Linear Algebra
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Lecture 8
Replacement Theorem Consequences

So, given a vector space with a basis consisting of finitely many elements say d, then we defined the dimension of the vector space to be equal to d, which is the size of our given basis. As a consequence of the replacement theorem, we had proved that, if there is a basis of size d in a vector space than any other basis should also have the same number of elements. In fact, this was proved by noticing that, if there is a vector space which has a basis of finite size say d, then any set which has size greater than d should necessarily be linearly dependent and any set which has size less than d cannot be a spanning set.

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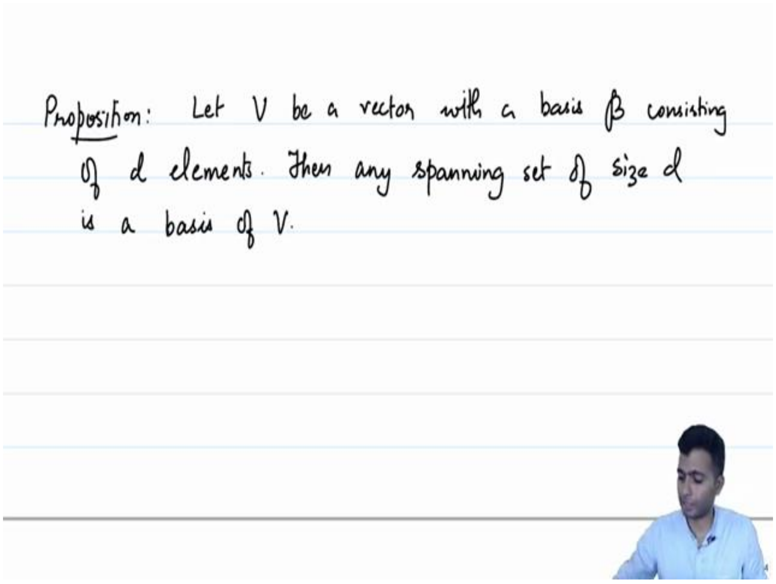

So let us look at many more consequences of the replacement theorem in this video. So, we will start with a proposition. So, let V be a vector space, which has a basis, be a vector space with a basis beta consisting of d elements, then any spanning set of size d should necessarily be a basis, any spanning set of size d is a basis of V, let us give a proof of this statement.

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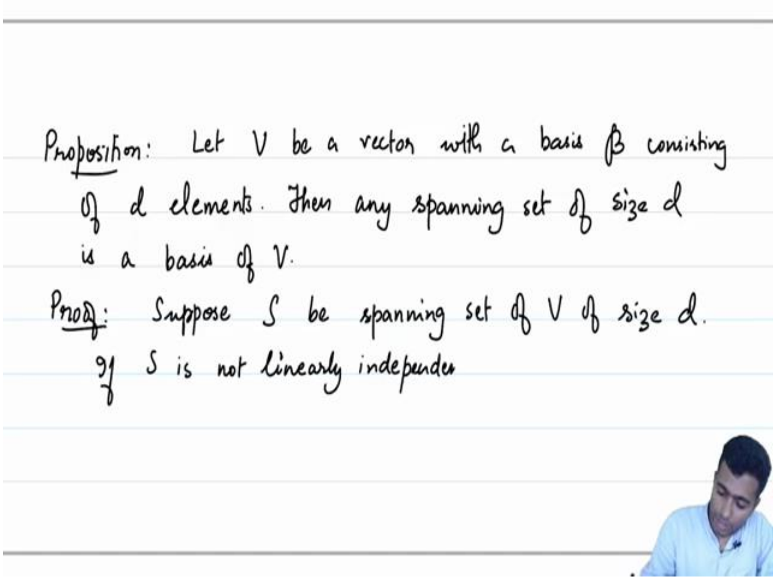

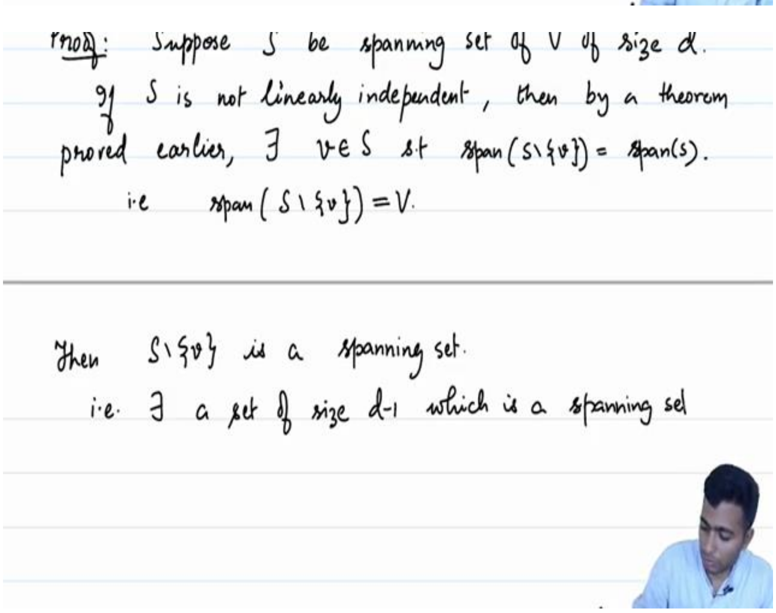

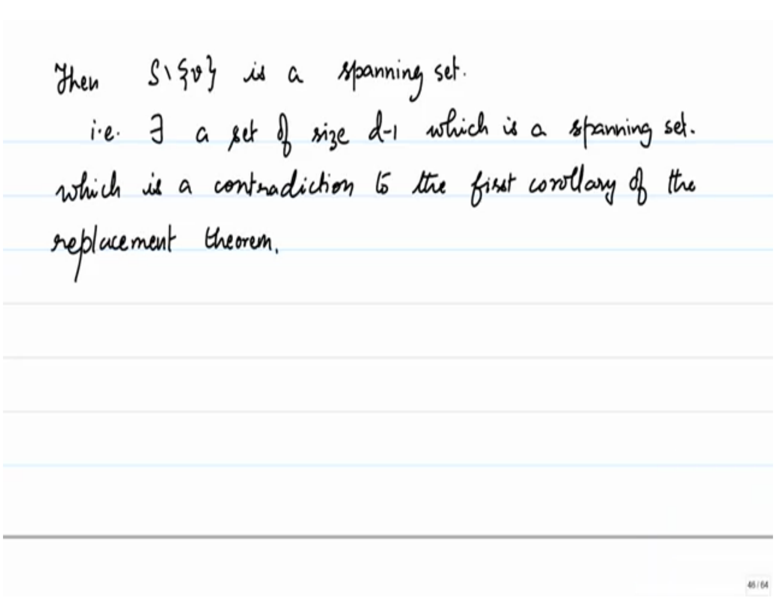

So, we have beta, which is of size d and suppose L or rather S be spanning set of V of size d. Then suppose our set is not linearly independent, if it is linearly independent then we have done it, it is a basis. So, if S is not linearly independent, then what happens is we call by a theorem which we proved in the second video of this week, if a set S is not linearly independent, then there exists some vector V in S such that span of S minus V is the span of S minus V is equal to be span of S.

Then by a theorem proved earlier, there exists an element V in capital S such that span of S minus V is equal to the span of S, but what is span of S? Our S is a spanning set, that means that span of S minus V is equal to V, but then this implies that S minus V is a spanning set, but what is the cardinality of S minus V?

In other words, what is the size of the set S minus V? It is d minus 1, then i.e. there exists a set of size d minus 1 which is a spanning set, which is a contradiction, because one of the corollaries we proved to the replacement theorem said that, any spanning set should have size at least d, which is a contradiction to the first corollary I think of the replacement theorem, we just proved in the previous video. And therefore, our assumption has to be false, therefore S has to be linearly independent.

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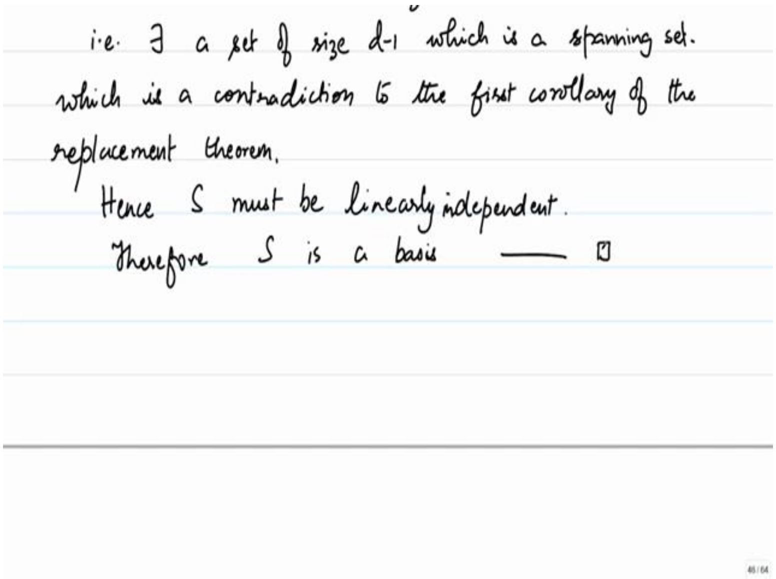

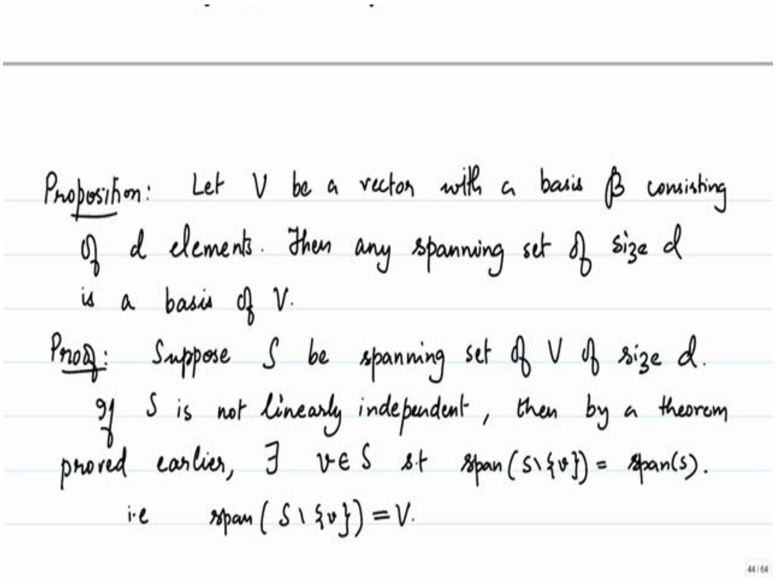

But S was a spanning set to begin with, it is also now linearly independent, therefore S is a basis and we have proved the preposition. So, what have we just just look at the statement we have just proved the proposition says that, if we have a vector space containing a basis of size d, then any spanning set of size d is a basis of the V. We can also prove a do a statement of it, let us note it and then give a proof of it.

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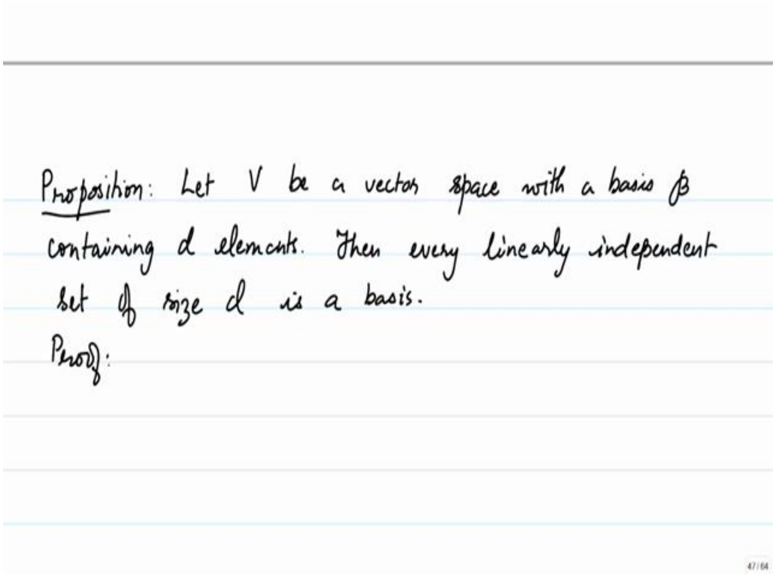

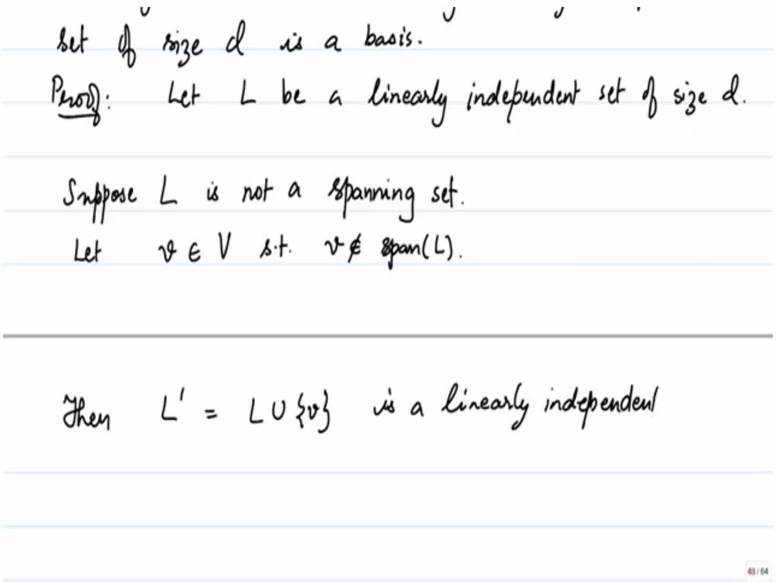

So, let again V be a vector space with a basis with a basis beta containing B elements, so we has a basis, which has d elements. In any linearly independent set of size d should necessarily be a basis, then every linearly independent set of size d is a basis, let us give a proof of this. So, let L be a set of size d which is linearly independent, so let L be a linearly independent set of size d. Suppose L is not a spanning set, again we will come to a contradiction and therefore, L has to necessarily be a spanning set and spanning set which is linearly independent must be a basis.

So, let us come to a contradiction by assuming that L is not a spanning set, what does it mean to say that something is not a spanning set? Means span of L will not contain some element of V. So, let V be an element in capital V such that v does not belong to span of L. Then by the theorem which we proved in the last video, L union V will be a linearly independence set. Then L tilde equal to L union V, may be not L tilde let me call it L prime equal L union v is a linearly independent set.

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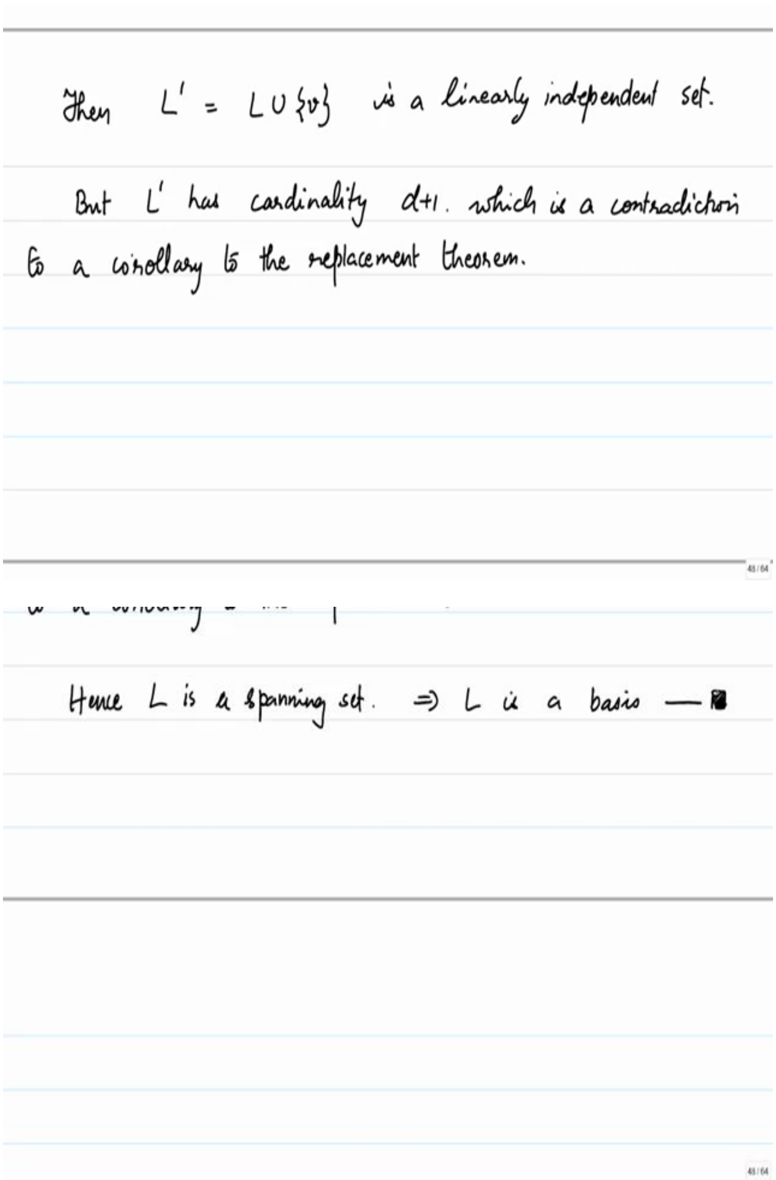

But, what is the size or cardinality of the set L prime? L prime has cardinality or size d plus 1 and the corollary that we have proved to the replace one of the corollaries we proved to replacement theorem said that, any linearly independent set should have size less than or equal to d, which is a contradiction, so this is a contradiction to I think the second corollary, so I will just write a corollary to the replacement theorem.

Therefore, our assumption is false that L is not a spanning set. Therefore, hence L is a spanning set which is also linearly independent which gives that L is a basis and hence we have proved the (())(9:38). So, we have just proved that any linearly independent set of size d should necessarily be a basics. So, any linearly independent set of size d should be a basis, any spanning set of size d should also be a basis, where d is the dimension of our given vector space.

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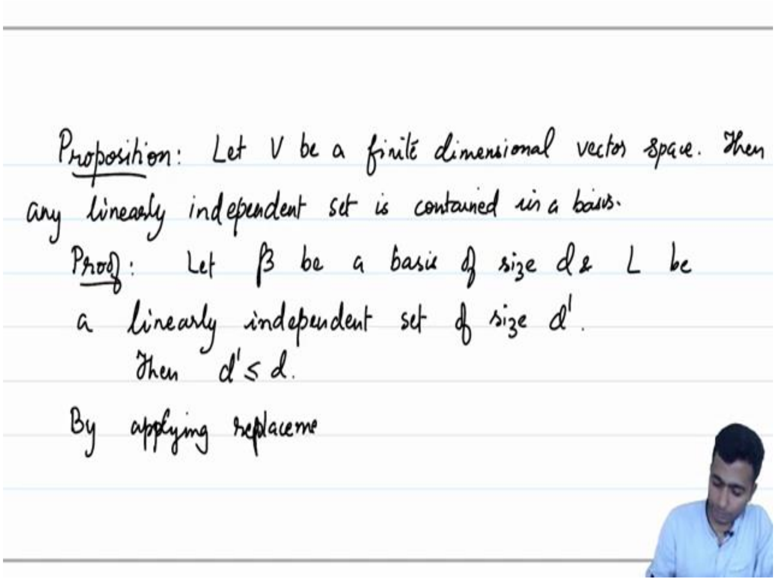

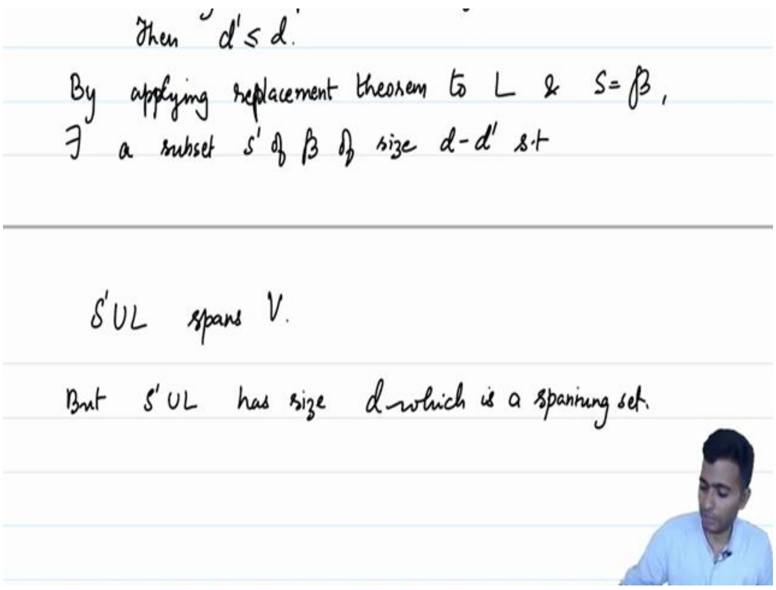

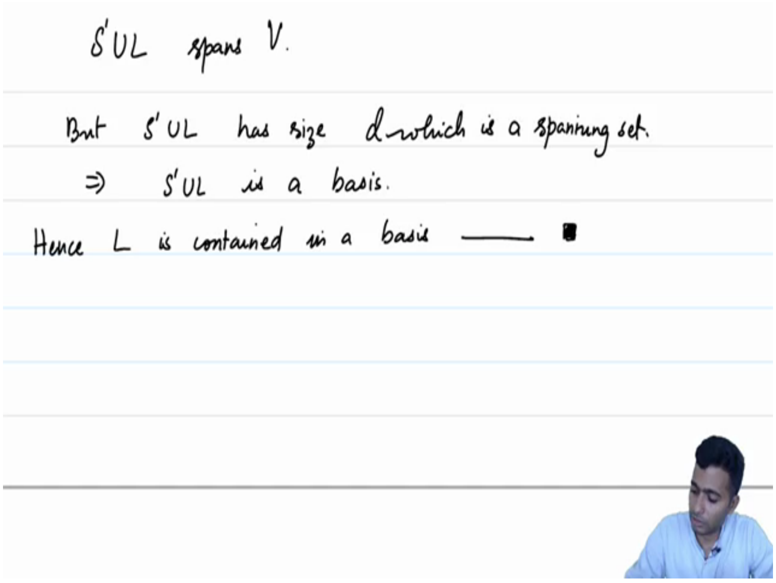

Okay, let us look at more consequences. So, the next proposition tells us that every linearly independent set is contained in a basis, so, let us look at the proposition. So, from now, I think maybe I will just write it, let V be a vector space finite dimensional vector space, V be a vector space let me slowly start writing that, V be a we finite dimensional vector space then any linearly independent set is contained in a basis, then any linearly independent set is contained in a basis.

So, let us give a proof, the third proposition today. So, if you start with a finite dimensional vector space, what does it mean? Where x is a basis, so let beta be a basis of size d, d is the dimension of our vector space, and let L be a linearly independent set. And L be a linearly independent set, of say size d prime. Now we know that D prime has to be less than or equal to d by one of the corollaries to be replacement theorem.

Now, we will apply replacement theorem to L and S given by beta, okay, by applying the replacement theorem to L and S equal to beta. So, beta is the spanning set S of size N in that case here it is d, L is the linearly independent set of size M here it is d prime. And the replacement theorem tells us there exits a subset S prime of beta of size d minus d prime such that S prime union L spans V, but that is good, because S prime union L has size equal to d, is not it?

S prime union L has cardinality or size d prime plus d minus d prime which is d and this is a spanning set, which is a spanning set. What do we know about spanning sets of size d? We just proved that any spanning set of size d in a d dimensional vector space should necessarily be a basis. Which implies that S prime union L is a basis and that is precisely what we wanted, we wanted to realize L as a subset of a basis. Hence, L is contained in a basis. So, in a finite dimensional vector space, you startup with any linearly independent set, you should necessarily be sitting inside a basis, not necessarily a unique basis of course, it could be sitting inside many, many business but there is certainly at least one basis (())(14:11) L is sitting inside it, L is contained in it.

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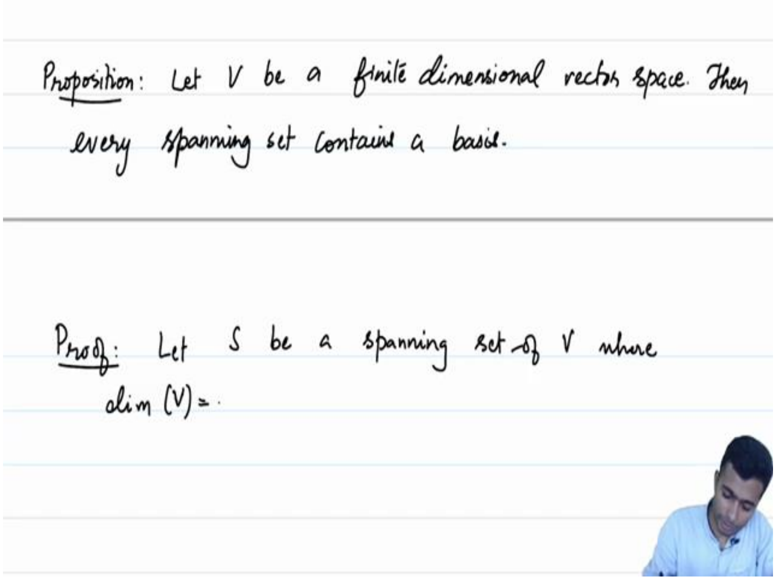

And as is to be expected, there should be a duel statement. The proposition next fourth proposition today is going to be a dual statement, which says that if there is a spanning set, it should necessarily contain a basis. So, let V be a finite dimensional vector space, then every spanning set contains a basis, every spanning set contains a basis, let us give a proof of the proposition.

So let us start with some spanning set S, so let S me as be a spanning set. We know that the cardinality of S should be at least d, so, if we can manage to get hold of d, linearly independent vectors then by one of the propositions proved earlier today, a linearly independent set of size d is necessarily a basis, where d is the finite dimensional, sorry dimension of. So, of V where dimension of V is equal to d. So, let us fix the dimension to be d,that means there exists a basis of size d of V.

So, we just managed to get hold of a subset of S which has d elements, which is linearly independent, then it should be a basis. And the theorem would have been proved because then we would have obtained a basis sitting inside it. Suppose, we are not able to get a linearly independent subset of S which has d elements. Of course, any linearly independent set should necessarily have less than or equal to d elements by one of the propositions one of the corollaries to the replacement theorem, but suppose we do not have linearly independent set sitting inside S which has size d.

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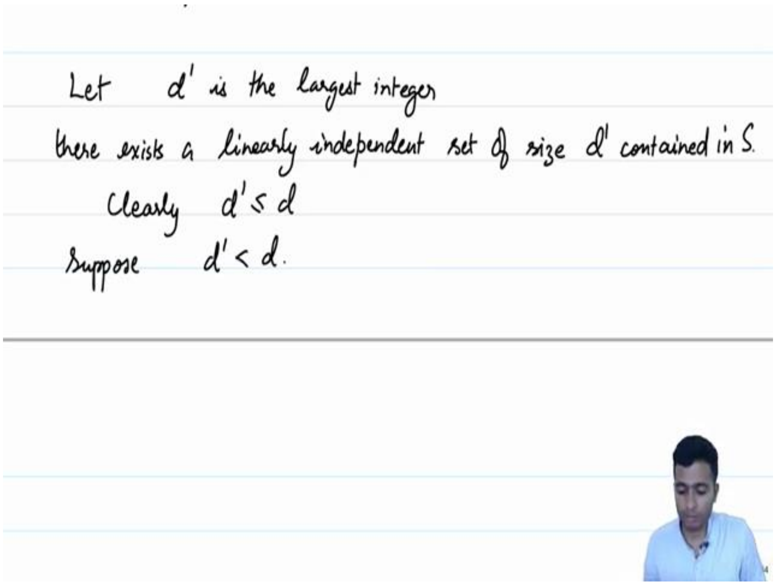

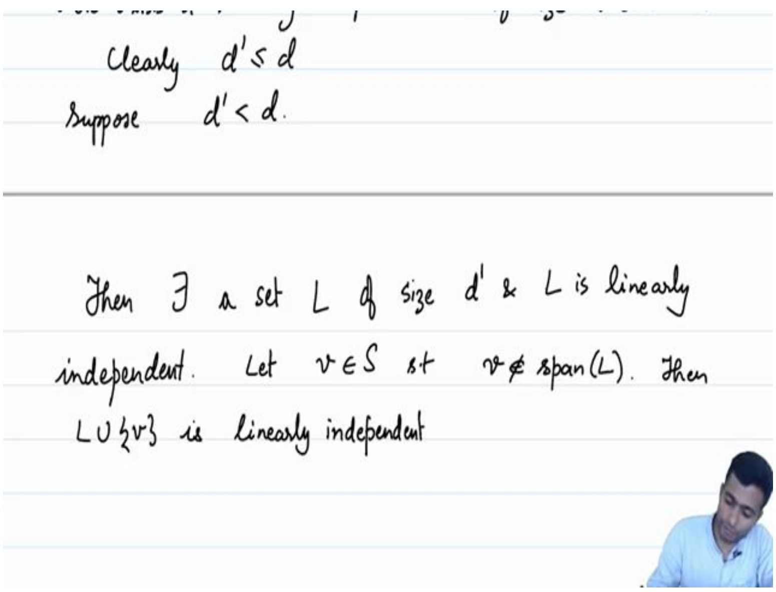

Suppose, d prime is the largest integer such that d prime is less than d and there exists a linearly independent set of size d prime. In fact, let us do one thing, let d prime be the largest integer such there exists a linearly independent subset, set of size d prime contained in V, sorry contained in S. So, if d prime is less than d, that is our contention, d prime cannot be greater than d, because any linearly independent set should have cardinality less than or equal to d.

So, if this is not suppose, let d prime, so d prime clearly is let just note that clearly d prime is less than or equal to d suppose, d prime is strictly less than d. That means that, there exists some then there exists a set L of size d prime and L is linearly independent, but then because the size of L is less than d it cannot be a spanning set, because you have already noted that any spanning set should have size at least equal to d.

So, let us look at some vector V. So, let V be an element in capital S such that v does not belong to span of capital L. Notice that, this is necessarily the case because otherwise span of S will then be the span of L which is the entire vector space. There certainly exists one such V, such that v does not belong to the span of L, but by one of the theorems we proved in the previous video, L union v is linearly independent, and what do we know about L union V.

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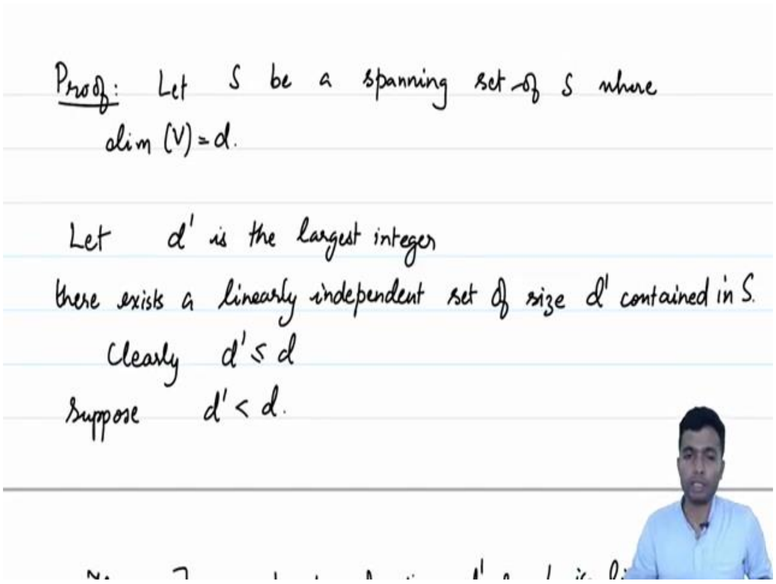

Then $\exists$ a set L of size d' & L is linearly independent. Let $v \in S$ s.t. $v \notin span(L)$. Then $L \cup \{v\}$ is a linearly independent subset of S of size $d'+1$ which is a contradiction to our assumption

Let d' is the largest integer there exists a linearly independent set of size d' contained in S.

Clearly $d' \leq d$

Suppose $d' < d$.

Then $\exists$ a set L of size d' & L is linearly

independent. Let $v \in S$ s.t. $v \notin span(L)$.

independent. Let $v \in S$ s.t. $v \notin span(L)$. Then $L \cup \{v\}$ is a linearly independent subset of S of size $d'+1$ which is a contradiction to our assumption

Hence $d' = d$.

i.e. L is a basis (Contained in S). — ∎

So, linearly independent subset of S of size d prime plus 1, which cannot happen, because d prime which is a contradiction to the fact or to the assumption that d prime is the largest integer such that, there exists a linearly independent set of size require, yes, so let me just note it, which is a contradiction, I will just write to our assumption, just go back and have a look at what our assumption was, contradiction to our assumption.

So, the assumption was that let me just draw it in green for you, this is our assumption, let me just underline it in green for you. Supposed d prime is less than d, all the problem is coming up because of that contradiction is coming up because. Hence, d prime is equal to d, but then that is precisely what we wanted i.e. L is a basis, why? Because linearly independent set of size d should necessarily be a spanning, it should be a basis and this is contained in capital S, we start off with this (())(20:59), hence we have done.

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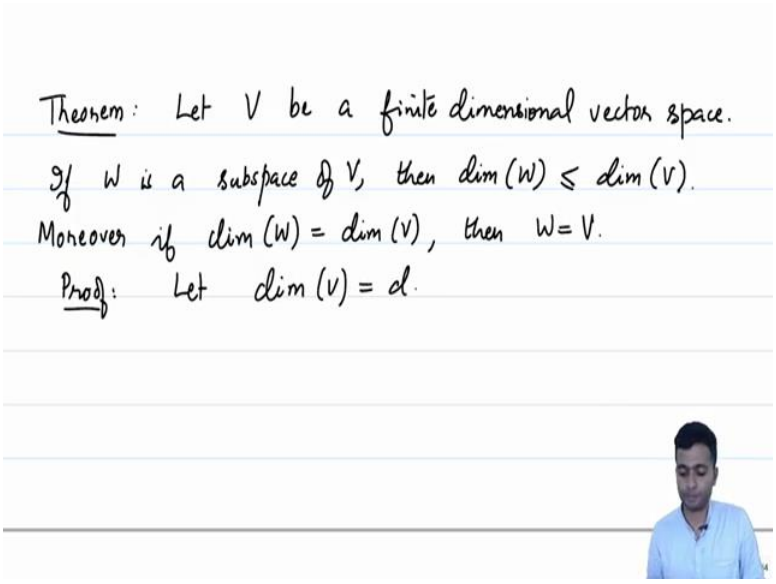

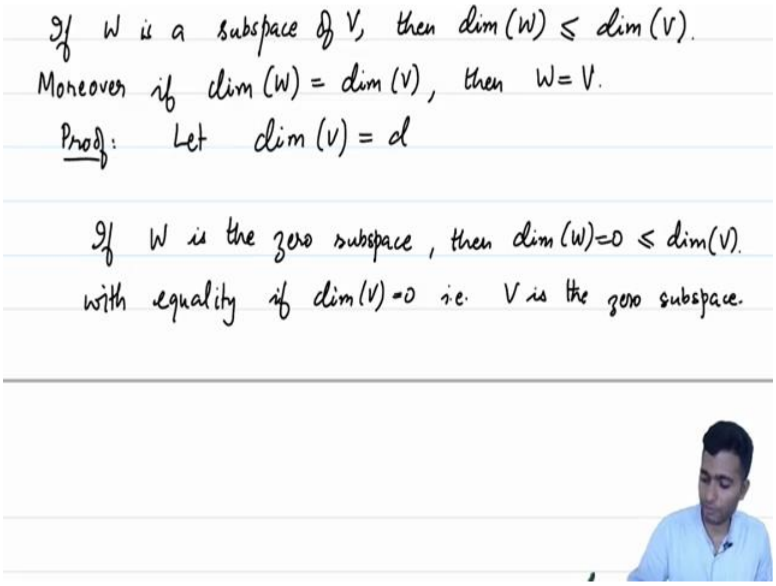

Let us next explore, what we can talk about the dimension of a subspace of a given vectors. So, let V be some vector space with W as its subspace. So, let us call it a theorem now, so let V be a finite dimensional vector space, if W is a subspace of V then dimension of W should necessarily be less than or equal to the dimension of V. If W is a subspace of V, then dimension of W should be less than or equal to the dimension of V. Moreover, if the dimension of V is equal to the dimension of W then W is equal to V. If dimension of W is equal to the dimension of V, then W is equal to V, it is not a proper subspace, it has to just let V be entire subspace.

So, let us give a proof of this, so suppose dimension of V is equal to d, so let dimension of V is equal to d, so finite dimensional vector space. So, if W is the empty subspace, then we are done, if W is the not the empty subspace, I am sorry, if W is the zero subspace, if it is zero subspace, the empty set is the basis. Then dimension of W is zero, which is clearly less than or equal to be dimension of capital V. So, this is equal only if V is also the zero subspace with equality if dimension of V is equal to zero, i.e V is the zero subspace.

So, let us now look at the case when W is not the zero subspace, and clearly V is also not the zero subspace, any element of W should be an element of V. So, let W be some arbitrary subspace, let us look at the other case. So, let us start let us try to get hold of a linearly independent set in capital W, notice that any linearly independent set in capital W will also be a linearly independent set in V, in the vector space V, okay.

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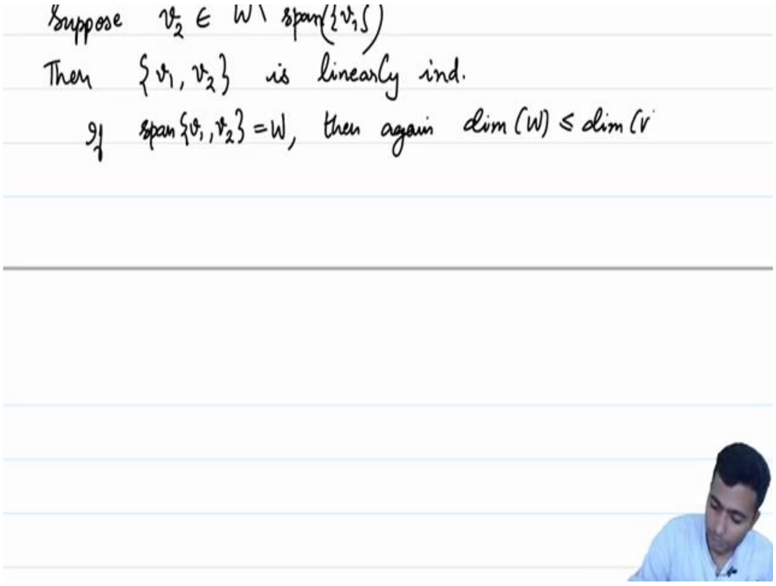

So, let v1 be a nonzero vector in W, be a nonzero vector in capital W. So, if span of v1 is W1 , so, if span of v1 is equal to W, not W1 , W then dimension of V is equal to, 1 sorry dimension of W is equal to 1, which is clearly less than or equal to the dimension of V, clearly dimension of V has to be greater than or equal to 1, because v1 is also in capital V and this is a linearly independent set, it has to necessarily sit inside a basis and therefore, the dimension of V should be at least 1, so this is clearly true.

Suppose not, suppose, the span is not equal to W, suppose v2 be in W minus span of v1, so, let us pick a factor in W2 which is not in the span of v1 . Then we know that v1, v2, I will slowly start referring to the theorems which we are using, you should be now quite used to all the theorems we have done, because v2 is not in the span of so, v2 is in W minus span of v1, so, v2 is not in the span of v1. Because it is not in the span of v1 , the set v1, v2 is linearly independent.

Again by one of the theorems we have already proved, the linearly independent set should be sitting inside a basis of V and therefore dimension of V is at least 2. If v1, v2 spans W then dimension of W is 2, which is less than or equal to dimension of V. So, if this it is if span of v1, v2 is equal to W, then again that then again dimension of W is less than or equal to dimension of V.

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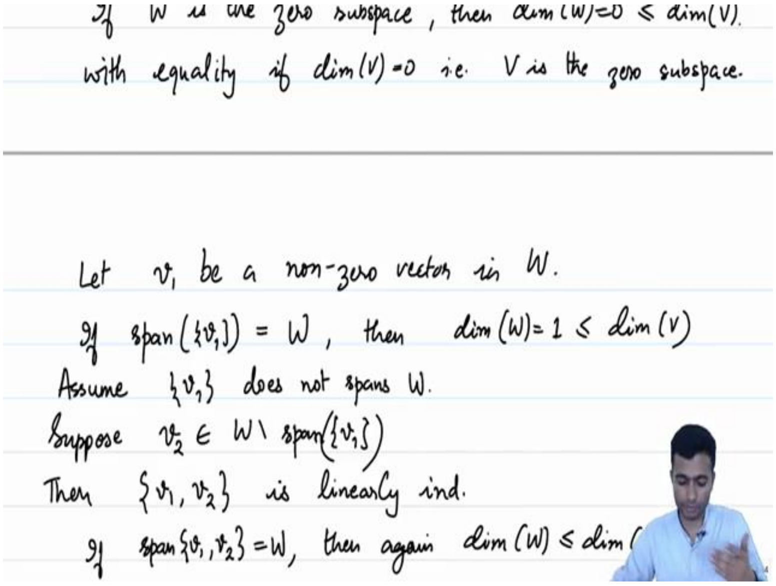

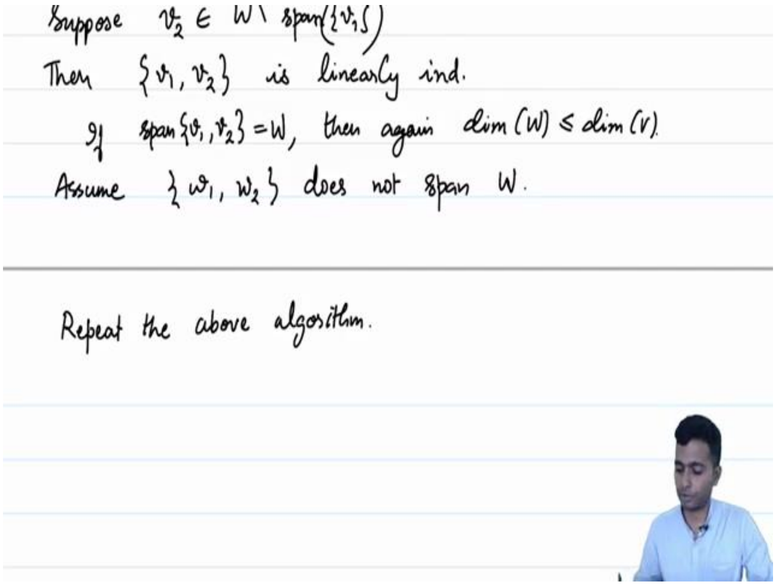

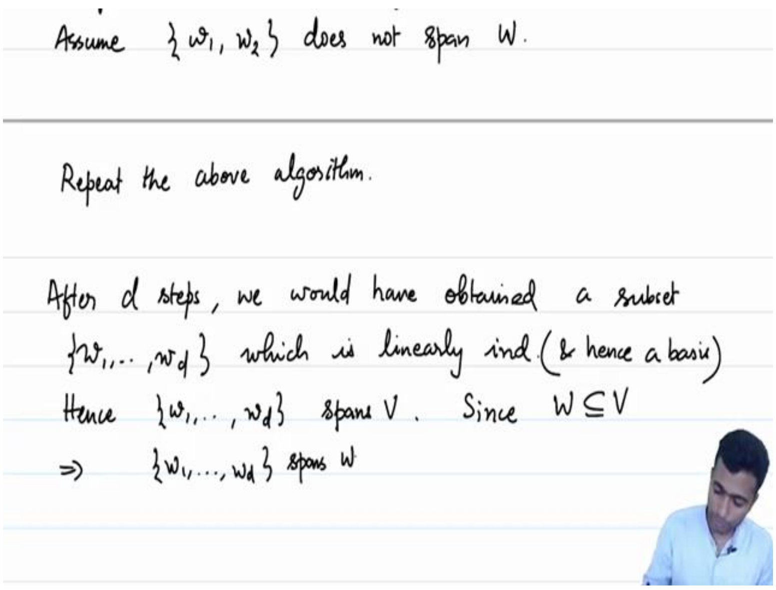

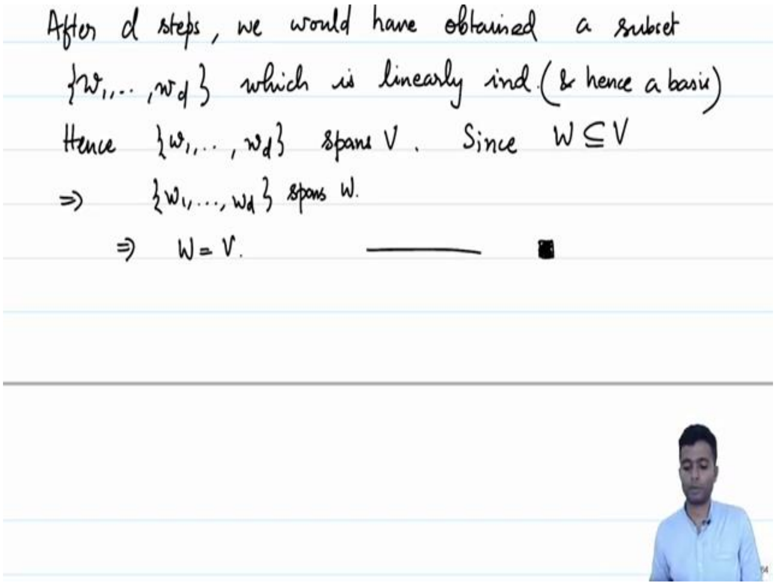

If v1, v2 does not span, let me just before the suppose let me write, let assume v1 does not span, spans W then there is nothing more to do, if it does not span (())(26:47) we are looking at this. And assume in this case that w1 , w2 this does not span W, then there exists some V, W3 three which is not in the span of w1 , w2 and it belongs to capital W.

Follow the same procedure, repeat the above process with the above algorithm, but this process, this algorithm has to stop after d steps, after d steps, if the algorithm does not stop, then after d steps we would have obtained a subset w1, w2 up to wd, which is linearly independent. If that is the case, the linear independence of w1, w2 up to wd in capital W, implies the linear independence of w1 to wd in capital V, and we know that V has dimension d.

Hence, w1 to wd spans V, because it is a and hence a basis here, because it has it is a linearly independent set of size d therefore, it has a basis, but if w1, w2 up to wd spans V, then it also necessarily spans w because W is a subspace of V.

Since W is a subspace of V, W is contained in V, this implies w1 to wd spans W, but then w1 to wd spans V as well, this implies W is equal to V, taking care of the case when dimension of W is equal to dimension of V and hence we have proved this. So next week we will discuss linear transformations, which is one of the most central topics in the study of linear algebra.