Algebra - I
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ALGEBRA I

1. Lecture 14: Problem solving

Exercise 1.1. *What are the conjugacy classes in D_4 ?*

D_4 is a subgroup of S_4 now suppose we have that $gxg^{-1} = x'$ for g, x in D_4. That means, that x, x' are also conjugate in S_4; they have the same cycle type. (Consult table from Lecture 14) The first kind of element that we have is the identity element with cycle type is $(1, 1, 1, 1)$.

And then we have rotation by 180 degrees with cycle type $(2, 2)$

And then we have two kinds of reflections: about axis that pass through edges with cycle type is also $(2, 2)$ *and then we have reflection about the vertical axis with cycle type* $(2, 2)$

And reflections about this diagonal axis with cycle type $(2, 1, 1)$ *is the cycle type and there are two such elements.*

So, certainly elements with different cycle types cannot be in different conjugacy classes and all we have to find out is whether elements of the same cycle type are in the same conjugacy class or not.

Let us look at this rotation 1 goes to 2, 2 goes to 3, 3 goes to 4 and then we have another rotation which is 1432. *So, to conjugate this, we need to find a permutation which takes somehow the vertices of the former to the vertices of the latter. So, we take* $w = 1432$. *These 2 elements also form a single conjugacy class.*

Consider $x = 2143$ *and the other one is* $x' = 4321$ *and now we can write* $w = 1432$. *So, indeed these 2 elements are in the same conjugacy class.*

Let us look at reflection about the diagonal. Consider $x = 1432$ *and* $x' = 3214$.*Then* $w = 2143$. *So, these definitely form a conjugacy class.*

So, the identity element is in a conjugacy class by itself the two rotations by 90 degrees are in a conjugacy class by themselves and these reflections about diagonals are in a conjugacy class by themselves and the only thing we do not know is the rotation by 180 degrees, which we call $r^2 = 3412$ *and the other element is* $x = 2143$ *and we want to know whether they are conjugate. So, now if you have permutation w which conjugates them then it must take the cycles of r square to the cycles of x. You may check that no element of D_4 can do this, so these elements are in separate conjugacy classes.*

Exercise 1.2. *What are the subgroups of D_4?*

Note that the order of a subgroup of a group divides the order of the group so its subgroups can be of order 2, order 4 or order 8 and of course, there is the trivial subgroup of order 1. So, basically we need to figure out what are the subgroups of order 2 and the subgroups of order 4. Now a subgroup of order 2 would just consist of the identity and one more element of order 2*. So, we have now reflections about vertical and horizontal axis, denoted* s_1 *and* s_2 *respectively form subgroups* $\langle s_1 \rangle$, $\langle s_2 \rangle$ *of order 2. Rotation by* 180 *degrees, denoted* r^2 *also generates a subgroup of order* 2*. Similarly we have the subgroup* $\langle rs_1 \rangle$*, we have the subgroup generated by* $\langle sr_1 \rangle$*, these are the 5 subgroups of order 2. What about subgroups of order 4 ? So, the subgroups of order 4 well we know that a group of order 4 is either cyclic of order 4; that means, it is just generated by one element of order 4 or it is z mod 2 cross z mod 2 and so, it is going to be generated by 2 elements of order 2 which commute with each other and so, the only element of order 4 in* D_4 *is the element r. So, the subgroup of order 4 is* $\langle r \rangle$ *that is isomorphic to* $\mathbb{Z}/4\mathbb{Z}$ *and then there are if you look at these elements r s and s r well they commute r s into s r is r squared as is s r into r s. So, they generate a subgroup of order 4 and then s and* sr^2 *also generate a subgroup of 4 elements.*

There is the obvious subgroup of order 1 which is just singleton identity and the subgroup of order 8 which is all of D_4.

Exercise 1.3. *which subgroups of* D_4 *are normal ? All subgroups of index* 2 *(*$|G|/|H|$*) are normal. So all subgroups of order* 4 *are normal.*

Obviously, the trivial group is normal. The subgroup generated by r is also normal it is a union of conjugacy classes.

Find all subgroups that are union of conjugacy classes- these are the normal ones.

Exercise 1.4. *Which subgroups of the quaternion group are normal? So, this was a group which had the elements* $\pm e \pm i \pm j \pm k$ *and we had worked out the conjugacy classes to be* $e, -e, \pm i, \pm j, \pm k$*. There were 3 subgroups of order 4 they were generated by each of the elements* i, j, k.

So, from this information you should be able to see which subgroups of Q are normal.

We find that the quaternion group is an example of a non abelian group all of whose subgroups are normal.