Linear Algebra
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Lecture 3.1
Linear Transformations

So, till now we have seen what are vector spaces? What is meant by linear combination of vectors, linear independents of vectors and spanning set and so on? However, all these notions are restricted to within a given vector space. So, in this video we will see how if there are two different vector spaces say, V and W how they interact with each other. The tool which the tool with which we will be studying this interacting is called as linear transformations.

Linear transformation is a function from given vector space to another which satisfies certain nice properties which we will see in a moment.

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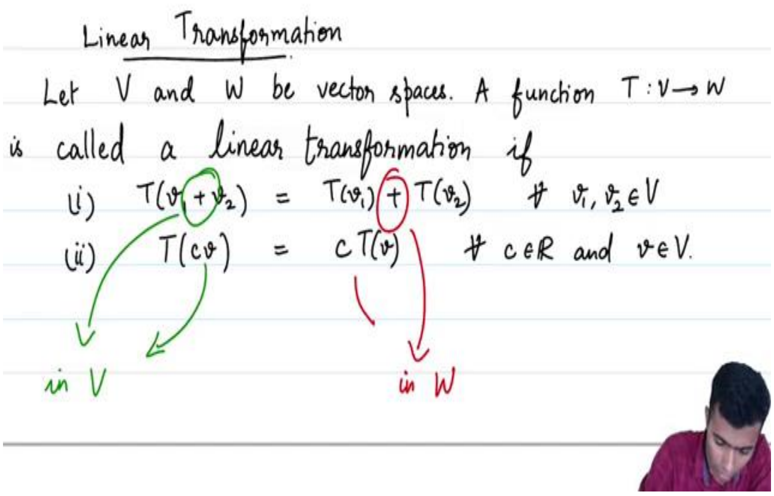

Let me start with a definition, so linear transformations are central to the study of a linear algebra in particular those particular course. We will be studying various properties of a linear transformations for a long time, not specific linear transformation but generally of linear transformations. So, let me first define what a linear transformation is? Linear transformation, so let V and W be vector spaces.

Till now, W was a vector subspace of V in various cases that is how these notations were being used. However, that is not the case now, V and W are two different vector spaces. As needless to say, I am always considering vector spaces so over R here. A function T let us call T for transformation from V to W is called a linear transformation if the following are satisfied. So, it is a function in particular it satisfies the following. 1, the addition in V is preserved in other words if we take two vectors look at the sum and look at its image, it will be the sum of the images, so let me write it down, it will become clearer.

T of v1 plus v2 is equal to T of v1 plus T of v2 and the second condition states that the scaler multiplication is also preserved. So, in other words T of cv is equal to c times T of v, so this is for all v1 and v2 in capital V the first case and this is for all the second case, c real number and v in capital V. So, let us spend a couple of minutes trying to see what is happening here. Let us use the green colour to denote, so let me circle the vector addition v1 plus v2. So, I would like to point out that this is happening in V, this is a vector addition which is happening in V.

Now, red is being used to circle out vector addition however, remember that this is a vector addition in W even though I am using the same notation. I am not specifying where it is but the context makes it clear, so it is your job to slowly start identifying where which operation is happening. So, as you can see this red is a vector addition which is happening in W, why? Because T of v1 and T of v2 are vectors in W. So, the vector addition has to be in the vector space W, is not it?

Similarly, this scaler multiplication that is also happening in V, c times v is a scaler multiplication in which happening in V at the same time, this scaler addition this is happening in W because T of V is a vector in W and c times T of v is happening in the vector space W. So that is what it says, It is in some sense preserving the structure the structure of the vector space, the structure in V and the structure of W are in some sense compactible through T. So, let us look at some examples of a linear transformations.

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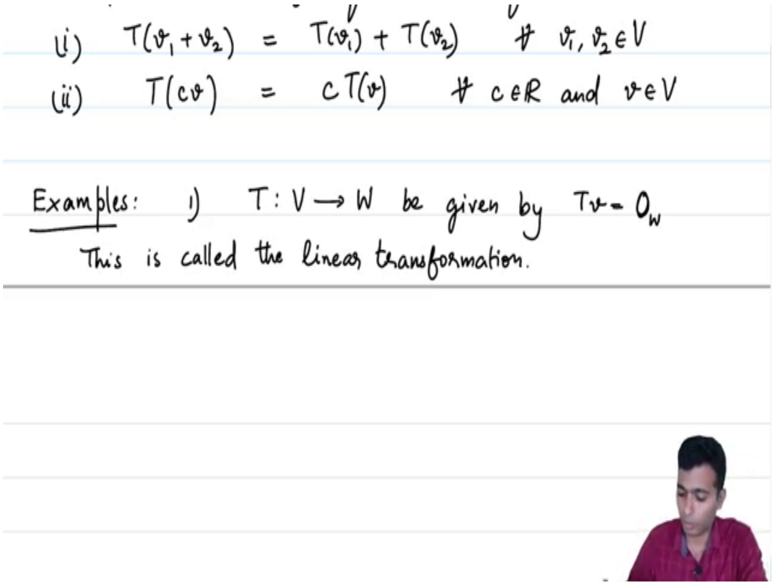

Examples, the first example and the, I should say the simplest example should will be the map T from a vector space V into another vector space W be given by, T of v is equal to as in 0 of the vector space W. Let me just write it once or twice to point out that this is where all these things are happening. So, this is the 0 vector in W, every element of V is being mapped with the 0 vector so this is called the zero transformation-linear, this is called linear transformation is called the why is it zero linear transformation we have to check that.

That is quite straightforward; let me do that for a couple of cases. Before we proceed, I would like to may be state a lemma which I will leave as an exercise.

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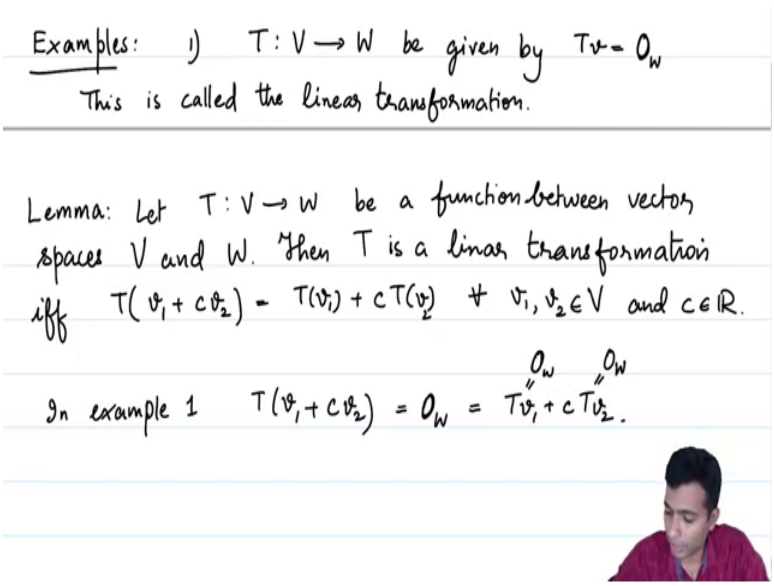

So, when is a function from a vector space V to W a linear transformation? It is a linear transformation, if both these conditions are satisfied. What we will do through this Lemma is too simplified, we will just put one condition to check for a particular function being a linear transformation. So, let T from V to W be a function between vector spaces V and W, then T is a linear transformation if and only if, T of v1 plus cv2 is equal to T of v1 plus c times T of v2 for all v1, v2 in capital V and c, a real number.

So, I will just leave this as an exercise it is quite straight forward, you have to show both sides here. If T is a linear transformation you should show that T of v1 plus cv2 is equal to T of v1 plus c times T of v2 for all v1, v2 in V and c in R. And similarly, the converse would be if T is satisfying this to that it is again these two conditions are satisfied, these two conditions of a linear transformation. With this Lemma let us start looking at more examples, before we go into a second example observe that in example 1, T of v1 plus c times v2 is equal to the 0 vector W.

And this is just the same as T of v1 plus c times T of v2, why is that the case? This is also the 0 vector in W, this is also the 0 vector in W. A scaler times 0 vector is 0 and the sum of 0 vector is again 0 vector. So yes, this is certainly a linear transformation.

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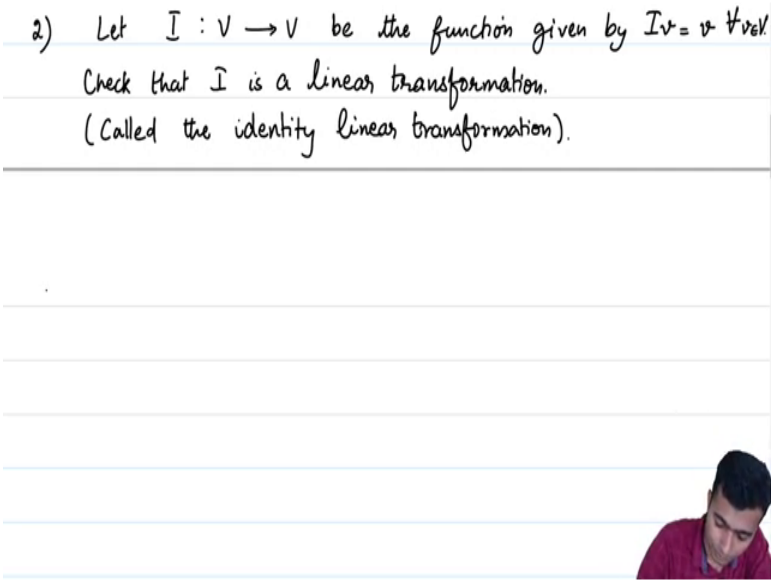

Let us look at the second example, let us call this particular map I so T is not necessarily the only alphabet you should be using I from V to itself be the function, be the map given by I of v is equal to v for all V in capital V.

Again, check that I is a linear transformation. This particular linear transformation I is called the identity linear transformation. So, these are two very important and very straight forward examples that we should immediately look ahead and let us now look into slightly more interesting examples.

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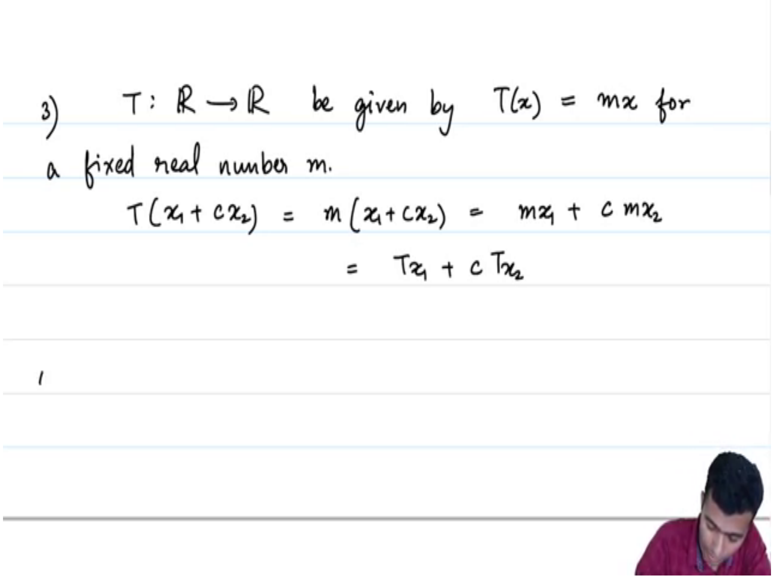

Let us see T be a map from R to itself be given by T of let us call it x an element in R is sent to m times x for a fixed real number m. Then, T will be a linear transformation, so just too quickly check that, T of let say x1 plus cx2 just the equal to m times x1 plus cx2 which is equal to mx1 plus mcx2 which I will write it as cmx2 which in particular as T of x1 plus c times T of x2.

So, when there is just one vector or one element x1 on which T is acting I will slowly stop writing the brackets around it. So, T of x1 and Tx1 should be in the same thing. So this proves that T is a linear transformation so in other words dilations or scaling by a fixed number is a linear transformation. What could be a good example next?

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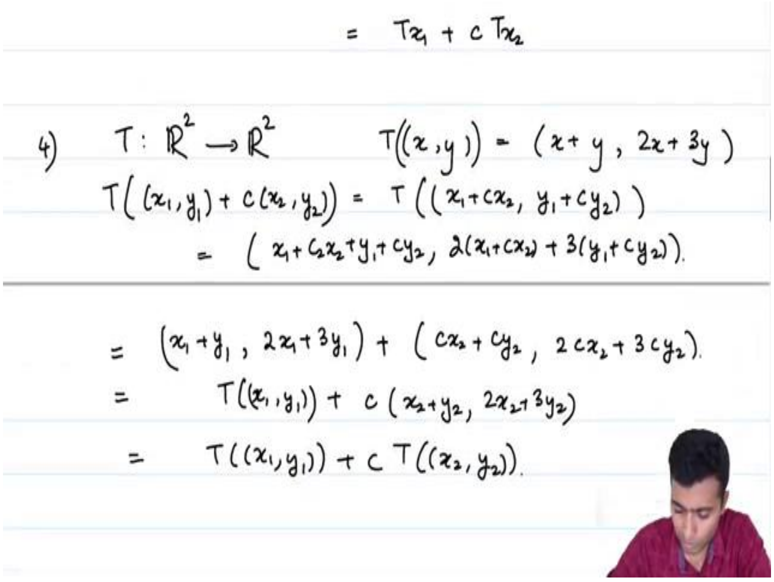

Let us look at T of so T of,T be a map from R2 to itself, where T of some vector let say x1, x2. So, T of this is equal to x1 plus x2, 2 x1 plus 3 x2 this is a map from R2 to itself, so one can quickly check so maybe I should do the check at least in the initial stages.

Let us see if x1, x2 plus say c times x1, y1 and x2, y2 so may be here as well, let me change it to so it was x plus y, 2 x plus 3 y. So, this is equal to what it would be equal to? This is equal to T of x1 plus cx2 comma y1 plus cy2 which is equal to x1 plus cx2 plus y1 plus cy2 comma 2 times x1 plus cx2 plus 3 times y1 plus cy2. Let see what happens when we expand this out. This is equal to, so finally what do we want? We want this to be T of x1, y1 plus c times T of x2, y2 that is what we want.

So, this is going to be x1 plus y1 comma 2 times x1 plus 3 times y1 plus cx2 plus cy2 comma 2 times cx2 plus 3 times cy2 which is equal to, so this is already equal to T of x1, y1. And in the second case this is going to be c times just take out the c common here because this is actually multiplication by, okay I will just write it down. This is x2 plus y2, 2 x2 plus 3 y2 which in particular is equal to plus c times T of x2, y2. So, I will slowly stop doing all these checks for every case and slowly start giving them as exercises to you.

So, this in particular proves that this particular map is a linear transformation. The previous example was from R to itself, this was from R2 to itself, so let us now just look at some example where the domain and the range are different.

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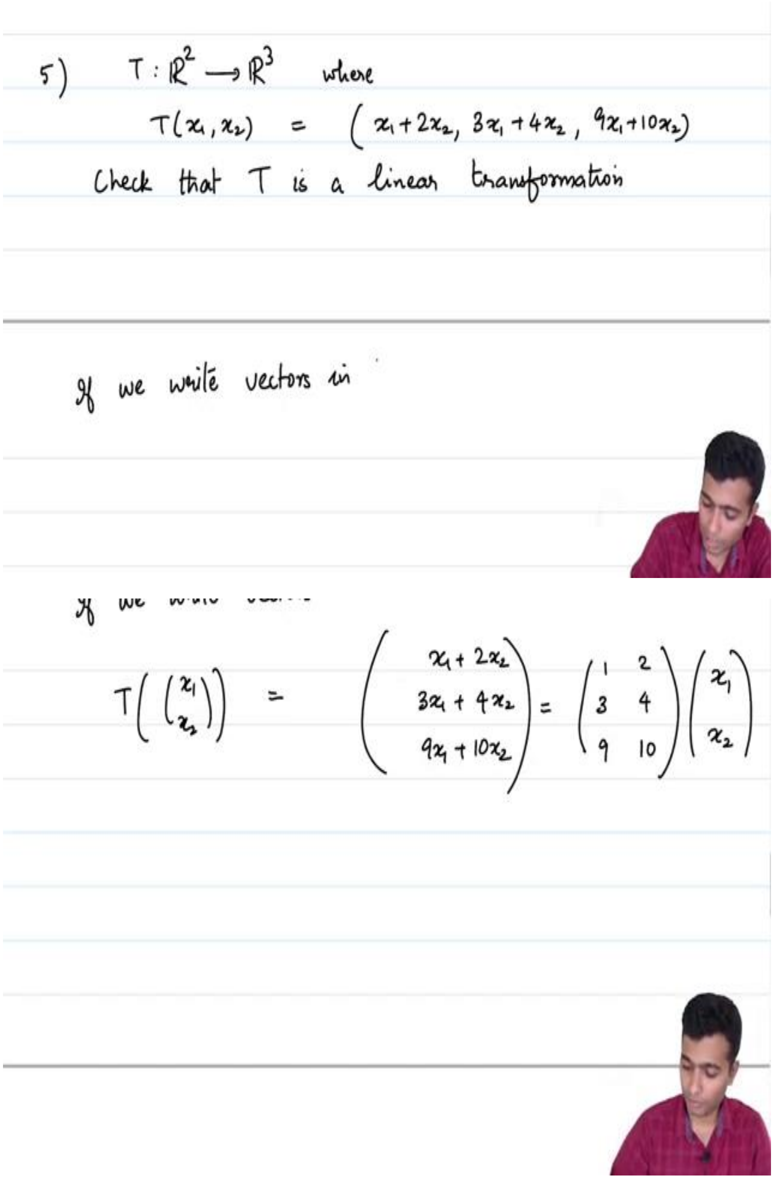

Let T of x1, so T be a map now from R2 to R3 where the map is given by T of x1 comma x2 is equal to x1 plus 2 x2, 3 x1 plus 4 x2, 9 x1 plus 10 x2. Check that T is indeed a linear transformation. At this point let us focus on how we can rewrite this in order to see if it can be realised as multiplication by a matrix.

I am sure you should be familiar with matrix multiplications. So, if vectors in say R2 and R3 can be written down as columns. So, writing if we write vectors in Rn as columns, Rn in general as columns instead of rows, what can we write this as? This is a basically telling as that T of x1 x2 this is the column representation of the vector x1, x2. This is equal to x1 plus 2 x2, 3 x1 plus 4 x2, 9 x1 plus 10 x2 this is the column representation of the image. What is this? This if you observe carefully is 1, 2, 3, 4, 9, 10 this matrix multiplied to the column vector x1, x2.

So, what we have done is we have realised that there is a matrix which is corresponding to this particular linear transformation. So, that naturally asks that naturally motivates is to ask the following question which will be our next example.

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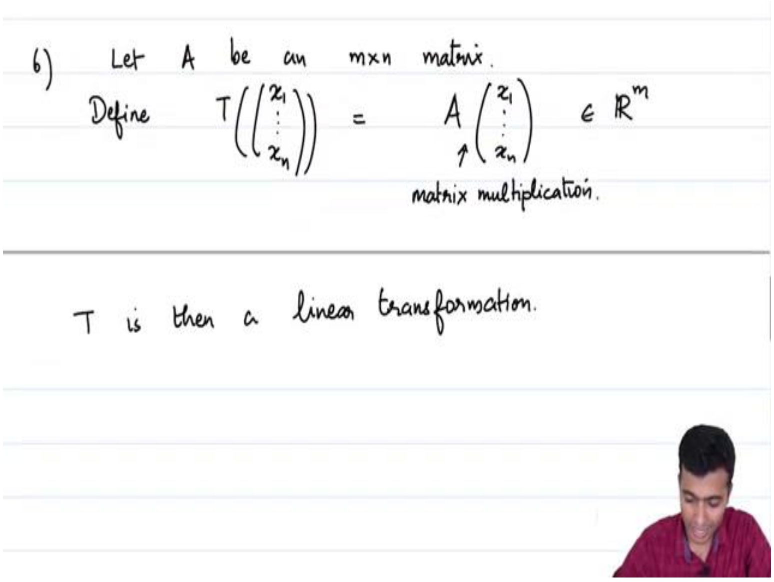

The next example being let A be an m cross n matrix with coefficients in R of course. Then, define T of an N vector, so right now we will be treating all these as a columns. So, this is x1 up to xn to be equal to A times the vector x1 to xn. What will be the solution here? What will be the result here? This will be an element of Rm in the column representation, so where this is the matrix multiplication.

What we will finally be able to see is that, T is then a linear transformation, we will come back to this later certainly. So, if you can take the effort to painstakingly write down the explicit expression of T of x1 to xn here in terms of the coefficients in A. It is very straight forward to indeed check that T is a linear transformation. So, I will not going to details there. The point here is to note that right now we are trying to realise matrixes, so as linear transformations. So, as I said later we will realise that, we will see that every linear transformation T can be realised as a matrix.

So, we will come to all that later but it is crucial to make this observation right now that matrix multiplication is indeed a linear transformation. We are only looking at linear transformations in Rn but may be the next example will tell you that, that is not necessarily restricted to just Rn obviously not.

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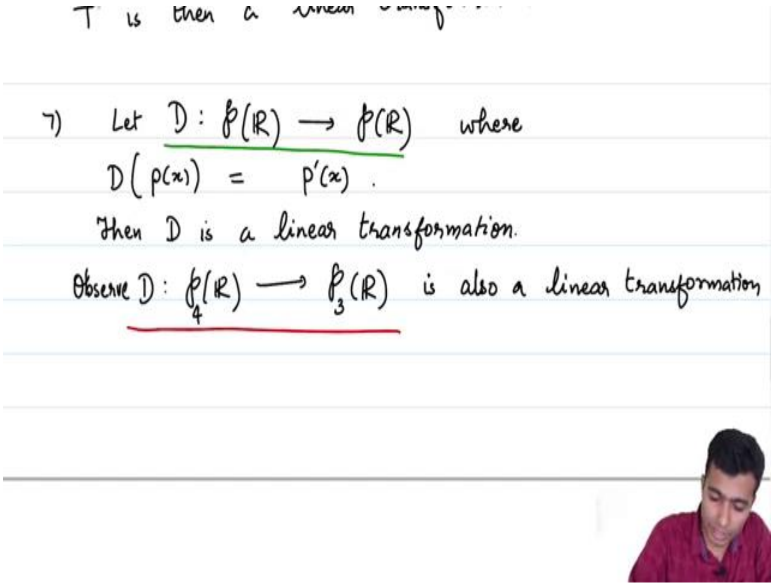

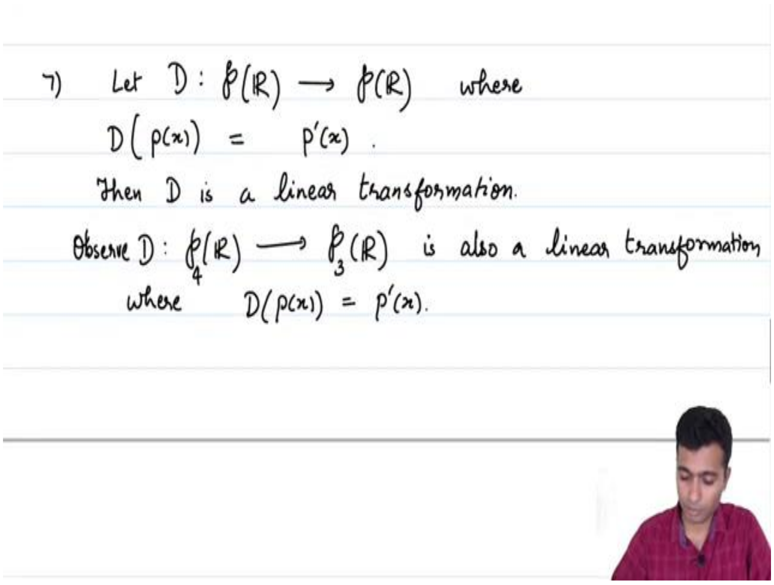

So, let T be the map let us call it something else, this actually has a classical notation to it. Let, D be the map from P of R to itself, where D of a polynomial is its derivative, P prime of x. Then check that from calculous you should know that D is a linear transformation.

The reason why I mentioned that you should know this from calculous is because; we know that if you look at the derivative of sum of two polynomials, It will be the sum of the derivatives, and similarly if you look at the derivative of a constant times a polynomial, it is going to be the constant times the derivative of the polynomial, so yes it is indeed a linear transformation. So, note that this is not necessarily a map from P of, it is certainly a map from P of R to itself but we could also think of D restricted to say a P 5 of R or P 4 of R.

Observe that D from say P 4 of R, if you take a degree four polynomial and look at its derivative you will end up with a degree three polynomial, so this is a polynomial in P 3 of R is also a linear transformation. I would at this point make the distinction that the example which I am underlining in a green is technically different from the example which I have underlined it red because the vector spaces from which the linear transformation underlined in green is P of R and it is range is also P of R that is not the case in the, case where it is underlined in red. It is a map from say linear transformation from P four of R to P 3 of R.

So, even though the map is the same, I did not write the map but the map is exactly the same. What is the map? The map is basically where D of P of x is the same thing, the derivative but it is important to observe that the moment we talk about a linear transformation, there is a domain and there is a range and they are important. Even though it is technically the same derivative that we are looking at. So, I will just note this as a different, okay let me not confused too much made my point with words so that is enough here.

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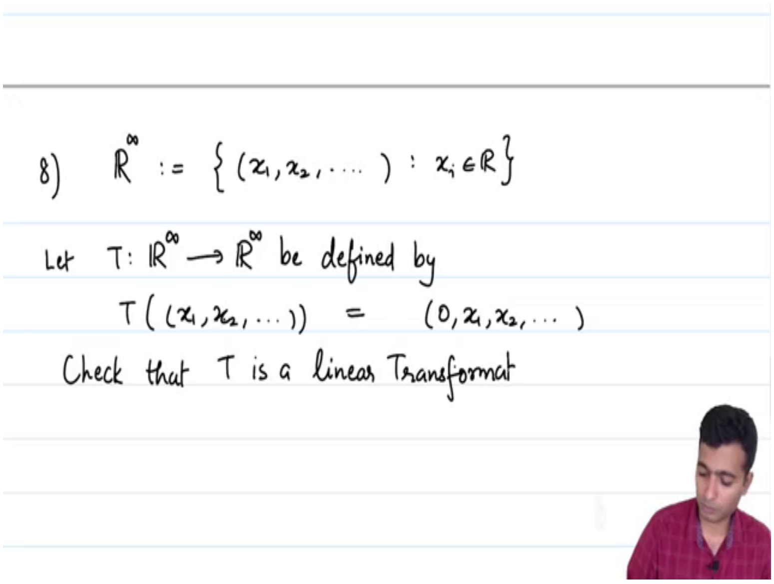

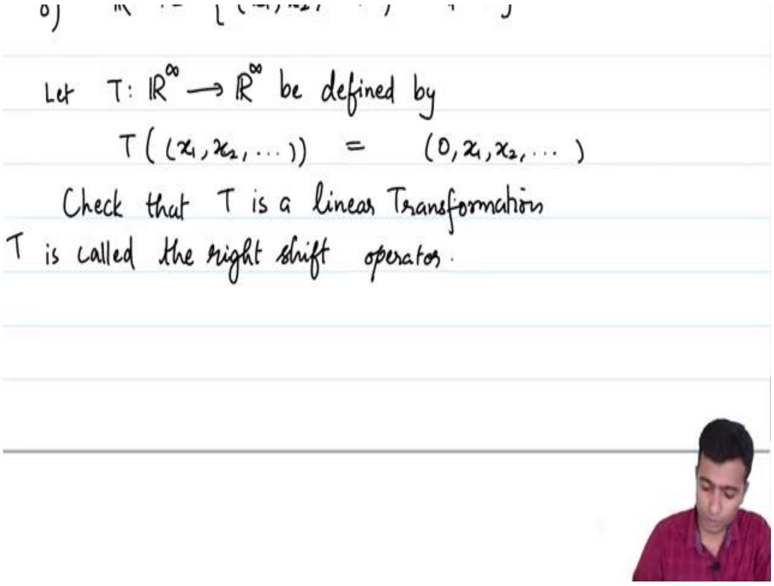

Another example might be to look at the example of the vector space R infinity, let me recall that this is the vector space x1, x2 dot-dot-dot the infinite sequences. Where xi is are all elements in R and T from R infinity to itself the defined by, T of say x1, x2 dot-dot-dot this is mapped to by shifting it to the right. Why are that two brackets not needed so, check that T is a linear transformation. T is called the right shift operator, it shifts the vector to the right, so it is called T is called the right shift operator. So, we have seen quite a lot of examples now and we will be seeing many-many more examples in the next few days.

And let us right now stop with examples and start studying the properties of linear transformation.

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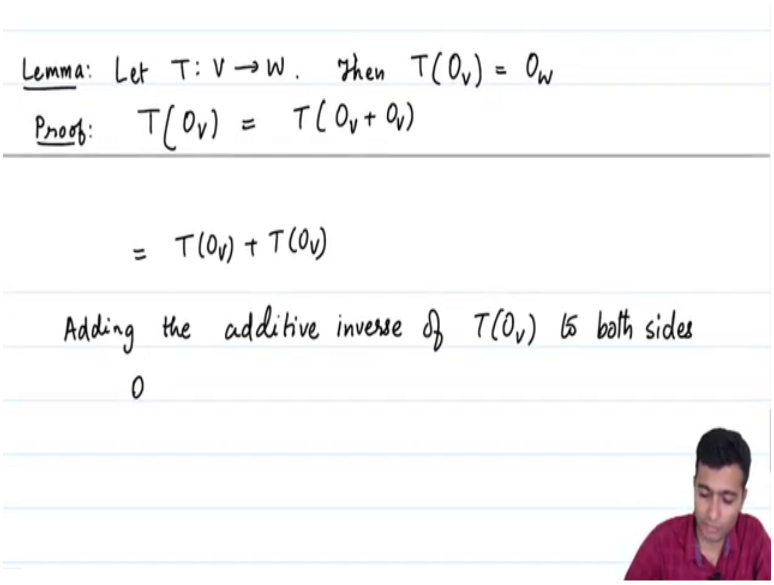

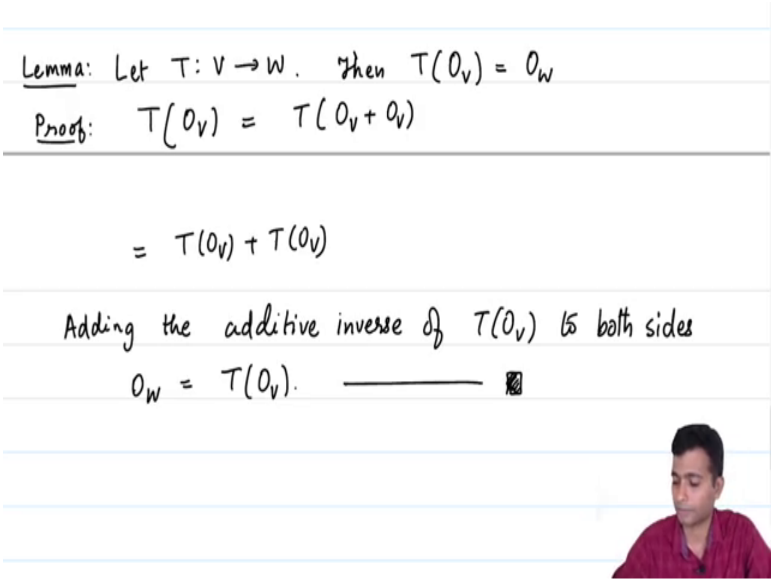

So, the first Lemma it is a simple result, the first Lemma which we would like to prove here would be that every linear transformation necessarily maps the 0 vector. So, let T from V to W so before I just go ahead observe that, go through the examples which were listed again and observe that we are not restricting our definitions and examples to finite dimensional vectors spaces.

As noted P of R to P of R is a linear transformation, that is an infinite dimensional, it is not a finite dimensional vector space. Same is the case with the last example which was given which was from R infinity to R infinity which is not which actually is an exercise to you, to show that R infinity is not a finite dimensional vector space.

And that this is an example of a linear transformation from an infinite dimensional vector space to itself. So, this Lemma T from V to W this Lemma states that, every vector every linear transformation from a vector space V to W should necessarily map 0 to the 0 vector of W, 0 of V to the 0 vector of W.

Then T of the 0 so after this Lemma I will slowly stop writing the subscripts V and W which will be clear from the context but, let me write the subscripts in the proof of at least Lemma. So, let us give a quick proof of this statement. So, what do we have this is the same trick which we have been using for proving these kind of results. We are interested in T of 0 of V and this is nothing but T of 0 V plus 0 V, but T is a linear transformation so this is equal to, T of 0 V plus T of 0 V.

Now, let us add the additive remember that T of 0 V is some vector in W adding the additive inverse of this vector. T of 0 V to both sides, what do we have? We have 0 of W because the left hand side has only a T of 0 V if you added to the additive inverse of T of 0 V in W we get the additive additivity of W which is the 0 W that is equal to I will leave it for you to check that this is 0 W plus T of 0 V which is equal to T of 0 V and that completes the proof.