Algebra - I
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ALGEBRA I

1. Lecture 15: The Orbit Counting Theorem

How many different necklaces (beads in a circle) can I make with 4 beads if I am allowed say two colours say red and blue? Now, what do I mean by different necklaces?

We can formulate this as a problem in group actions as follows the group $G = \mathbb{Z}/4\mathbb{Z}$ acts on the set X of all necklaces with 4 beads which are red or blue.

Lemma 1.1 (Burnside's counting lemma)**.** *Let X be a finite set, G a finite group, and G acts on X. Then the number of G-orbits in X is*

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where $X^g = \{x \in X \mid gx = x\}$.

Let us look at this lemma applied to necklaces. Imagine the four beads at the corners of a square . Let us take the identity element of $\mathbb{Z}/4\mathbb{Z}$: all possible necklaces are fixed by the identity element. There are 2^4 elements in total. which is 16 for i equals 0 ok. Now, suppose I have a necklace, and I want to know i is 1, so that is like clockwise rotation by 90 degrees. And I want to know what how many necklaces can I have?

The element 1 corresponds to a rotation by 90 degrees: so every bead in the necklace must be coloured red or every bead must be coloured blue: so 1 fixes two necklaces.

The element 2 corresponds to rotating by 180 degrees. So diagonally opposite beads must be coloured the same colour. Thus the element 2 fixes four necklaces.

The element 3 corresponds to a rotation by 270 degrees, which like the element 1 fixes two necklaces.

So by the lemma we have the number of distinct necklaces (i.e. orbits under the action of $\mathbb{Z}/4\mathbb{Z}$ is $\frac{1}{4}(16 + 2 + 4 + 2) = 6$.

If you had 3 colours, then it would be very similar to this. We have for each calculation three choices for bead colour instead of two as we did previously.

Consider a two-coloured necklace with 6 beads where flipping the necklace (bead 1 to 6, 2 to 5, 3 to 4) is considered the same necklace. That is, we choose D_6 instead of $\mathbb{Z}/6\mathbb{Z}$.

We apply Burnside's lemma for each of these 12 elements in the group. For the identity rotation, all 2^6 necklaces are fixed.

For the rotation by 60 degrees, there are as before two choices.

If you rotate by 120 degrees, it fixes 4 necklaces.

If you rotate by 180 degrees, 8 necklaces are fixed.

If you rotate by 240 degrees (120 degrees counterclockwise) it fixes 4.

If you rotate by 300 degrees (60 degrees counterclockwise) it fixes 2.

There are 3 possible axes for the reflections which pass through a vertex (and the centre) and reflection around each of them fixes 2^4 necklaces.

There are 3 possible axes that pass through the midpoint of two edges (and the centre) and reflection around each fixes 2^3 necklaces. The final answer is 13 different necklaces.