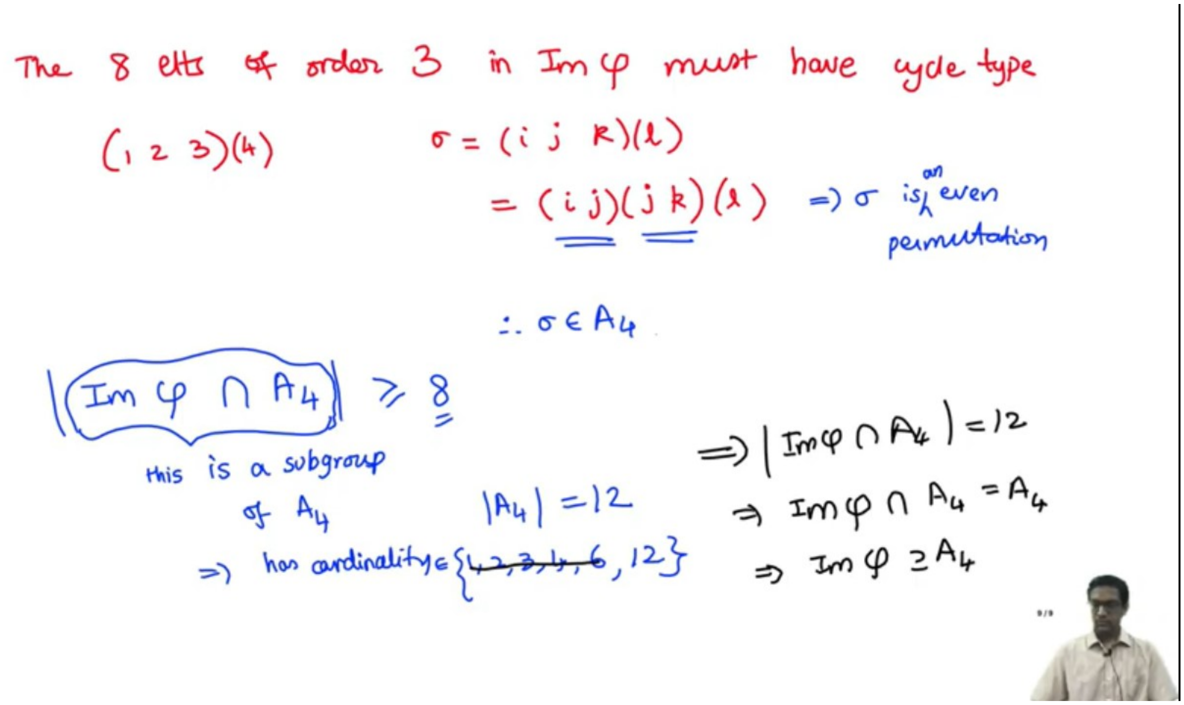
So, an injective map preserves orders. So, that is what I wanted to say here let me let me get rid of this, let me just go on. So, the image is under ϕ that is what I want to understand there are 8 elements of order 3 ok.

Now, let us let us look at the these symmetric group S_4 itself and ask what are the elements of order 3 in S_4. What do they look like ok and question, what do the elements of order 3 in, what are the elements what are the elements of order 3 in the group S_4 ok.

And the answer is lies in in trying to understand the various cycle types. So, let us recall what are the various possible cycle types of elements of S_4. So, these are cycle types in S_4 you can have say the element, the identity element whose cycle type is (1)(2)(3)(4) or I can have a two cycle and two 1 cycles , I can have two 2 cycles or I can have an element which is a 3 cycle with a 1 cycle or of 4 cycle . So, these are the five possible cycle types and in each case it is very easy to see what the order of an element is of that cycle type.

So, let us tabulate the orders. So, if an element is identity of course, it has order 1. So, this is order 1 this is a transposition. So, I mean if I square this element I will get identity, this again its a product of 2 transpositions, if I mean which commute with each other, here again if I square this element I get the identity, now in this case here is a 3 cycle. So, if I cube this element I get the identity. So, the order is 3 and this element has order 4 ok.

So, here is the interesting bit that there is actually exactly one possible way in which you can get an element of order 3, one possible cycle type which will give you order 3 and that cycle type is 3 cycle with 1 cycle ok. So, what we conclude therefore, is that these 8 elements that we know which lie in the image of phi, these 8 elements must have this as their cycle structure ok. It is got to be 1,2,3 with A_4 3 cycle, but the the proof more or less is done.



Now, the key point is that elements of this cycle structure must all be even permutations ok, are necessarily, even permutations that is what we are going to claim.

So, observe now that the 8 elements of order 3 in the image must have cycle type like this (123)(4) ok. So, in other words have the cycle type meaning they should all look like so, every one of these 8 elements therefore, looks like sum three cycle $(ijk)(l)$, let us say here where $(ij)(jk)(l)$ are some (123)(4) in some model. So, this is what any one of these 8 elements every one of them has this form ok.

Now, the key point is suppose I take an element of this form this can be written as the the product of the following transpositions. So, I can write this as let me get the order right. So, this is $(ij) \cdot (jk)$. So, this is the product of the transpositions (ij) and (jk) and (l) of course, is is not a transposition.

So, it is a fixed point. So, observe when I write sigma as a product of transpositions I get exactly two transpositions as (ij) and (jk) ok and so, which means that sigma is an even permutation and even permutation in other words it belongs to A_4 therefore, $\sigma \in A_4$ ok.

Now, what does that tell us. So, it tells us that if I look at the image of ϕ image of this homomorphism it was an injective homomorphism and I want to prove that the image is is exactly A_4 , but at least I know this that there are 8 elements inside this image. The image of $Img\phi \cap A_4$, at least these these 8 elements are certainly in A_4 ok.

Now, again its it is a simple counting argument solves left. So, this guy observed is; however, a subgroup this is the image of a homomorphism is a subgroup . So, this is therefore, an intersection of two subgroups of S_4. So, this is a subgroup of A_4 , if you wish this is a subgroup of A_4 it is a subgroup whose cardinalities are at least 8.

Now, observe $|A_4| = 12$ ok which means that this this subgroup should have cardinality which divides 12 ok. This is a subgroup of A_4 which implies has cardinality which divides 12, which means its 1, 2, 3, 4, 6 or 12 . These are the only possibilities has cardinality belonging

$$|\mathrm{Im}\,\varphi| = |G| = 12 \qquad \underset{12}{G} \xrightarrow{\varphi} \mathrm{Perm}(X) \approx S_4$$

$$\mathrm{Im}\,\varphi \supseteq A_4 \qquad |A_4| = 12$$

$$\Rightarrow \boxed{\mathrm{Im}\,\varphi = A_4.}$$

to this set ok, but we know it is at least 8, because of this this previous argument which means that all the other possibilities do not exist. None of these is possible the cardinality has to be actually 12.

This means that $Img\phi \cap A_4$ has to have cardinality 12 in other words its its equal to A_4, $Img\phi \cap A_4 = A_4$ ok; that means, that the image contains A_4 , but the the size. So, recall; however, that the cardinality of the $Img\phi$ is the same as the cardinality of the original group G ok, because what was this ϕ was an injective map from G to S_4 to the set of permutations of X right.

So, ϕ was an injective map. So, it the if G has cardinality 12 then of course, the image of G under phi also has cardinality 12. So, this guy actually has exactly cardinality 12 ok. So, it has cardinality 12 and on the other hand the image contains the group A_4 whose cardinality is also 12 which means that these are actually equal to each other. So, the $Img\phi$ is in fact, equal to A_4 ok. So, if you identify the permutations with S_4 in any which way what we have shown is that the image is exactly the subgroup A_4 ok.

So, that completes the the thing that we set out to prove that either H is normal or if not, the group itself is isomorphic to the group A_4 ok. So, along the way we have had occasion to use many different facts that we have learned until now about groups and actions and homomorphisms and so on ok .