Linear Algebra
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Lecture 7
Dimension

So, we have already seen the notion of basis of a vector space. If you recall, a basis was defined to be a subset of a given vector space which was linearly independent and a spanning set at the same time. However, in a example, we noted that there could be multiple basis for a given vector space.

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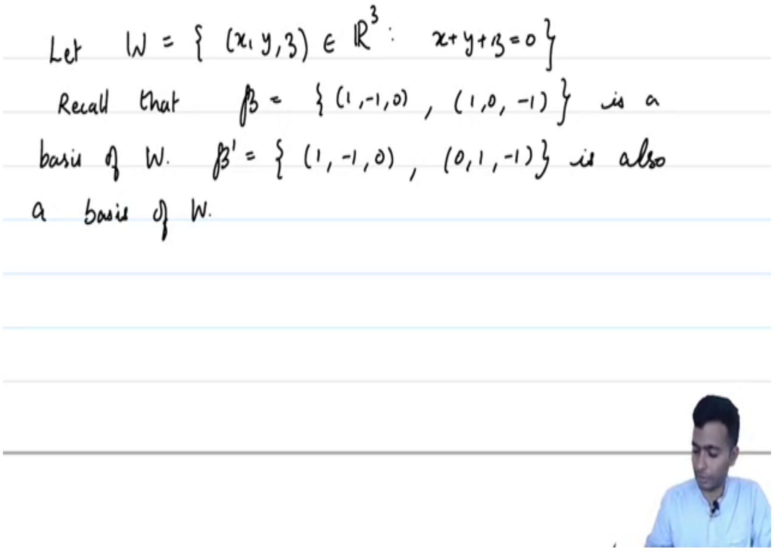

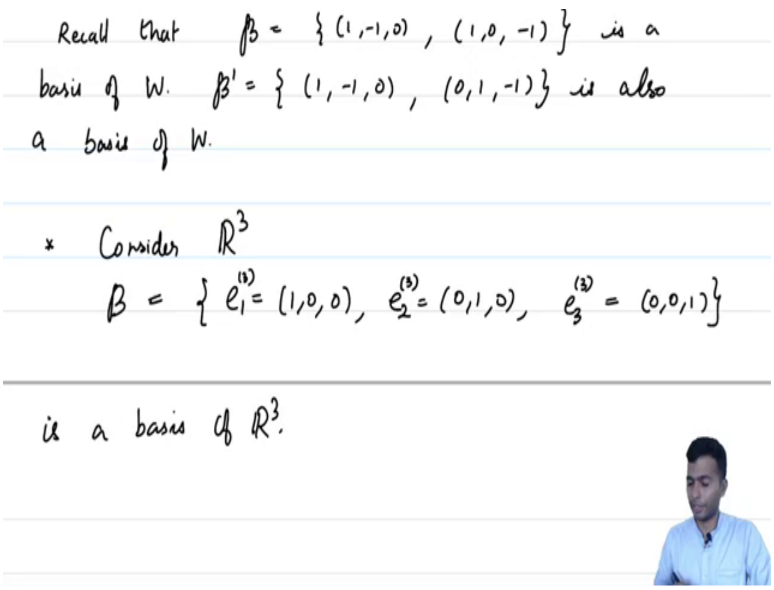

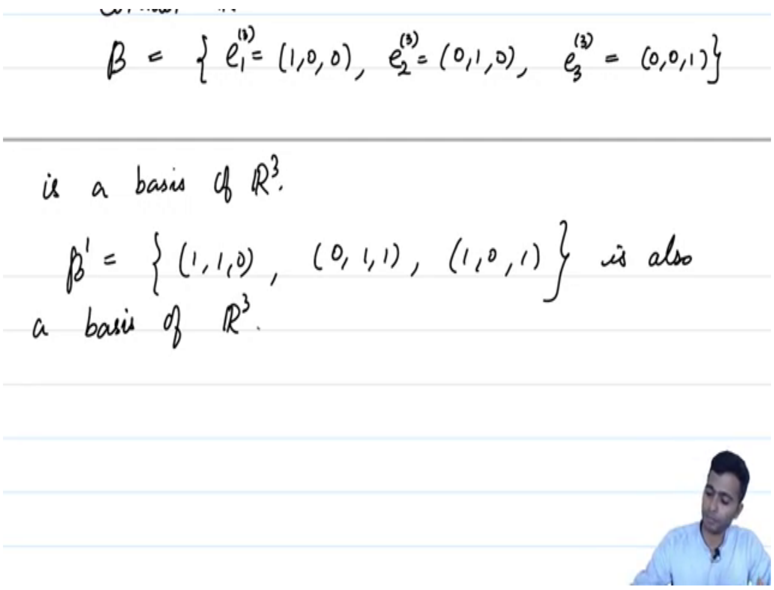

So, for example, recall that we were looking at this example, let W be the set of all x, y, z in R 3 such that x plus y plus z is equal to 0. So, this is a subspace of R 3 and by one of the theorems, we have proved already subspace is a vector space, so this vector space. This is a vector space and we did calculate on it. We did find out a basis for this vector space. So, recall that B of beta equal to 1 minus 1, 0 and 1, 0 minus 1 is a basis of W.

But then if you try to recall how we got hold of this basis, there was a choice involved. And based on the choice we would have ended up with a different basis. So, for example, beta prime which is a given by 1 minus 1, zero and say 0, 1 minus 1 is also a basis. In fact, we could get multiple, many-many basis of W using the same procedure.

But no matter, what basis you take beta prime, you will observe that it has exactly two elements. Let us just consider R 3, in fact, so let us consider R 3 and R 3 and let beta be e 1 equal to 1, 0, 0, e 2 equal to 0, 1, 0. There should be a 3 here, denoting that this is in R 3 but we will slowly drop this superscript 3 which is defined to be 0, 0, 1 is a basis of R 3. We already noticed that there could be multiple basis in R 3 as well.

So consider beta prime, which is say something like 1, 1, 0, 0, 1, 1 and 1, 0, 1. This is also a basis of R 3, well you can spend your time to construct many-many fancy basis of R 3, but again, no matter what basis you come up with, you will always notice that the number of elements in the basis is 3 in the case of R 3.

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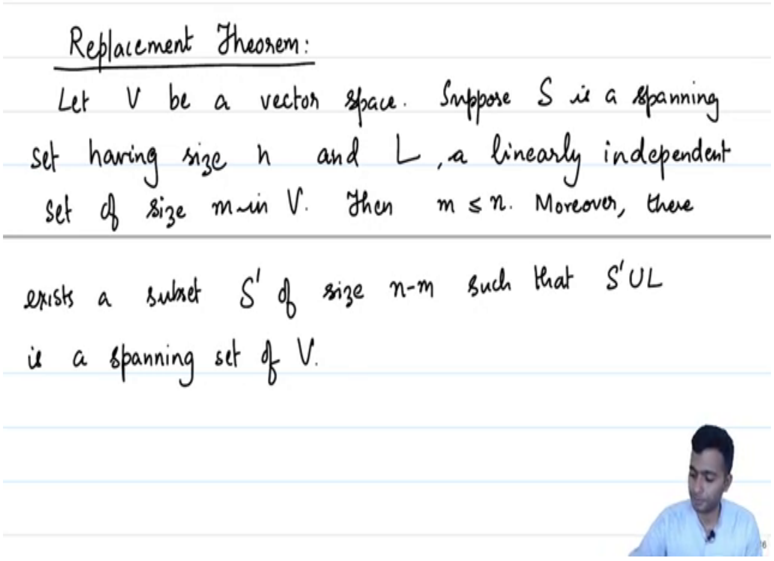

So, this stems from a deeper fact that if you have a vector space, which has a basis consisting of finite linear elements in every basis should have the same number of elements. So this deep statement for will follow a technical reason which we will now state. The result is called the replacement theorem. The technical result is called the replacement theorem. So what does the replacement theorem tell you?

So, let V be a vector space, suppose S is a spanning set having size n and L linearly independent set, independent set of size M. So, suppose we have a spanning set which has size n and linearly independent set of size m in capital V. These are all both are subsets of V. S is the, S is a spanning set of V having size n, n elements are there, L is an independent linearly independent set of size m in V.

Then the conclusion tells us that m is less than or equal to n. Moreover, it is not enough. It is not over, there is a more moreover, there exists a subset S prime of size n minus m, such that S Prime union L, which now has size n minus m plus m and which is equal to n, is a spanning set of V.

So, the theorem might look bit complicated, but one thing which we can immediately conclude from this replacement theorem is that, no matter what linearly independent set you take, and no matter what spanning set you take, the size of a linearly independent set should be less than or equal to the size of a spanning set. That is the first conclusion in this.

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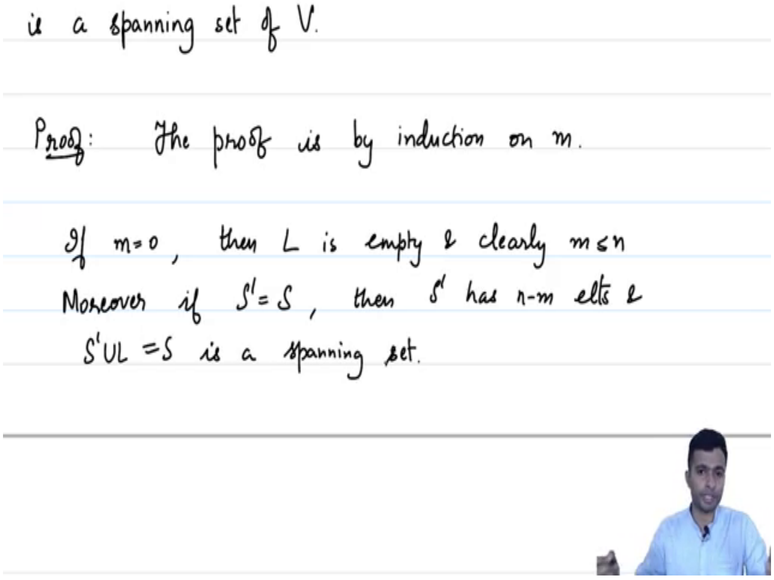

And then the second part says, that the spanning set you can replace many elements with elements in capital L, and get hold of a new set, which will continue to be a spanning set. Let us give a proof of this statement, it goes in many steps. Let us patiently go through all of them. So, the proof is by induction on m. So, proof is by induction on m, suppose m is equal to 0, what does it mean? If m is equal to 0, that means that L has no elements, it means L is empty, then L is empty.

And clearly, m is less than or equal to n, there is nothing to prove there. Moreover, if S is equal to S prime is equal to S, then S Prime has n minus m elements because m is 0, S prime has n minus m elements and S Prime union L is equal to S is a spanning set because S is a spanning set. So, the base case when m is equal to 0 is trivially getting satisfied.

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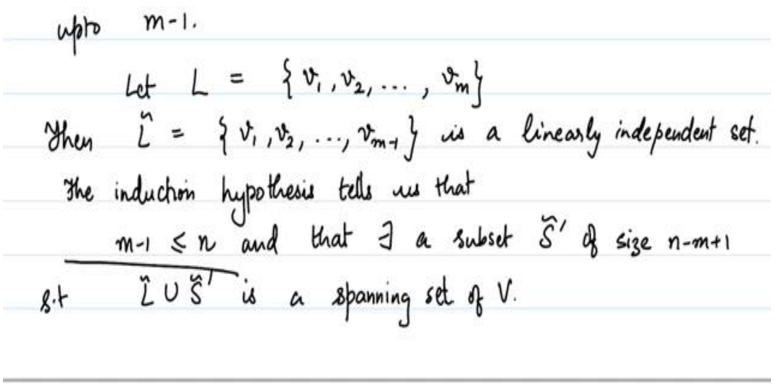

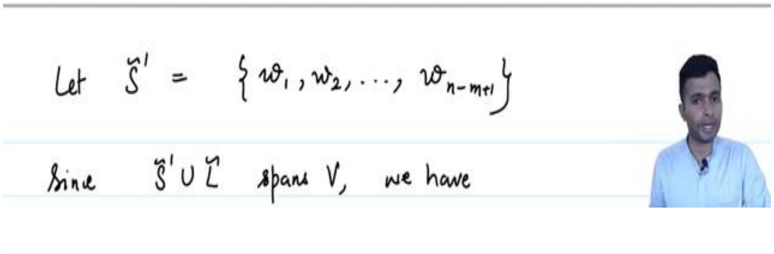

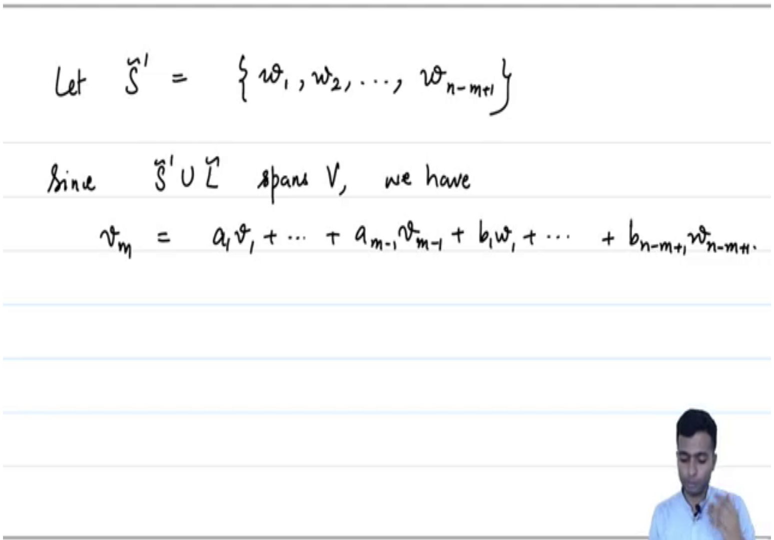

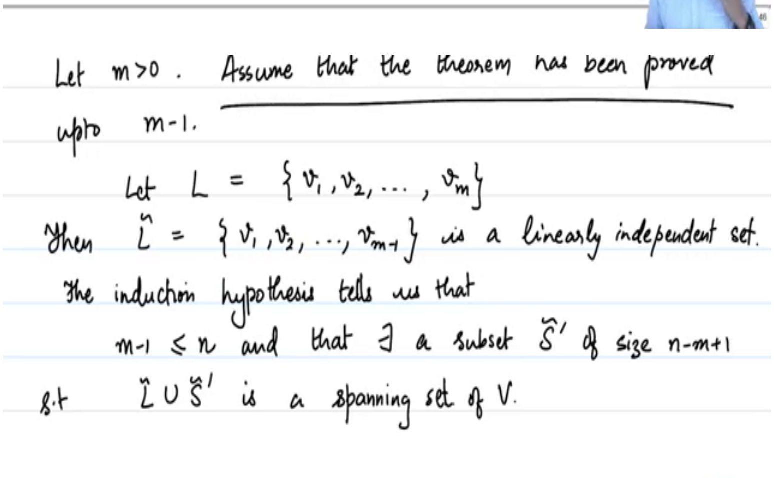

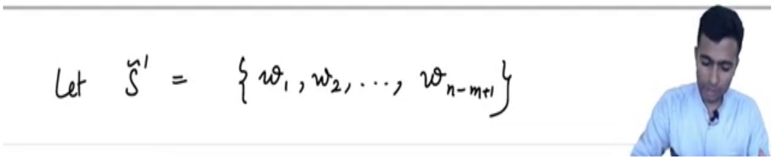

So, let us assume that the theorem has been proved. So, let m be greater than 0, so let m be some positive number and assume that the theorem has been proved for up to m minus 1. So, assume that the, that is the induction hypothesis, the theorem, the replacement theorem has been proved up to m minus 1.

So, in other words if there is an, if there is a subset of size m minus 1 which is linearly independent, then m minus 1 is less than or equal to n, and then there is a subset of S which when of size n minus m plus 1, which if you take union with a given linearly independent set, it will be a spanning set.

That is what the theorem says, for a set up to m minus 1, but we are now proof it for a linearly independent set of size m. So, let L be, let us list down the elements of L. So, let L be equal to v 1, v 2 say up to v m. L is a linearly independent set, then L tilde if you consider just the first m minus 1 elements of it, which is v 1, v 2 up to v m minus 1 is a linearly independent subset because it is the subset of linearly independent subset, this is a linearly independent set.

But we have the induction hypothesis telling us that the result is true for up to m. So, in particular for L tilde, we know that the result is true. So, the induction hypothesis tells us that m minus 1 is less than or equal to n and that there exists a subset. So, corresponding to S, L tilde S tilde prime of size n minus m plus 1 such that L tilde union S tilde prime is a spanning set, this is what the induction hypothesis will tell us.

Because for any linearly independent set of size m minus 1, we have assumed that the result is true. All right, so let us enumerate the vectors in S tilde prime. So, let S tilde prime be the set w 1, w 2 up to w n minus, m minus 1 which is m plus 1.There are n minus m minus 1 elements in S tilde. We know that S tilde prime union L tilde that is a spanning set of V, so this is spanning set of V.

So, in particular, since S tilde prime union L tilde, spans V, we have V m is an element of the span of this. And therefore, V m can be written as something like a 1 v 1 plus dot dot dot, a m minus 1, v m minus 1 plus b 1 w 1 plus dot dot dot, plus b n minus m plus 1, w n minus m plus 1. Let us know prove that m should be less than or equal to n.

So, we have already noticed that m minus 1 is less than or equal to n by, yes. So, notice that we have already assumed that the theorem is true for up to m minus 1 and the moment we considered L tilde, where was L tilde, L tilde we have m minus 1 is already less than or equal to n. Let us prove that m minus 1 cannot be equal to n. If m minus 1 is equal to n, let us come arrive at some kind of a contradiction and that would force m to be less than or equal to n, is not it?

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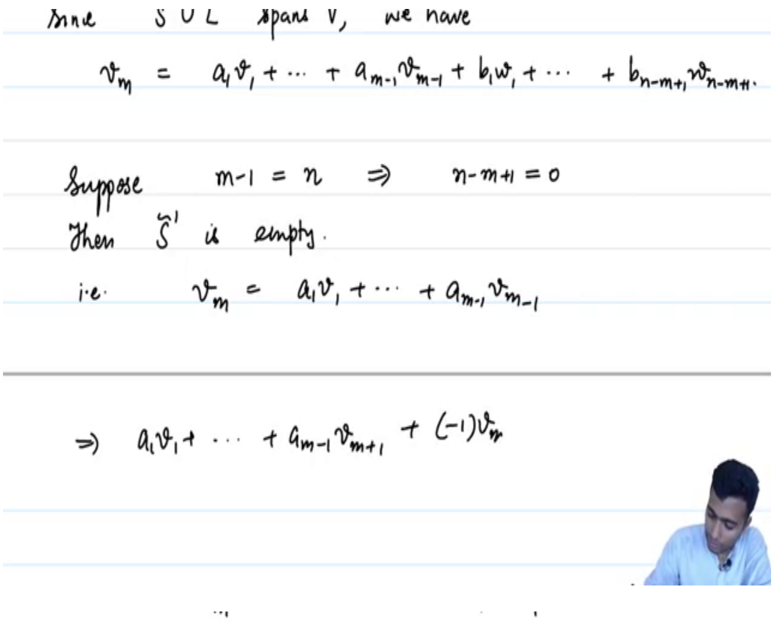

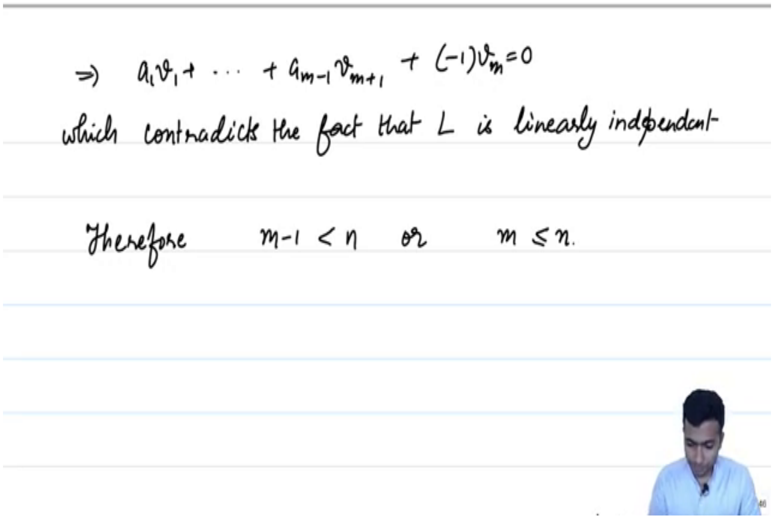

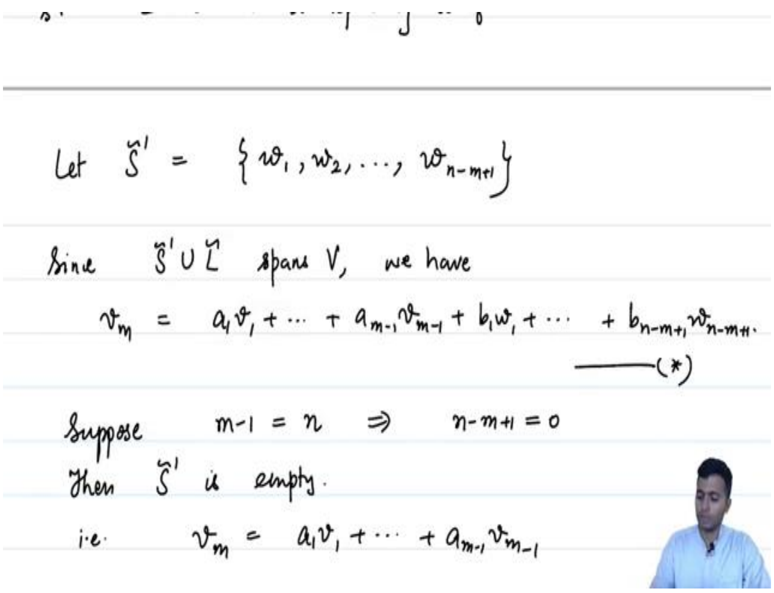

So suppose m minus 1 is equal to n, we already know that m minus 1 is less than or equal to n, but that would imply that n minus m plus 1 is equal to 0. That would mean that the set w 1, w 2 up to w n minus m plus 1, the set S tilde prime is empty i.e. V m is equal to a 1 v 1 plus dot dot dot, up to a m minus 1, v m minus 1, which would imply that a 1 v 1 plus a m minus 1 this is m minus 1 (())(14:15), v m minus 1 plus minus 1 times v m is equal to the 0 vector.

But that is a contradiction, because this is a non-trivial linear combination of v 1 v 2 up to v m, which means that S is linearly dependent. This is a contradiction, which contradicts the fact that L is linearly independent. And therefore, n minus m cannot be equal to m, or in other words m minus 1 is strictly less than n or m is less than or equal to n, they are all integers, m is less than or equal to n. So, we have established that much.

So, now let us consider what would happen, what is the next thing that we would like to have? We would like to show that S tilde, so we would like to get hold of a subset of S, S prime. So is that S prime union L is the, is the spanning set, and S prime has size n minus m. Let us revisit the equation here, so let us call this equation something let us call it star. If you notice, not all b 1, b 2 up to b n minus m plus 1 in star can be 0.

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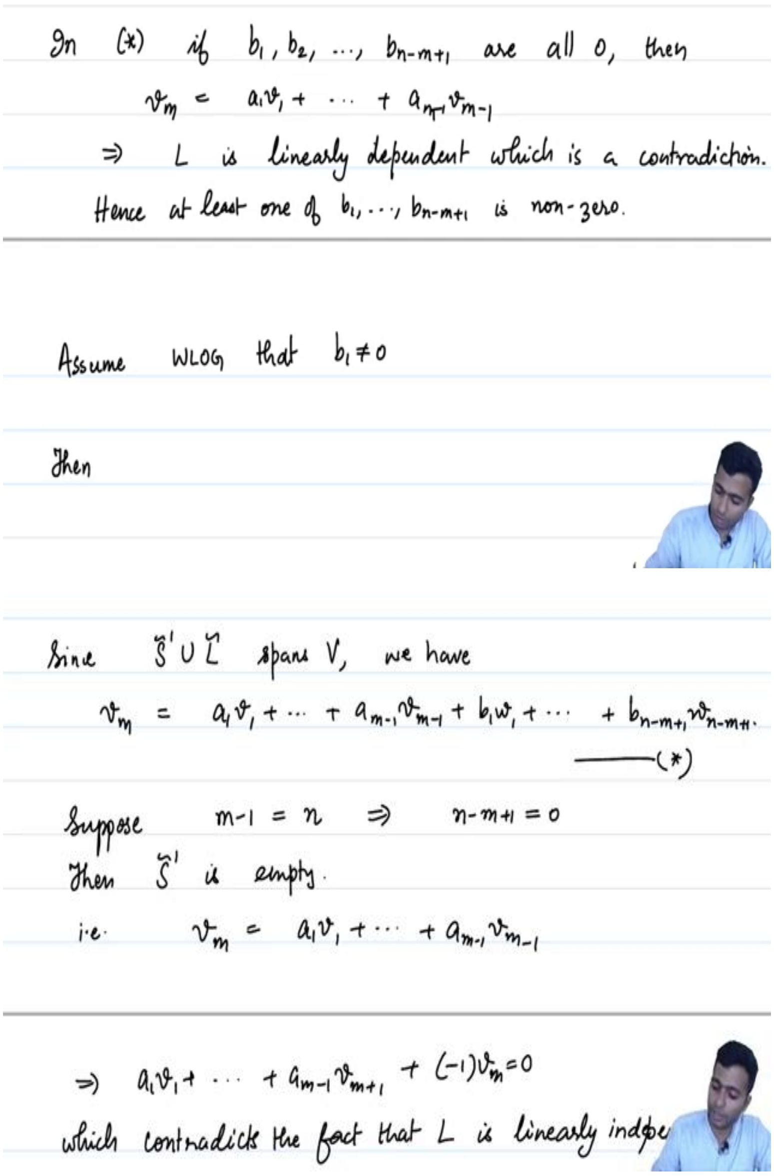

So, let me just note that, notice that in star, so let me show you a star once more. The equation says that v m is equal to a 1 v 1 plus a 2 v 2 plus up to a m minus 1 v m minus 1 plus b 1 w 1 up to b n minus n plus 1 w n minus m plus 1, sorry.

This equation, my claim is that not all of b 1, b 2 up to b n minus 1, b n minus m plus 1 is 0 because if all of them are 0, let me note it down if b 1, b 2 up to b n minus m plus 1 are all 0, then yet again v m would be equal to a 1 v 1 plus a 2 v 2 up to a n minus 1, sorry, m minus 1 v, m minus 1, but then yet again, we are at the same contradiction, but this implies that L is linearly independent.

This implies that l is linearly dependent, which is a contradiction. Therefore, not all of them can be 0, not all of b 1, b 2 up to b n minus m plus 1 is 0. So, that one of them is 0, and after remembering if needed, assume without loss of generality that b 1 is not equal to 0. Assume without loss of generality that v 1 is not equal to 0. So, here this concludes that hence, at least one of b 1 to b n minus m plus 1 is nonzero. That is what we have concluded and assuming that we can do a remembering of the indices without loss of generality assume that b 1 is not equal to 0.

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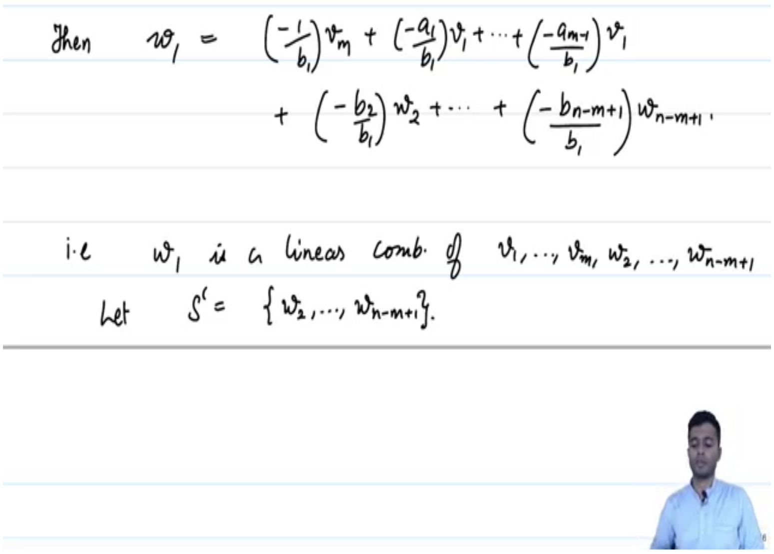

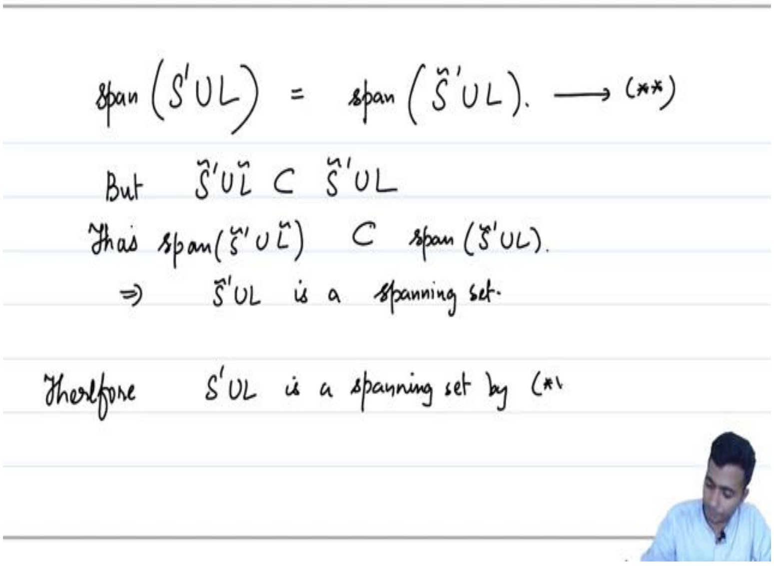

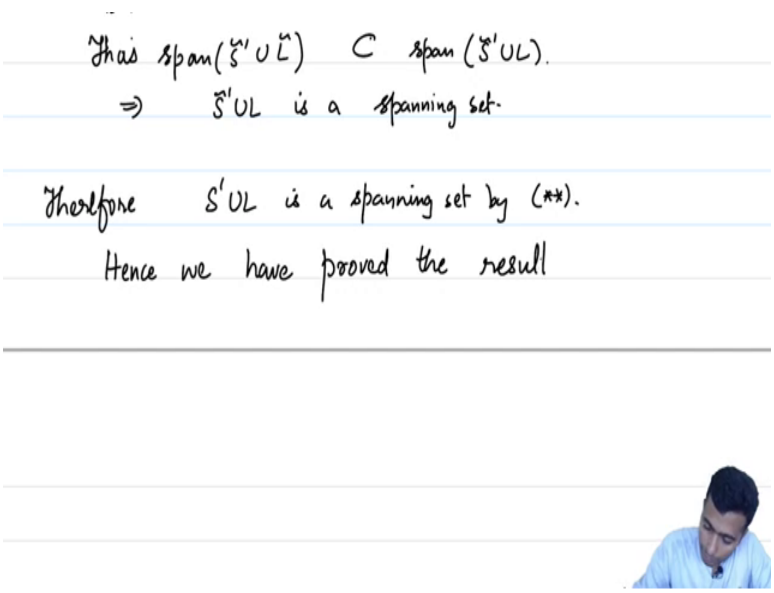

And what does that imply? That means that then, keeping star in mind, let me write down an equation w 1 will then be equal to minus of 1 by b 1 times v m plus minus of a 1 by b 1 times v 1 plus dot dot dot, minus of a m minus 1 by b 1 times v 1 plus minus of b 2 by b 1 times w 2 plus dot dot dot, minus of b n minus m plus 1 by b 1 times w n minus m. w 1 is an element in the span of L tilde. That means w 1 can be written as a linear combination of v 1 to v m and w 2 to w n minus m plus 1.

So, let S prime now be equal to the set w 2 up to w n minus m plus 1. Then notice that S prime union L that is a set which contains, let us try to see what is the span of S prime union L? S prime union L has the same span as span of a first claim is that this is S tilde prime Union L because w 1, if you notice is in the linear span of v 1, v 2 up to v m w 2 up to w n minus m plus 1 and by using a theorem from the previous video, because v is in the span, the set is linearly dependent and span after removing it is preserved.

But S tilde prime union L is a superset, but let me just note that S tilde prime union L tilde, union L tilde is a subset of S tilde prime union L. And therefore, span of thus span of S tilde prime union L tilde, which by the way is equal to the vector space V is contained in the span of S tilde prime union L, which implies that S tilde prime union L is a spanning set.

Therefore, S prime union L is a spanning set by maybe star star, spanning set by star star. So, if you notice we have proved the result because S prime has n minus m elements, and it is a subset of capital S and moreover, S prime union and it is a spanning set. So, we have proved every statement which we had written down in the statement of the replacement theorem. So, there exists a subset S prime of size n minus m is that S prime union L is a spanning set of v. So, hence we have proved the result.

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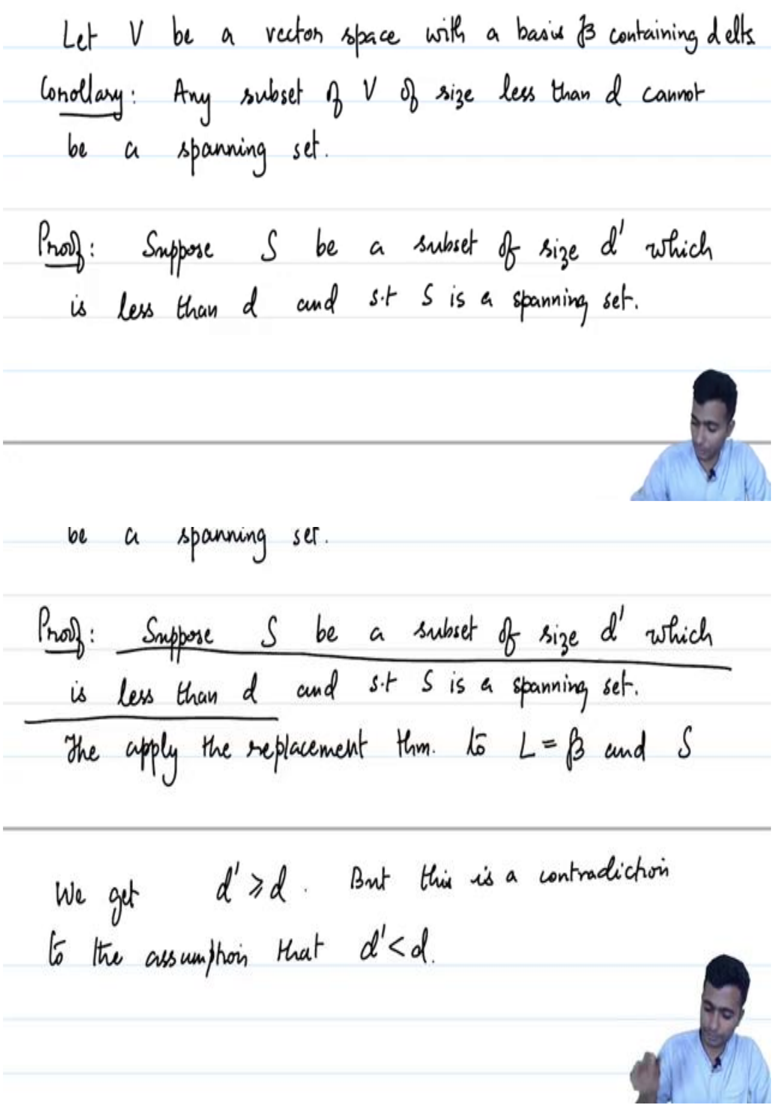

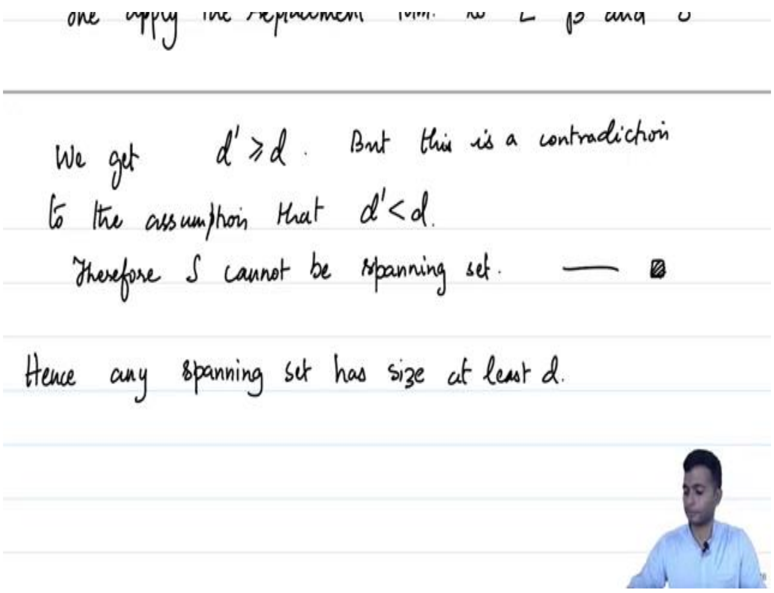

So, let us look at a few corollaries. So, to do that let V be a vector space with a basis beta containing d elements. So, we are given vector space and the basis of it which has d elements. So, the first corollary tells us that if you take any subset of V, which has size less than d then it cannot be a spanning set.

So, let me first write the corollary statement any subset of V of size less than d cannot be a spanning set. So, let us give a proof of this. So, suppose when below suppose S be a subset of size say d prime, which is less than d and such that S is the panning set, will arrive at a contradiction. Saying that if you take any set which has size less than d cannot be a spanning set.

So, let us assume that there is one such set, set S which has a d prime, where d prime is less than d and S is spanning set. Then apply the replacement theorem to L equal to beta and S equal to S and S just S. So, for the linearly independent set, take the basis beta and a spanning set S is the spanning set we are given. Then by we get by the replacement theorem, d prime should be greater than or equal to d.

But that is a contradiction, because they have assumed that d prime is less than, but this is a contradiction with the assumption that d prime is strictly less than d, if you notice this is what our assumption began with, suppose S is a subset of size d prime which is less than d and such that S is a spanning set, so that cannot happen. So, there is this assumption going wrong and therefore, it cannot be a spanning set. Therefore, S cannot be a spanning set.

A rephrasing of the above statement hence tells us that any spanning set should have at least size d i.e., so we have finished the proof. Hence any spanning set has size at least d, any set of size greater than d cannot be linearly independent.

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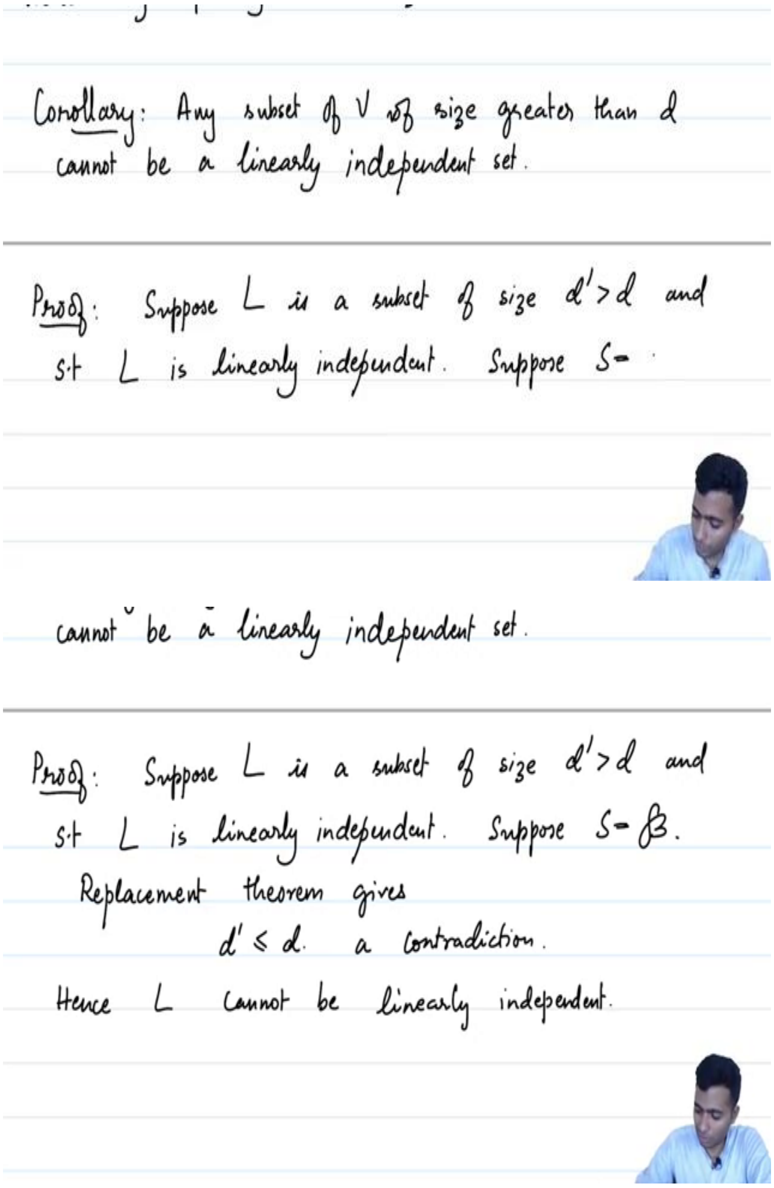

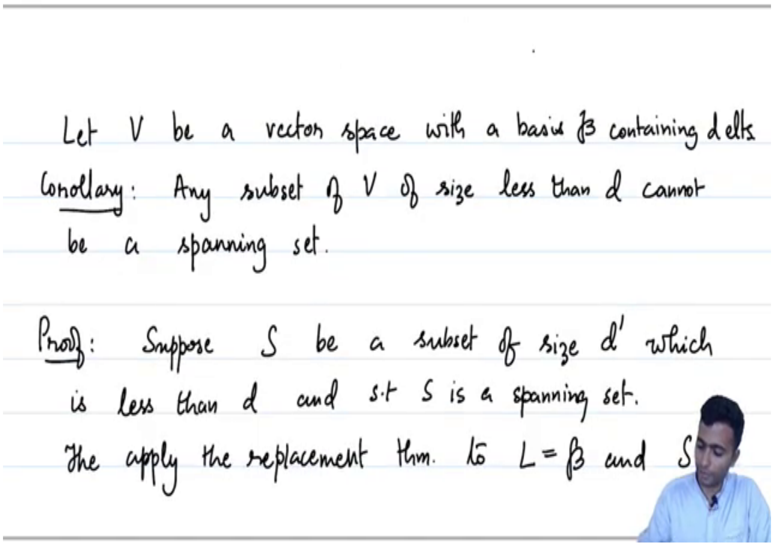

So, let us look at another corollary. It is in a similar way as in this, this said that any subset of V have size less than d cannot be a spanning set. Next corollary tells us that any subset of V which has size greater than d cannot be a linearly independent set. Any subset of V of size greater than d cannot be a linearly independent set. Let us first prove it, when S is finite, so, suppose again we will prove it by introduction, sorry contradiction. Suppose S is a subset of size d prime which is greater than d and such that S is, so let me just rename it, let me call it L, L is linearly independent.

Now, let us apply replacement theorem with S equal to beta. Let suppose, S is equal to beta the basis, we are given that beta is a basis and therefore, a spanning set. Now, when we apply replacement theorem to S and the L replacement theorem, L is a linearly independent set. Replacement theorem tells us, d prime is less than or equal to d. The size of a linearly independent set should be less than or equal to the size of a linearly sorry size of a spanning set. Therefore, d prime is less than or equal to d, which is a contradiction. Therefore, hence L cannot be linearly independent.

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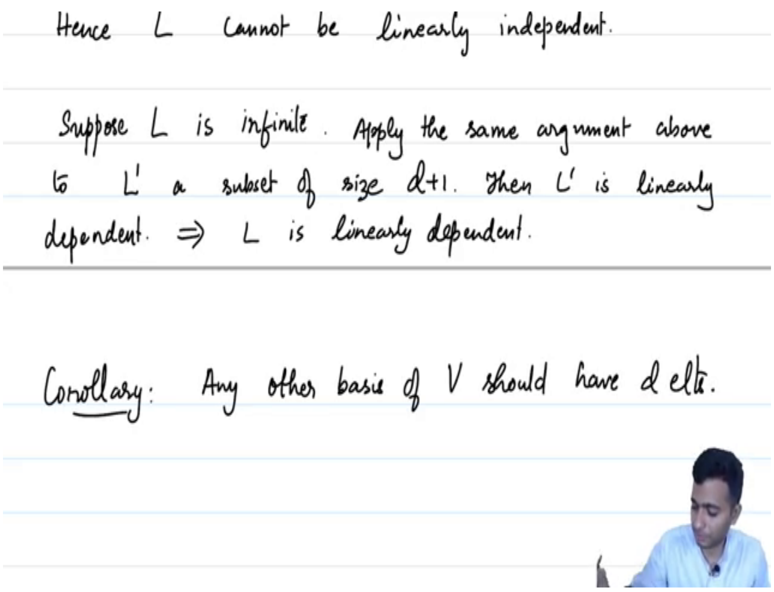

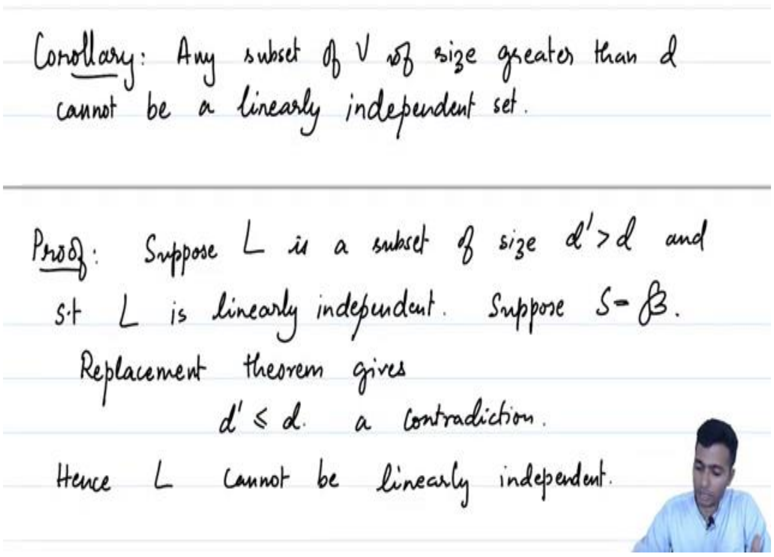

But we put in a strong assumption, not necessarily a strong assumption, but we have put in assumption that L is finite here, what will happen if L is infinite? We could not have obtained a d prime or we could not have applied replacement theorem here. So, this is the case, suppose, L is infinite. If you notice, the replacement theorem has in its assumption that the size of the linearly independent set is finite, the size of the spanning set is finite and then we conclude that the size of a linearly independent set is less than or equal to the size of the spanning set.

But it was not said in the statement of the corollary that it is finite. It is the same size greater, It could have been an infinite set, but then if it is an infinite set, there is no problem. Apply the same argument as above to L prime a subset of size, say d plus 1. To take a subset of L, which is of size say d plus 1, then L prime is linearly dependent, but if a subset is linearly dependent, the set has to be linearly dependent, dependent, which implies that L is linearly dependent and we are done.

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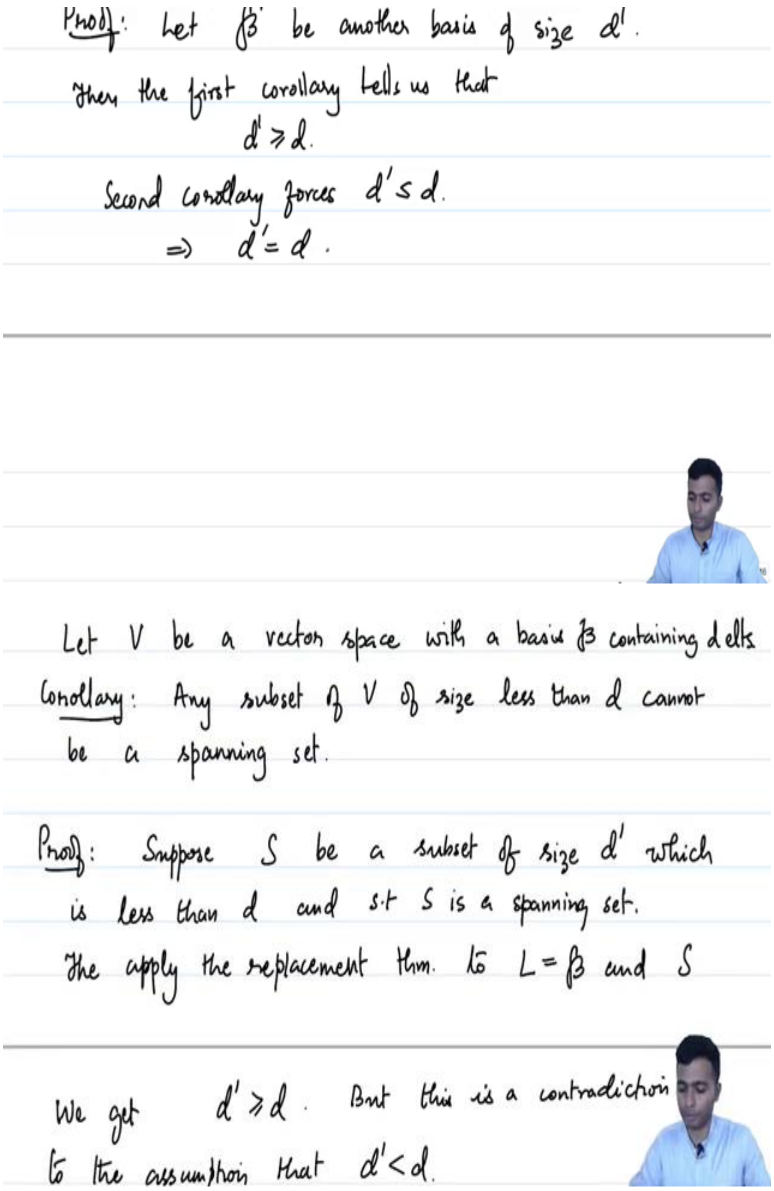

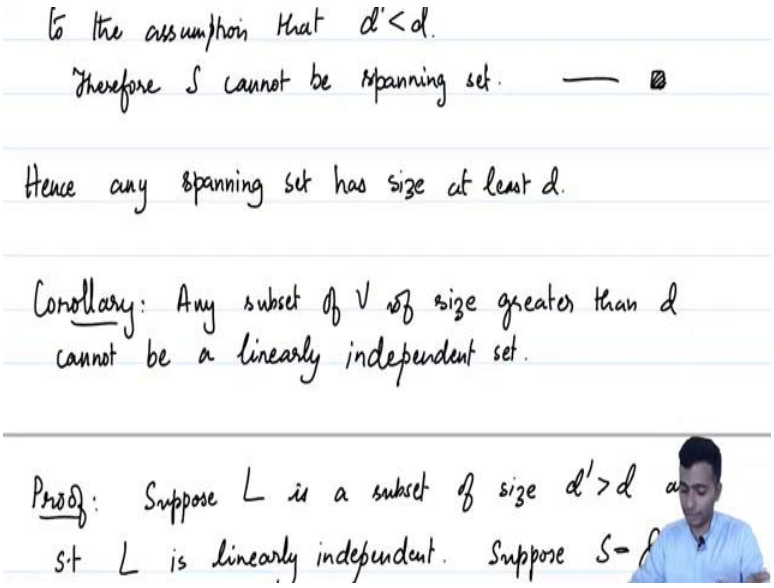

So, let us now look at one more corollary, any other basis of V should necessarily have d elements of V should have d elements. So, let us give a proof of this that will establish whatever we have set out to proof. So, beta prime be another basis, beta prime be another basis of size d prime. Then corollary one tells us that what is the first corollary? Let me show you the first corollary, any subset of size less than d cannot be a spanning set, but this is a spanning set, and therefore, d prime should be greater than or equal to d.

The first corollary, then the first corollary tells us that d prime is greater than or equal to d and how about the second corollary? The second corollary tells us that any subset of your size greater than d cannot be linearly independent, but our beta prime is a basis, it is linearly independent and hence its size should be less than or equal to d. The second corollary tells us that d prime is less than or equal to d. Second corollary forces d prime to be less than or equal to d, this gives both these inequalities will give that d prime is equal to d.

Therefore, we have established that if you start of with a vector space which has a basis of size d, any other basis should necessarily have the same number of elements. This number is special.

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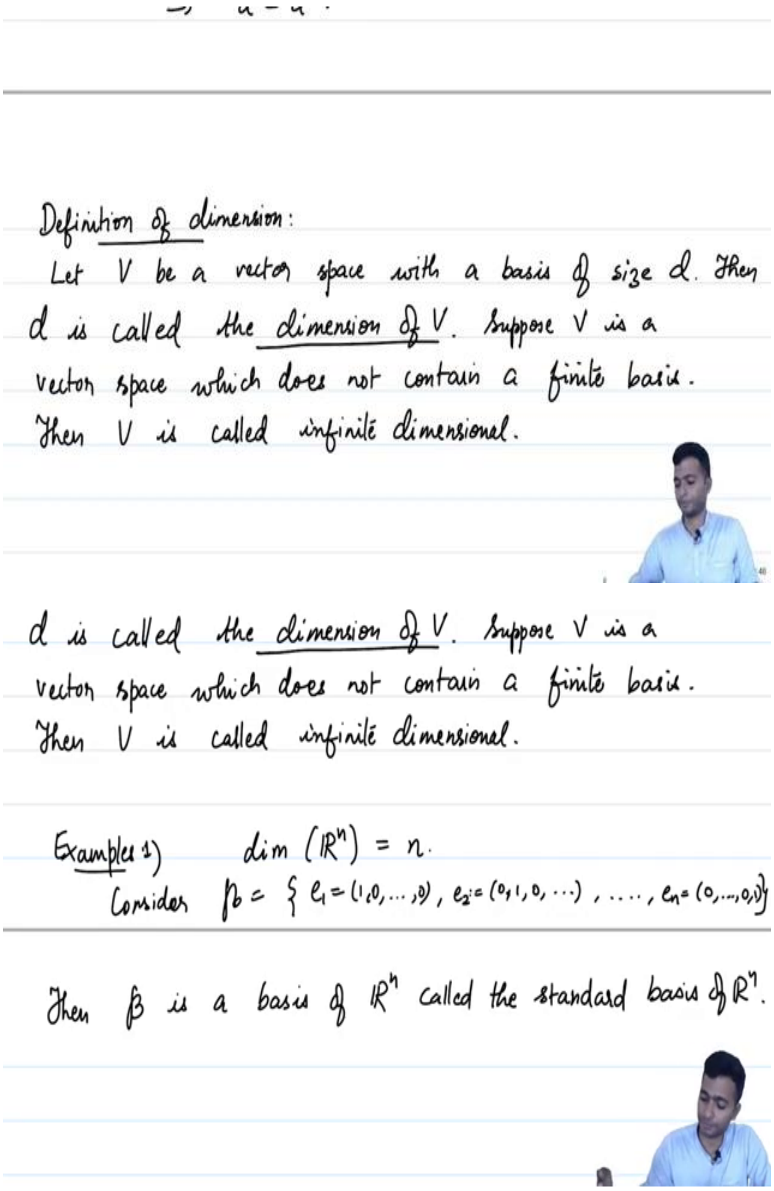

So, let us give a name to it, let us call it the dimension of the vector space. So, let V be a vector space with a basis of size d, then d is called the dimension of V. We just noted that this is a well-defined number, depending on the basis, the size will not change, you take any 2 basis, if there is one basis of size d, any other basis should also has size d. So, this is a well-defined quantity and this dimension it captures a lot of information about the vector space which we will see later.

Let us give however, before our going into all that, let us give the definition of infinite dimensional vector space. Suppose, V is a vector space which does not contain a finite basis then V is called infinite dimensional. Let us look at some examples, maybe the first example, could be of R n, dimension of R n is equal to n, why is that the case?

Consider beta to be the standard basis, so this basis considered beta to be the set given by e 1 equal to 1, 0, 0 and up to all zeros later, e 2 is equal to 0, 1, 0, so on, e 3 with 0 everywhere except in the third coordinate which will be 1 and e n is equal to 0, up to 0, 1. Then beta is called is a basis of R n called the standard basis of R n. Therefore, dimension of R 3 is 3 and dimension of R 5 is 5.

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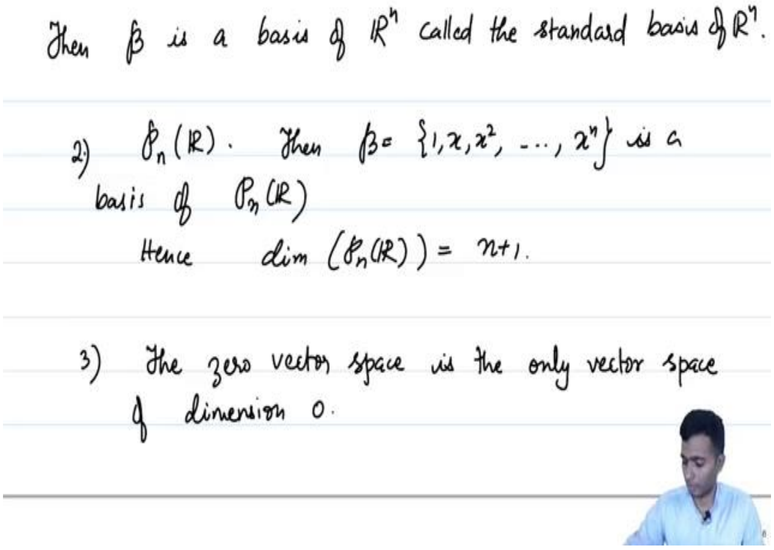

Similarly, we have checked that if you look at P n of R then beta which is given by 1, x, x square up to x to the power n is a basis of P n of R, which gives a dimension of P n of R is equal to n plus 1. So, that is another example. Another example is the 0 vector space, the 0 vector space let me allow you to guess 0 vector space as the empty set as its basis, is the only vector space in fact, of dimension 0.

Like by conventions, span of the empty set is 0, and empty set is linearly independent. It is all consistent with the conventions. That is the basis that is a basis of the 0 vector space that is only basis here. And that is the only vector space which will have dimensions 0, you should think about why that is the case.

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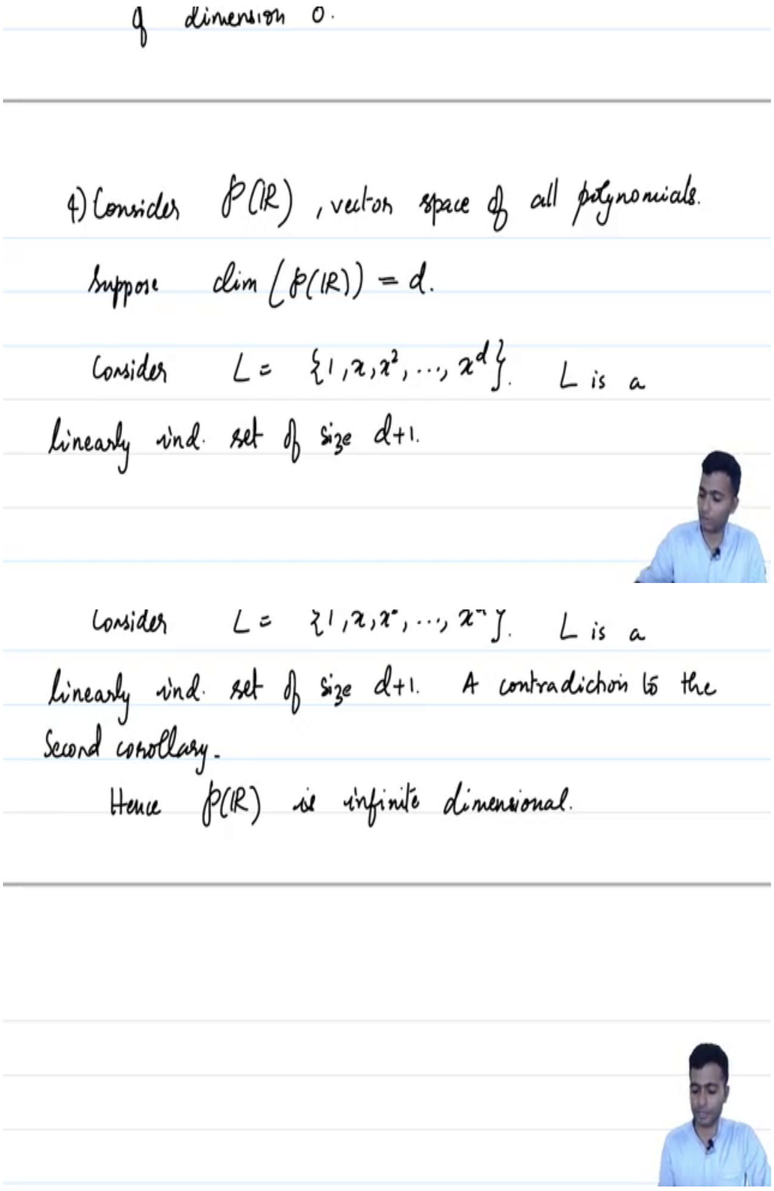

Let me now give an example of space which is not finite dimensional. So, consider P of R. If you recall, this is the space of all polynomials, vector space of all polynomials, vector space of all polynomials. Now, if this is a vector space of finite dimension, suppose dimension of P of R is say something like equal to d that means as a basis of dimension d. Consider S to be equal or L to be equal to 1, x, x square up to x to the power d. By the corollary, the size of L, which is a linearly independent set, should be less than or equal to d.

But this has L is a linearly independent set of size d plus one, dependent set of size d plus one and therefore, it is a contradiction, the contradiction to corollary, to the first corollary, contradiction to the second corollary. Therefore, P of R cannot be finite dimensional. Therefore, hence P of R is infinite dimensional.

Next video, we will look at some more consequences of the replacement theorem. We will also see how dimensions interact with the subspaces, subspaces of the given vector space.