Linear Algebra
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Lecture 6
Basis

Let us next discuss the notion of a basis of a vector space. So, a basis of a vector space is a set which is linearly independent and which is also a spanning set at the same time.

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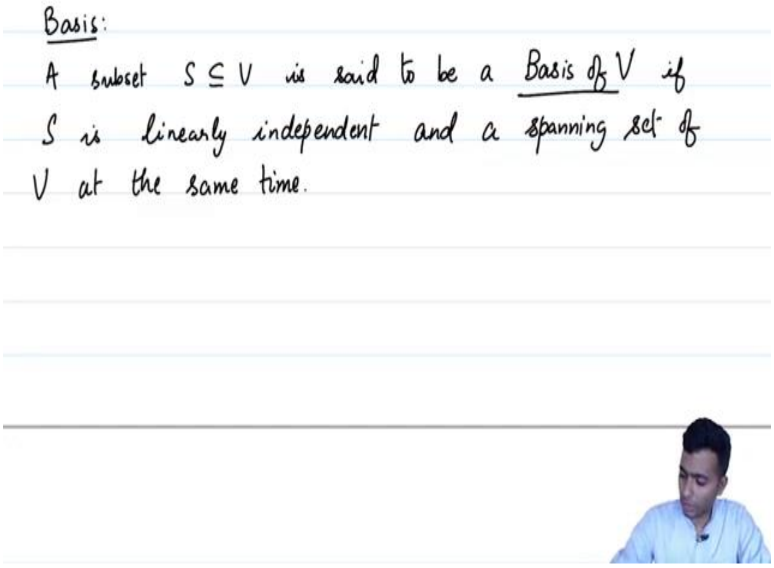

So, let me start with the definition here basis. So, a subset S contained in V is said to be a basis, a basis of V if S is linearly independent and a spanning set of V at the same time. So, notice immediately that a theorem from our previous video tells us that if you throw out even one vector from a basis, then the span will be a strictly smaller set, and therefore it cannot be the, it cannot maintain the spanning property. So, basis in some sense captures the bare minimum or the minimum number of vectors which is needed to span of a given vector space.

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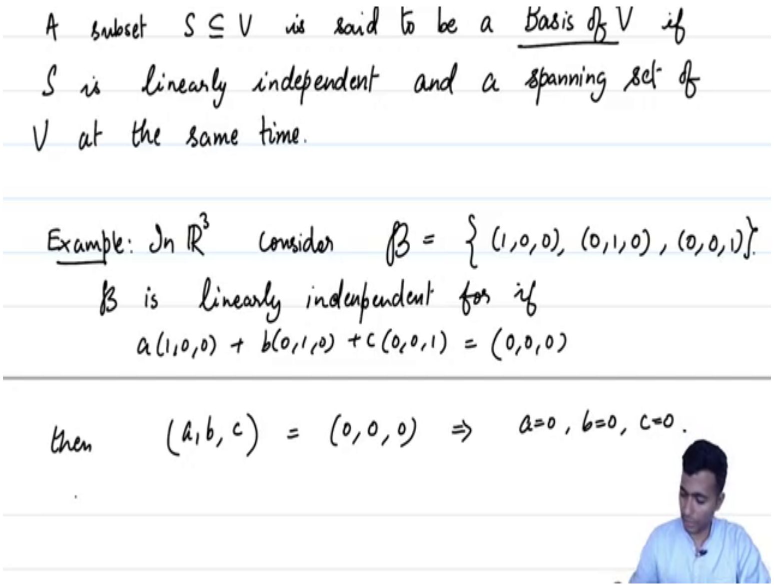

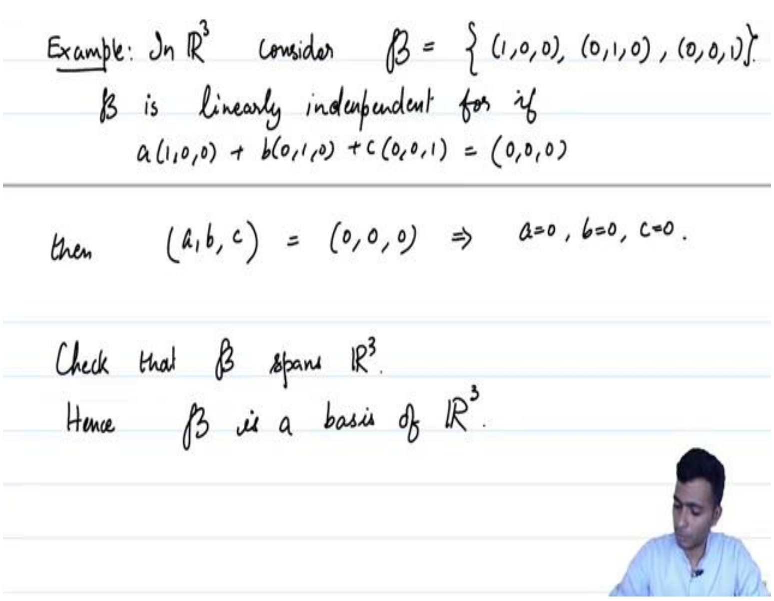

So, let us look at a couple of examples. First example is probably R 2, R 3 R 2 will have a similar set, in R 3, consider so, let me not use the symbol S, let me use B, which is equal to 1, 0, 0 0, 1, 0 and 0, 0, 1. So, it is quite straightforward to check that B is both linearly independent, and that it is a spanning set. So, maybe I will just show that it is linearly independent B is linearly independent, independent or if a times 1, 0, 0 plus b times 0, 1, 0 plus c times 0, 0, 1 is equal to the 0 vector there is a linear combination which.is equal to 0. Then the linear combination, the LHS will just turn out to be vector a, b, c being equal to the 0 vector.

And that means component wise it is, coordinate wise it is equal to the coordinate vectors of these 0 vector, therefore, we are done. We are done with the linear independence. I leave it as an exercise for you to check that B is a spanning set of R 3. So, check that B spans R 3, or beta spans R 3, hence B or beta is a basis of R 3. So, we now have one example of a spanning set.

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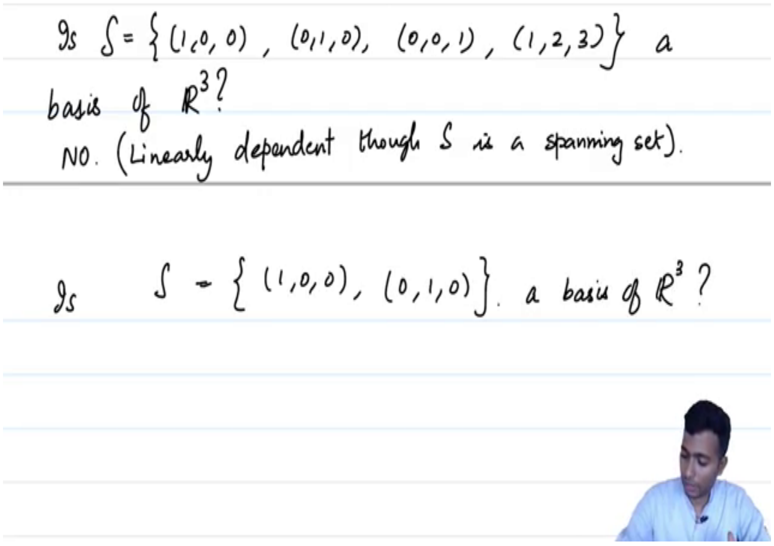

So Is 1, 0, 0 0, 1, 0 0, 0, 1 and say 1, 2, 3 is this a basis? Let me call it as Is is a basis of R 3. The answer is that, no it cannot be, because it is not linearly independent I will leave it as small exercise for you to get hold of the linear dependence of the set S answer is no, not, so, it is linearly dependent.

Similarly, if you consider Is the set S which is given by say 1, 0 0. So notice that in this case, when our S was 1, 0, 0 0, 1, 0 0, 0, 1 and 1, 2, 3, it is still has spanning set, it is though S is a spanning set. Though S is a spanning set, it is still a spanning set. So, one property is not getting satisfied of that of linear independent, it is not a linearly independent set. So it is a linearly independent set. That is why it is not a basis.

And let us now consider this let a collection of vectors, wherein we throw out 0, 0, 1 and 1, 2, 3 from the previous example, then the question Is S a basis of R 3.

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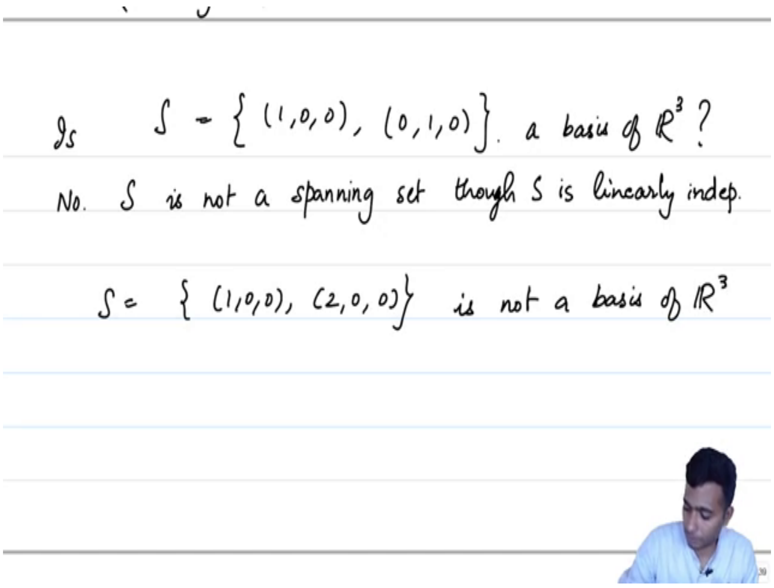

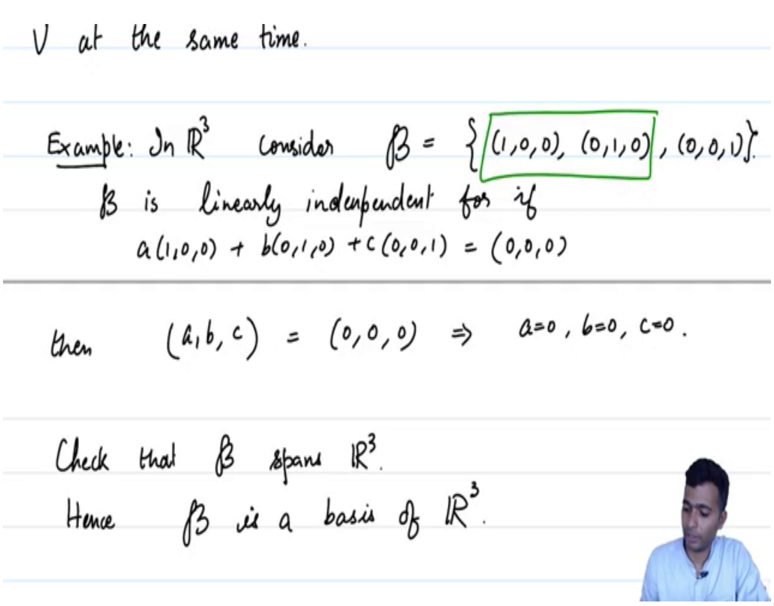

And before you go ahead, I will request you to think about whether S is a basis of R 3 or not on your own, and come to a conclusion that S is not a basis. No, S is not a basis, observe that S is contained in a basis B because it is after all, the first two elements in this basis and basis to begin with this linearly independent. Therefore, this set is also linearly independent.

A Subset of a linearly independent set will also be linearly independent. Therefore, S is certainly linearly independent, however, the previous week's theorem tells us that if you throw out an element from a linearly independent set, the span reduces strict, it will become a strict subspace of the span of S.

And in this case, the span of S was the vector, entire vector space, if you throw out one of them, then it has to be a strict subspace, in fact, we noticed that 0, 0, 1 will not be in the span of S here. So, this is not a spanning set, though S is linearly independent. So, these two examples are to highlight that both the properties, spanning's being a spanning set and that of being a linearly independent set, both should be satisfied. Even if one of them fails, it not be a basis. So, it is a basis if both the properties are satisfied. So yes, this is not a basis.

In fact, there are examples where say if you takes a 1, 0, 0 and a say 2, 0, 0. This is neither linearly independent nor a spanning set, is not a basis clearly of R 3.

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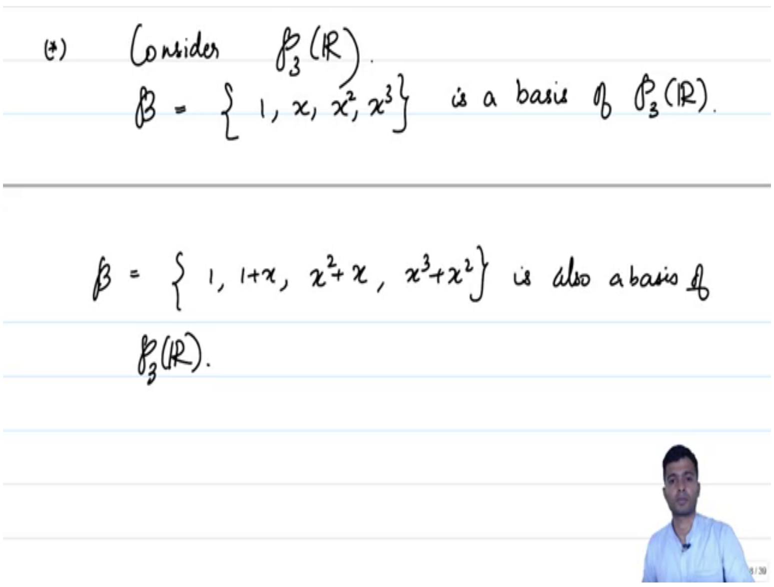

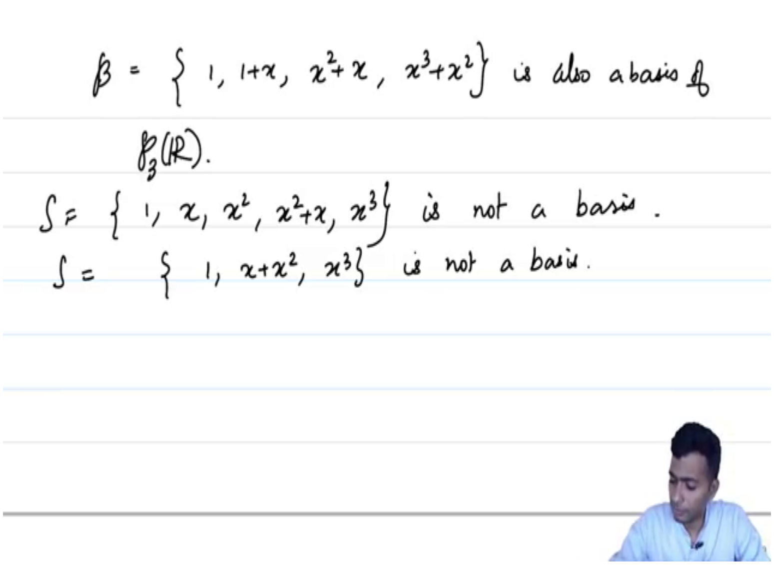

Alright, so let us maybe look at a different vector space, consider P 3 of, maybe P 3 of R, let us put 3, no problem. And I would like to claim that 1, x, x square and x cube, the monomials 1, x, x square, x cube. This is a basis of P 3 of R, let me not spend too much time to check that for you, it is quite straightforward, just like in the case of 1, 0, 0 0, 1, 0 and 0, 0, 1 in R 3, check for the linear independence and the spanning property of the set B here. We are also like to state that 1, 1 plus x, x square plus x and x cube plus x square is also a basis of P 3 of R.

And if I consider our set S to be say 1, 1 plus x or maybe 1, x, x square, x square plus x and x cube, this is not a basis though it is a spanning set, it is not linearly independent. Similarly, if you look at 1, x plus x square and x cube, this is also not a basis, though it is linearly independent because it is not a spanning set. All right, so we have seen a couple of examples and couple of vector spaces and many examples of subsets which could be which were basis and which were not basis for various reasons.

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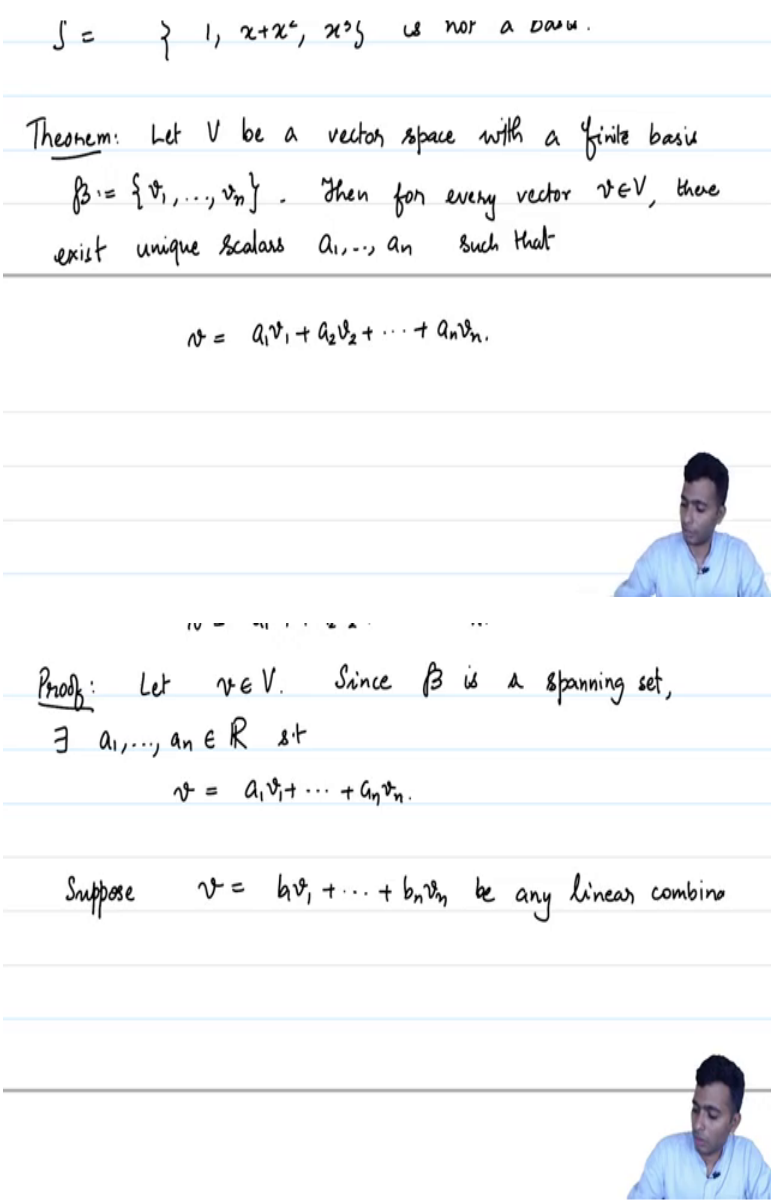

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n.$$

$$\Rightarrow \quad (a_1-b_1)v_1 + (a_2-b_2)v_2 + \cdots + (a_n-b_n)v_n = 0$$

But $\mathcal{B}$ is linearly independent.

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But $\mathcal{B}$ is linearly independent.

Hence $a_1-b_1=0, \quad a_2-b_2=0, \ldots, a_n-b_n=0$

$$\Rightarrow \quad a_i = b_i \quad \forall \ i=1,2,\ldots,n.$$

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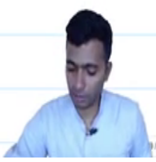
Hence every vector in V can be written as an unique linear combination in $\mathcal{B}$. —— ■.



Let us slightly make example a bit more complicated. Before we look at more examples, let us do a theorem which tells us why starting basis is worthwhile, it is actually quite special. So, theorem tells us that you take any vector in our vector space V, then the vector can be written as a unique linear combination of elements in the basis or vectors in the basis.

Every element can be uniquely written as a linear combination of vectors in a basis. So let us state the theorem, so let V be a vector space with a basis, a finite basis. Yes, let me impose the restriction of finite here, finite basis B which is defined as the set V 1 to V n. So, what the statement says is that we have a finite set V 1, v 2 up to V and, which is both linearly independent and which is a spanning set at the same time.

Then, for every vector V, in capital V any vector in the vector space, there exist unique scalars a 1 to a n, such that v is equal to a 1 v 1 plus a 2 v 2 plus dot dot dot a n v n. So, every vector can be uniquely written as a linear combination of the vectors v 1, v 2 up to v n. Let us give a quick proof of this theorem.

So, as I was mentioning, this is quite remarkable. So, given any vector, we have a 1, a 2 up to a n uniquely determined in terms of v 1, v 2 up to v n, where a1, a 2 up to a n are elements in R, so they are scalars. All right, so let us let v be in capital V, then since B is a basis, it is a spanning set. So they are certainly exists one linear combination of v 1, v 2 up to v n which is equal to, since B is a spanning set there exists a 1 to a n, in R such that v is equal to a 1 v 1 plus up to a n v n.

So certainly, the existence part has been ensured by this spanning property of v. The only question is, uniqueness. So, we will show that if we have any other linear combination which of v 1, v 2 up to v n which is equal to v, it should necessarily be the same as the linear combination in terms of a 1, a 2 up to a n. So, to do that, let us start with another linear combination, suppose v is equal to b 1 v 1 plus dot dot dot plus b n v n, be any linear combination of v 1 to v n equal to v, what does that mean? That means that we already have one linear combination here.

V is already equal to a 1 v 1 plus a 2 v 2 up to a n v n. Therefore, then a 1 v 1 plus a 2 v 2 plus dot dot dot a n v n, which is equal to v is also equal to b 1 v 1 plus b 2 v 2 plus dot dot dot plus b n v and, but we are in a vector space, and our vector addition and scalar multiplication satisfy all the properties 1 to 8 in the definition of our vector space. Using many of those properties, I will leave it as an exercise for you to check that this implies a 1 minus b 1 times v 1 plus a 2 minus b 2 times v 2 plus up to a n minus b n times v n is equal to be 0 vector in v, this is a linear combination of v 1 v 2 up to v n, which is equal to the 0 vector.

But B, recall is linearly independent, and the linear independence, why is it linearly independent? Because by definition B is a basis and basis is linearly independent and linear independence forces each of the coefficients to be equal to 0, because if even one of them is not equal to 0, then that is a linear dependence of v 1, v 2 up to v n. Therefore, hence, a 1 minus b 1 is equal to 0, 0 scalar, a 2 minus b 2 is the number 0 and so on, a n minus b n is equal to 0. This implies a i is equal to v i where all i equal to 1 to n.

Therefore, the expression, therefore, the linear there is a linear unique linear combination of B in terms of v 1, v 2 up to v n because you take any other linear combination equal to v, we have forced or we have shown that it is forced to be the same as the above. So, hence every vector in capital V has a unique that is what we have just shown, linear combination linear or let me just write it as can be written as a unique linear combination, can be written as an unique linear combination in B, therefore we have proved (())(16:52).

We are seeing a couple of examples, let us now look at more examples. Let us look at a more complicated example.

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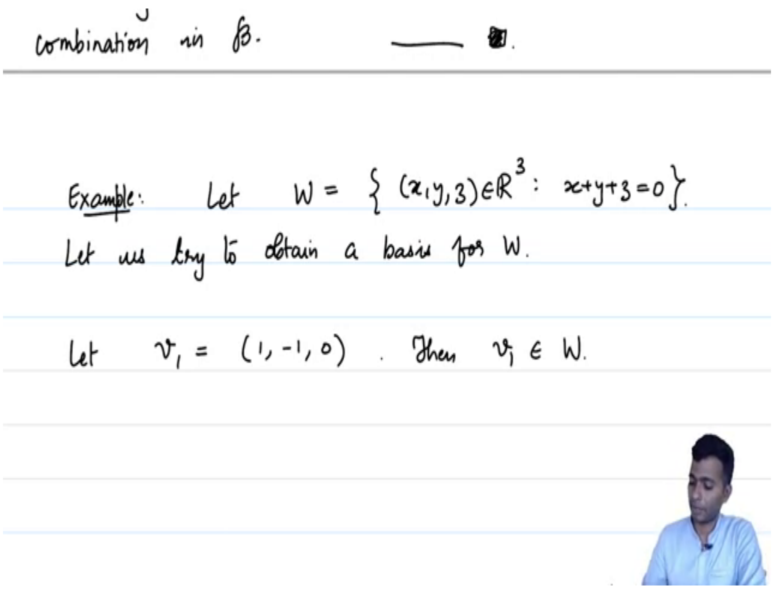

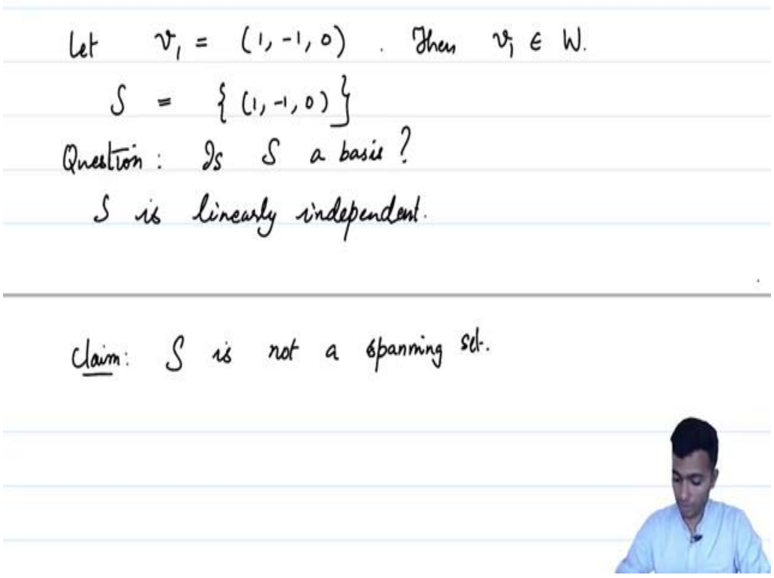

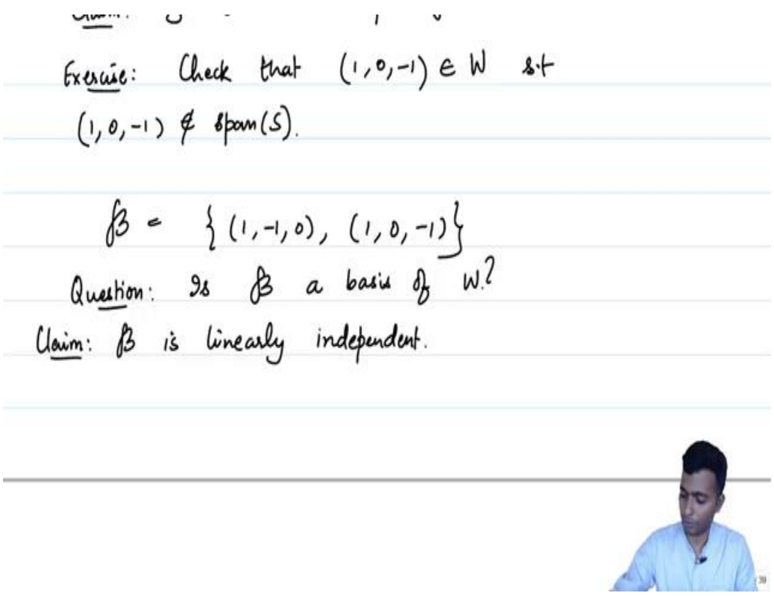

So, consider the example of a subspace you were considering. And we introduced a vector subspaces, which was namely let W be the set of all x, y, z in R 3 such that x plus y plus z is equal to 0. So, let us try to get hold of a basis for this subspace W. So, let us try to obtain a basis for W. So, should start somewhere, let us pick some vector in W, you notice what are some arbitrary vector 7, 6, minus 13 or 20, 30, minus 50. These are all vectors in R 3 which are in W. The sum of the coordinates should be equal to 0. Let us not make it complicated.

Let us start with v 1 to be the set to the vector v 1, minus 1, 0 1 minus 1 is 0 and then added to 0 it will be back 0, then v 1 belongs to our W, it is an element in our subspace W of R 3. So, is it a basis, so it is you just consider B to be equal to just the set consisting of 1 minus 1, 0, let me not call it B because we do not know what let say let S be this, this particular vector, question is S a basis? Well, it is set consisting of one nonzero vector, S is certainly linearly independent, S is linearly independent.

Now, the question that comes up next is, is it a spanning set? So, claim, S is not a spanning set. In fact, I will explicitly give you an, a vector in W, which is not in the span of S. So, let me just leave it as an exercise for you to check that our limit vector was 1 minus 1, 0. So check that 0, 0, minus 1, sorry, that is not in W, 1, 0, minus 1 that is not, is an element in capital W such that 1, 0, minus 1 does not belong to the span of S.

And therefore, S is not a spanning set, but let us not lose hope. We obtained one vector 1, 0, minus 1, which is not in the span of the previous set. So, let us now do one thing, let us append this vector and call our S to be, maybe S is already used, let me call it B, I am writing B, so you should get the clue here. Let us just put the two vectors and ask the same question. So, question is B a basis of W? So I will actually leave it as an exercise for you to check that if you have a linear combination equal to 0, then a plus b 0, let me just show it. B claim make a claim and prove it, B is linearly independent.

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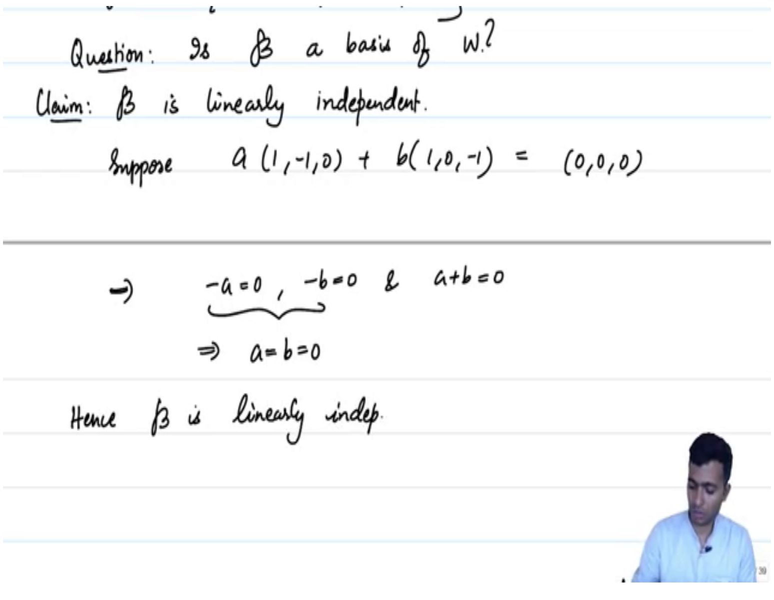

The easier claim actually to show that it is a basis, you have to check both, let us first check that it is linearly independent, suppose A times 1, minus 1, 0 plus B times 1, 0, minus 1 I guess these were the two vectors, yes, is equal to the 0 vector. This would immediately tell us that minus a is equal to 0, minus b is equal to 0 and a plus b is equal to 0. The first two already tells us that a is equal to b is equal to 0, and it is consistent with the third and hence yes, this forces a and b to be 0, which gives that B is linearly independent.

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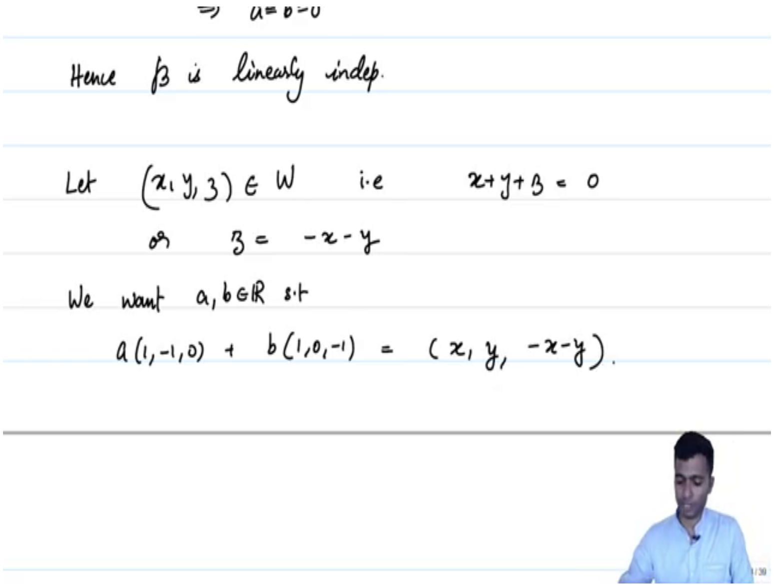

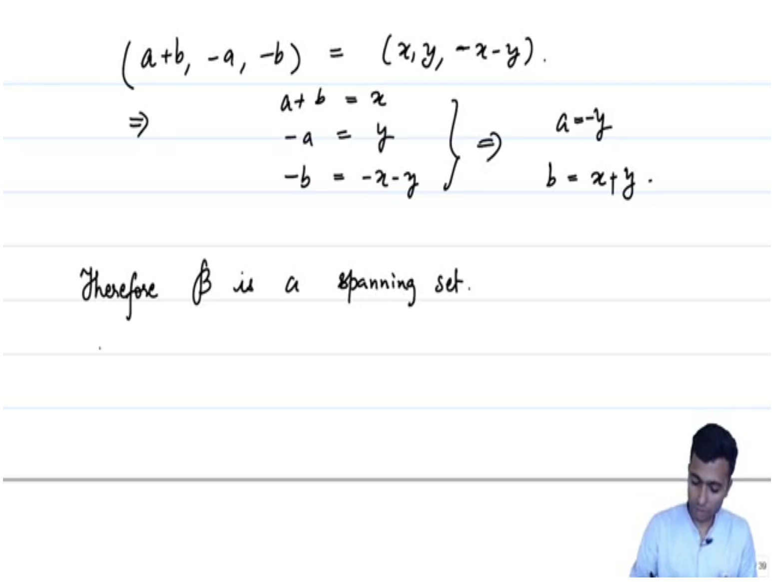

How about, how about some, how about the spanning property? So, let us take some arbitrary vector let x y, z be in capital W, but then what does that mean? i.e x plus y plus z is equal to 0 or let us write z to be equal to minus of x minus of y. So, suppose, so to check that B is spanning set, we need to get hold of some A and B such that, so we want scalars a and b such that a times 1, minus 1, 0 plus b times 1, 0, minus 1 is equal to x, y, minus x, minus y, just replacing the z by minus x, minus y because we know that z is minus x minus y.

But this is nothing but a plus b, minus a, minus b is equal to x, y, and minus x minus y, maybe I should have picked a bit more carefully. This implies a plus b is equal to x minus a is equal to y, minus b is equal to minus x minus y. Therefore, a is equal to y and b is equal to x minus y, a is equal to minus y and therefore, b is equal to x plus why, and therefore, yes we have a spanning property here. So, therefore B is a spanning set, therefore a basis and hence a basis.

So, let us carefully look at what we did? What we did here was to get hold of a basis in an algorithmic manner. We picked some vector at random, picked 1 minus 1, 0 and checked whether it was a spanning set, saw that it was not a spanning set, picked an element which is not in the span, appended it and therefore, after appending, we obtained a new set which was both spanning, as well as linearly independent.

If you notice carefully, there was nothing unique about the choices we made, we could have started off with, instead of 1 minus 1, 0 we could have just started off with 2 minus 2, 0 or 2, 3, minus 5, something like that, and done the same procedure and maybe we would have ended up with a basis.

So, this tells us that you know, this construction, of course, basis need not be unique at all and this tells us that the algorithm also need not give you a, the basis that could be many, many, many basis, plural of basis is bases, bases, you could get, you could end up with many, many number of basis through this algorithm. This was, however, done in a very particular subspace of R 3. Let us try to get clue from this example, let us take motivation from this example and try to give a general theorem which works for an arbitrary vector space.

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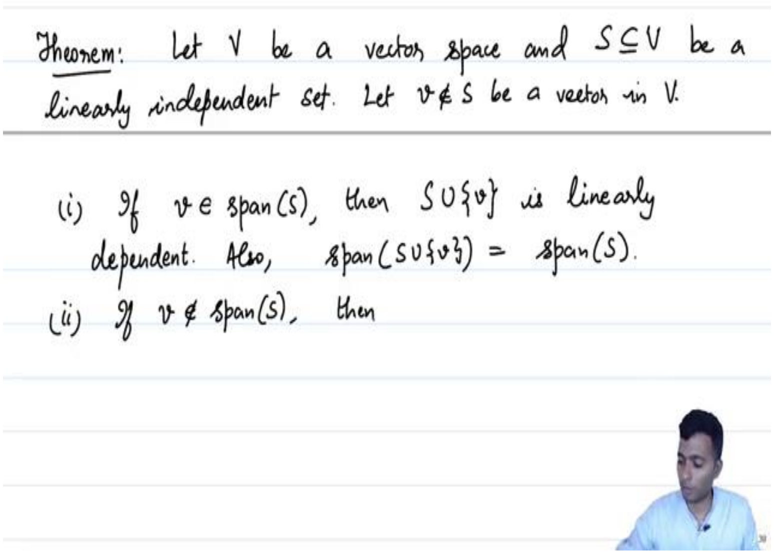

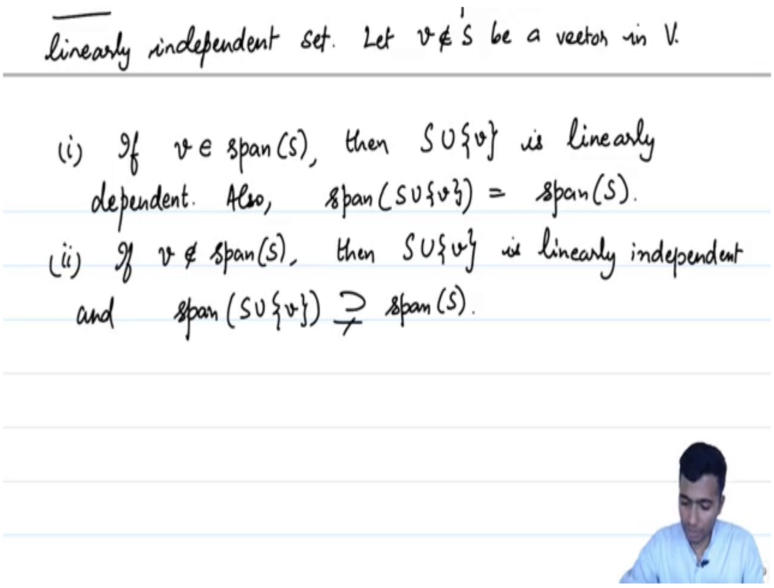

So theorem, so let me V be a vector space and S contained in V be a linearly independent set. Also let us pick some vector v, which is in the vector space capital V, which is not in S. So, let v not in capitalist S be a vector in capital V. Now, two cases can happen, either v is in the span of S, or v is not in the span of S. Let us consider both these cases separately, so, if v is in the span of S, let us see what happens then? Then S union v is linearly dependent, the appending of v to S destroys the linear independence. S union v is linearly dependent. Not just that, appending this vector is of no use in terms of considering the span.

Also, the span of S union v is equal to the span of this, that is what happens when v is the span of S. How about if v is not in the span of S. Then, as already noted in the example of the subspace that we were saw subspace of R 3 that we were considering, if v was not in the span, if you appended it, if you append it, in that case the linear independence was maintained, that is going to happen in any abstract vector space, then S union v is linearly independent.

Not just linearly independent, also span is strictly super space or the span of S is a strict subspace of span of S union v contains span of S. So, let us spend a couple of minutes trying to see the few implications. The first one we have already noted, that if you remove one element from the basis, then the whatever remains, it loses this spanning property. So, in sometimes the basis is the smallest set, smallest number of vectors which will be spanning on a given vector space.

Look at what is being set by one. If you start with a basis, that is already a spanning set, if you take any vector v, which is not in our basis, then it is obviously in the span of the basis because the basis is a spanning set, the given businesses is a spanning set. If you append this vector to our given basis, then one tells us that linear independence is lost. So, appending a vector destroys the property of v being basis, so basis in some sense is that optimal set which is linearly independent at the same time, a spanning set.

You remove one of them, linear spanning property is lost, if you add one of them, the linear independence is lost. So, one is quite deep in that aspect, the statement one and what about two? Two is telling us that if you do not have a basis already, let us not lose hope, pick some element which is not in the pick some vector which is not in the span append it, we get a bigger set which has a bigger span and which is again linearly independent. We have used two, already in the example prior to this, wherein we obtained a basis in an algorithmic manner.

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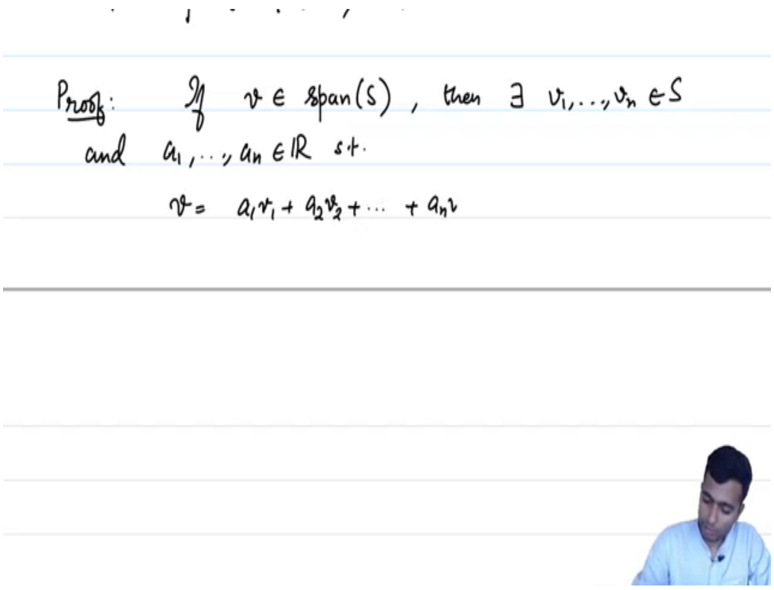

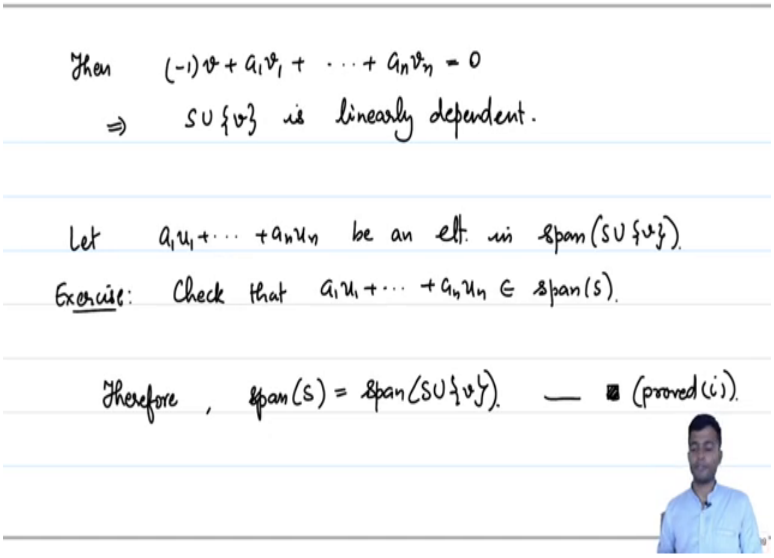

Alright, so let us prove both one and two. So, let us first prove one. So, if v belongs to span of S. What does that mean? That means that, then there exists, v 1 to v n and a 1 to, so the v 1 to v n in capital S and a 1 to a n in R, such that v is equal to a 1 v 1 plus a 2, B 2 plus dot dot dot a n v n. So, then by adding the additive inverse of minus v, we can write minus 1 times v plus a 1, v 1 plus a 2, v 2 up to a n v n is the 0 vector. That implies that S, you notice that v is not one of v 1, v 2 up to v n because v is not an element of S, and therefore S union v is linearly dependent.

That is what we had set out to prove, if you notice. You also have to show that S union v has the same span as S. So, let we have already done a similar argument. So, let us take some arbitrary element in the span of S union v, so let a 1 u 1 plus a n u n be an element in the span of S union v and by a very similar argument as earlier, let me give it as an exercise for you to check that a 1 u 1 plus up to a n u n is in the span of S. We have already done a very similar argument earlier. And therefore, we get that span of S union v is contained in span of S, but S is contained in S union v that was span of S is always contained in the span of S union v.

Therefore, if the with both the implications, span of S is equal to span of S union v and with that we have settled one.

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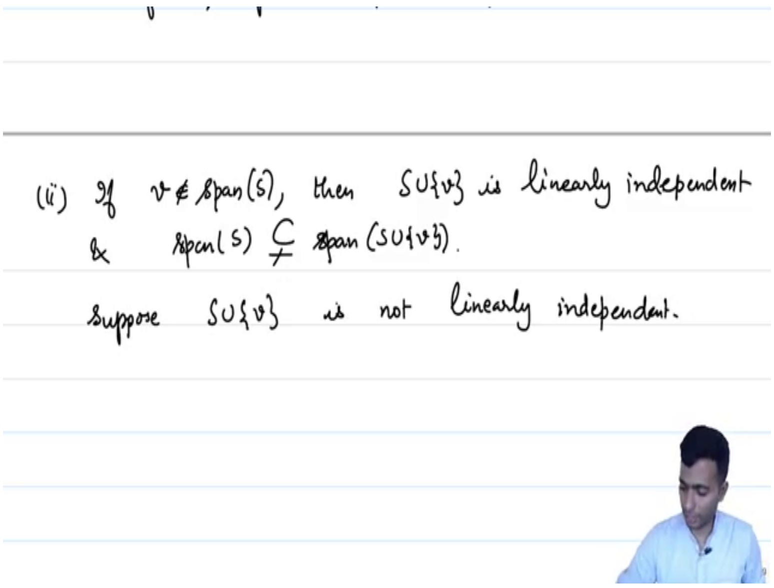

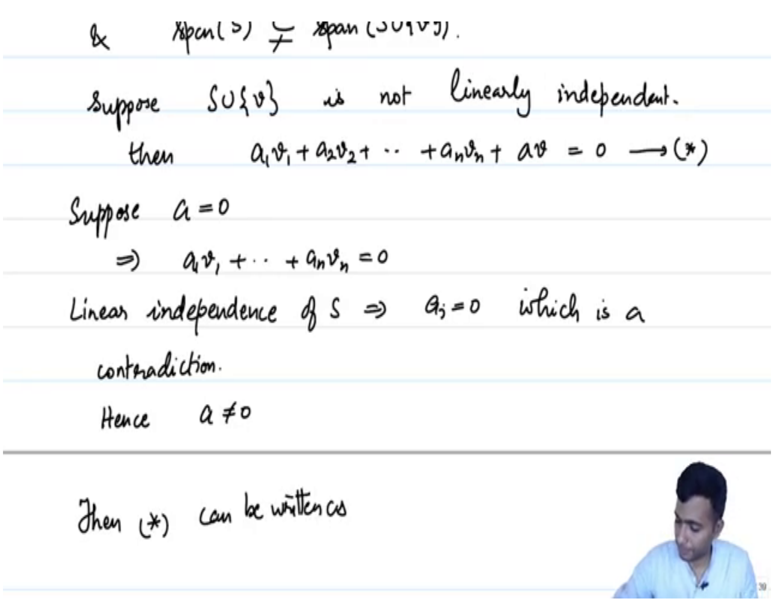

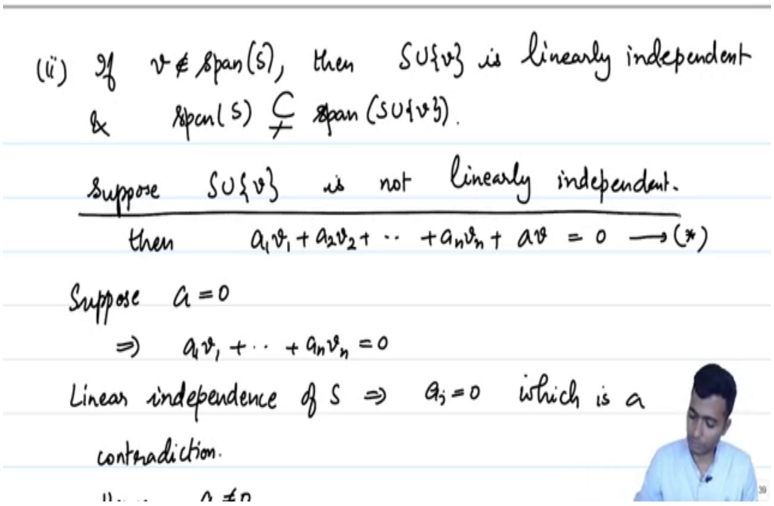

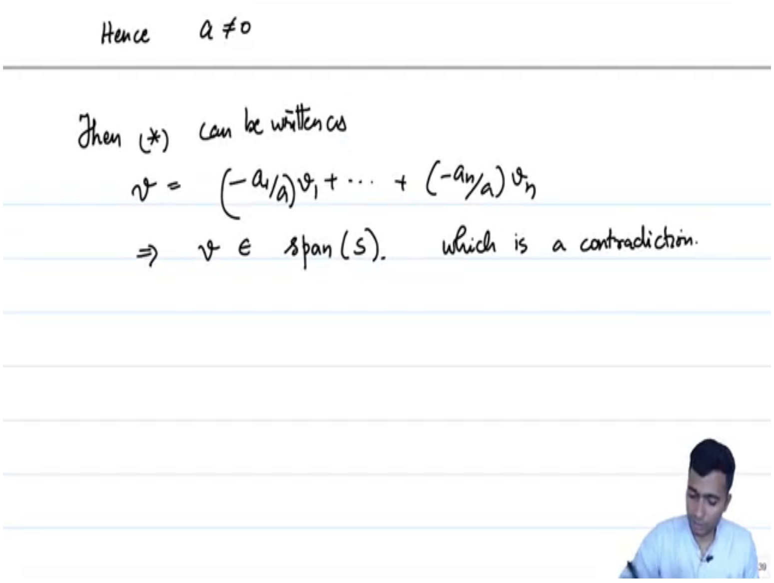

We are now left with proving second statement, if v is not in span of S then S union V is linearly independent. So, so let us let me just write it down. If v is not in span of S, we want then S union v is linearly independent and span of S is a strict subspace of span of S union v. Let us prove that, let us prove the first statement first, that S union v is linearly independent, if v is not in span of S.

So suppose it is not linearly independent, suppose S union v is not linearly independent, what does it mean to say that something is not linearly independent? Then a 1 v 1 plus a 2 v 2 plus a n v n plus a times v is equal to the 0 vector. That is what it means to say that where v 1, v 2 up to v n are not the vector v, they are different vectors.

There is some linear combination which is equal to 0, with not all a i's or and a together equal to 0. So, suppose a is equal to 0, let us look at what happens? This would imply that a times v is 0 and therefore, a 1 v 1 plus a 2 v 2 up to a n v n is equal to 0. But then, v 1, v 2 up to v n are vectors in capital S which is linearly independent, linear independence of S implies that a i is equal to 0, but that is a contradiction to our assumption that it is union v is not linearly independent. This actually implies that S union v is linearly independent, that is not the case, a cannot be zero.

But this is a let me just say that this is a, which is a contradiction. So, this contradiction has come up because we assumed a is equal to 0, hence a is not equal to 0 but if a is not equal to 0, then again, what is our assumption? v is not span of S. Then let us just write down, rewrite this equation star. Then star can be written as, v is equal to minus of a 1 by a times v 1 plus dot dot dot minus of a n by a times v n but that implies that v is in the span of S because v 1, v 2 up to v n is a a set of elements, vectors in capital S, which is again a contradiction, because we had assumed to begin with contradiction, we had assumed to begin with that v does not belong to span of S

So, something has to be wrong, so that means this assumption has to be false.

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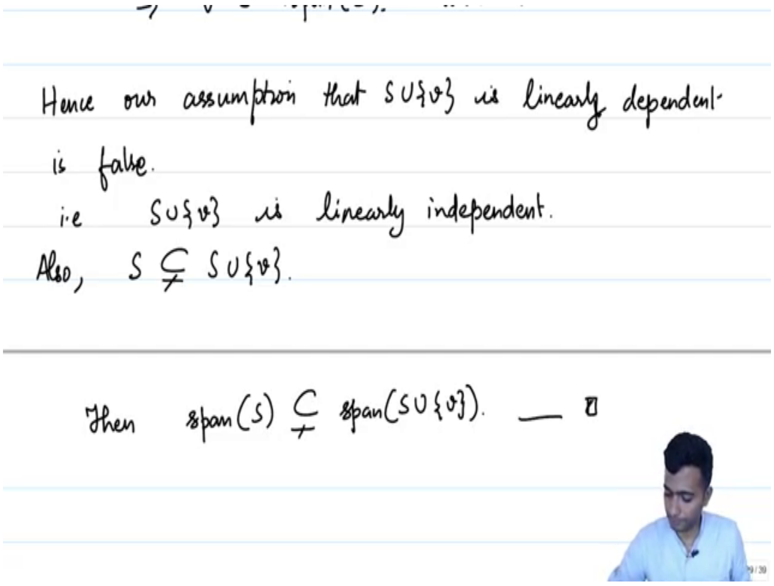

Therefore, hence our assumption that S union v is linearly dependent or not linearly independent is false because if we assume this, we are arriving at some contradiction i.e. S union v is linearly independent, false. That is good because what do we know about the subset of linearly independent sets? S is clearly a subset of a strict subset of S union v. Also, we know this. Why? Because we assume that v is not even in this span of S, not just S, it is not in the span of S.

Therefore, S union v is a strict super set of capital S, which is and we just proved that S union v is linearly independent. If you knock off one element from a linearly independent set, we have shown that the span by the theorem from the previous session span of S is a strict subset of span of S union v. That completes our proof.