Linear Algebra
Professor Pranav Haridas
Kerala School of Mathematics, Kozhikode
Lecture 5
Linear Independence

So, in the last week, we defined what are vector spaces. We saw many examples of vector spaces. And then we defined what is meant by a subspace of a vector space, and then there after given subset S of v, we talked about what is meant by a linear combination in S and what is the meaning of span of S. In this week we begin by discussing what is meant by linear dependence and linear independence.

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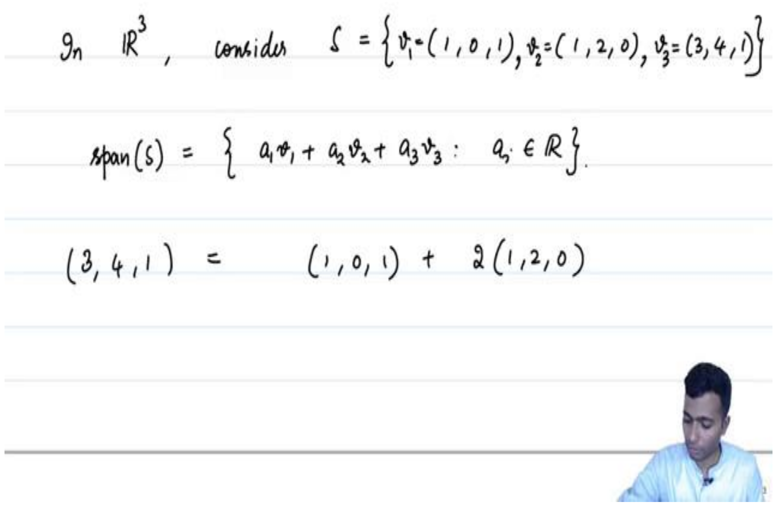

So in order to do that, let us start by looking into an example. So, in R3 consider the set S given by say 1, 0, 1 1, 2, 0 and then maybe 3, 4, 1. This is a subset of R3 and let us try to see what is the span of this set S. So, if you recall the span of S is the by definition, we have already defined it. This is the collection of all vectors of the type A, B, okay, so let me give a few names to this, this let us call it v1, let us call this v2 and let us call this v3.

So, span of S, is just going to be a1 v1 plus a2 v2 plus a3 v3, where ai belongs to scalars, the field of scalars or R is real numbers, but let us carefully look at the set s. The set S v1, v2, v3 are in some sense satisfying some relation, if you if I am to show it to you if you carefully observe this is just the following 3, 4, 1 is maybe if I am writing it wrong I will correct it. This is just 2 times 1, 2, 0 2 plus 1 is three, 2 times 2 plus 0 is 4, and 0 plus 1 is 1. Yes, exactly.

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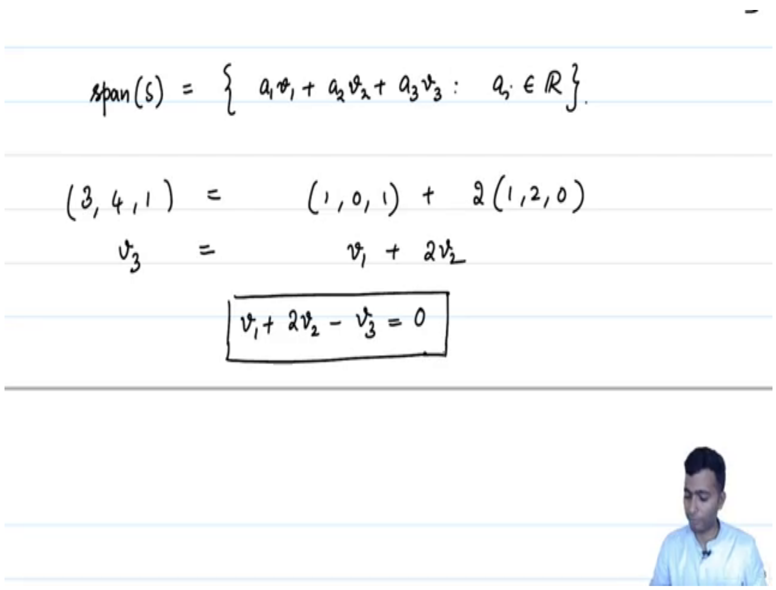

So, if we take some arbitrary element and span of s. And so this is basically v3 this is how the expression is, right? This is v1 plus 2 times v2 or let me put it in a box, v1 plus 2 v2 plus minus one times or minus v3 is equal to the 0 vector, 0, 0, 0, so let me put it in a box. So, this is exactly what our v1, v2, v3 is satisfying, and what is the implication of v1, v2, v3 satisfying such an identity.

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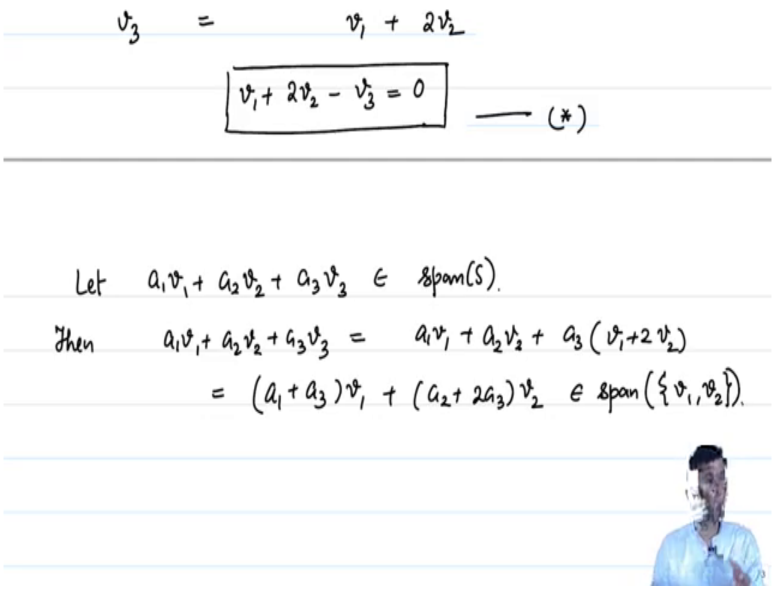

What happens is you take any element in span of S, so let a1 v1 plus a2 v2 plus a3 v3 be in span of S, we immediately observe that using this star, which I just noted, namely v3 being equal to v1 plus 2 v2, we will be able to write, then a1 v1 plus a2 v2 plus a3 v3 is equal to a1 v1 plus a2 v2 plus a3 times v1 plus 2 v2 by substituting for v3, and this is nothing but a1 plus a3 times v1 plus a2 plus 2 a3 times v2, which is an element of the span of the set v1, v2.

So, if we were to study the span of S here, we do not really have to study the span of the entire set S.

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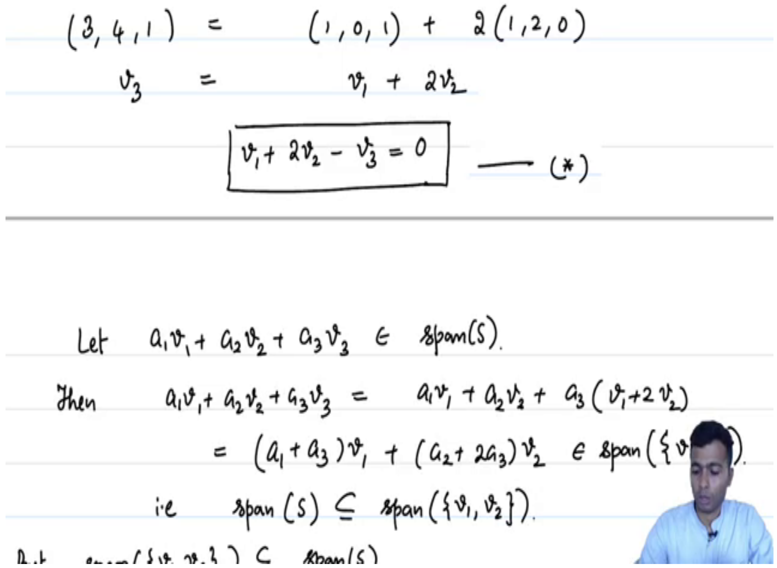

We just have to look at the span of v1, v2, so effectively we have shown that span of S is contained in span of v1, v2, but any venial combination of v1, v2 in particular is a linear combination of v1, v2, v3 with you know, the coefficient of v3 means 0 or some other linear combination, does not matter.

The point is that span of v1, v2 is always contained in span of S, because it is a bigger set. And therefore, what we have is that span of S for looking at the span of S, we have to only look at the span of v1, v2, but what does that mean? This means that to study the span of S, an element of S namely v3 is totally redundant.

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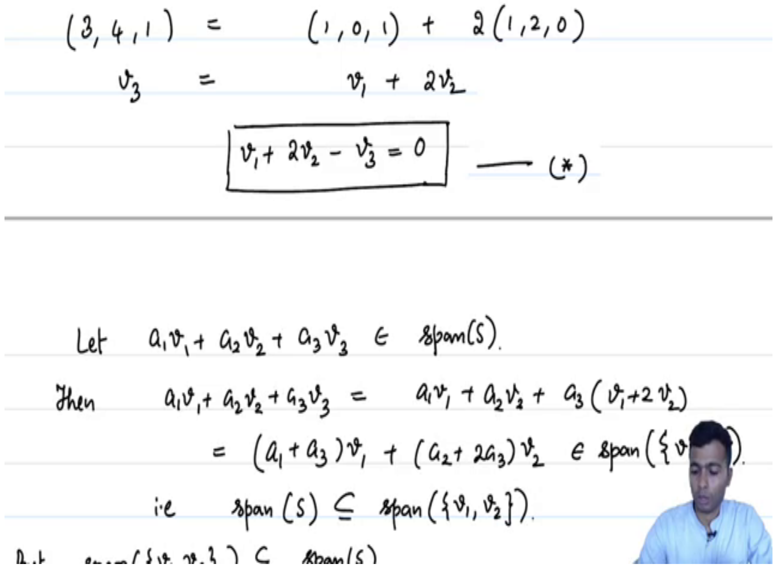

And that is because of what is happening in this box. v1 plus v2, v2 minus v3 is equal to 0. This is what is called as, this is what is referred to as a Linear dependent of v1, v2, v3.

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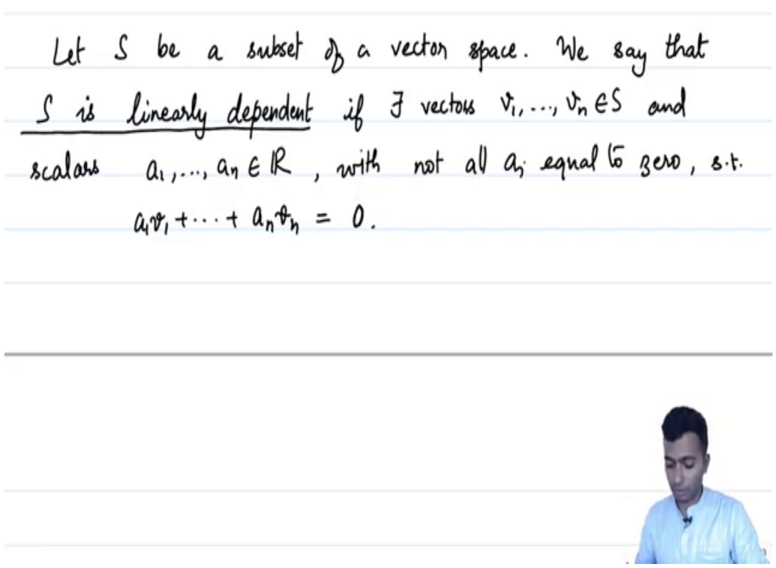

So, let us capture that in a definition, linear dependence. So, let us give the definition in a more general setting. So, let S be some collection of vectors in a vector space v. So, let S be a subset or a collection of vectors of a vector space v. We say that S, is linearly dependent, so let me just underline, this is linearly dependent, if there exists vectors v1 to vn in S and scalars a1 up to an in R, so R scalars here is real numbers with not all ai equal to 0.

So, that means that at least one of the ai's will be non-zero, such that, we put a1 v1 the linear combination of v1 to vn with respect to with coefficients ai's. This gives as the 0 vector. So, notice that 0 is always linear combination of finite set of vectors.

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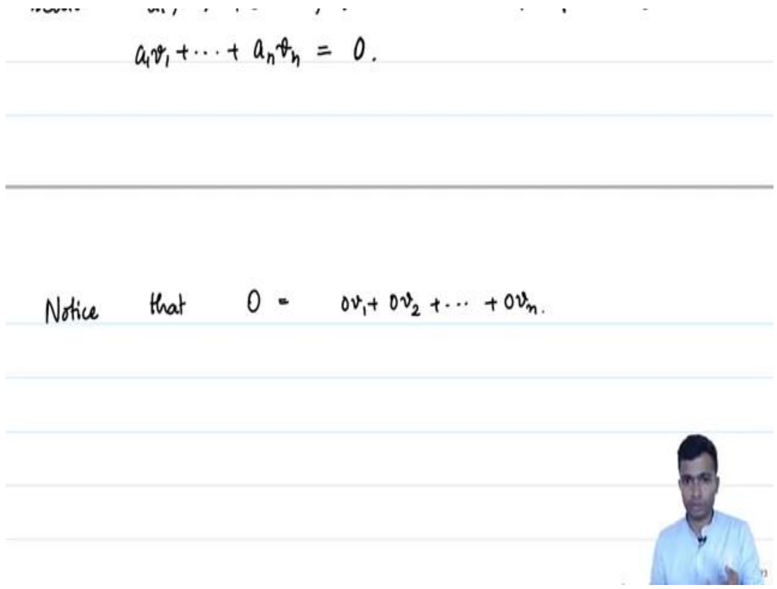

So notice that 0 is always equal to 0 times v1 plus 0 times v2 plus dot dot dot0 times vn for any v1, v2 up to vn because the scalar 0 or the number 0 times any vector is 0.

And if you add the 0 vector, it will give you back 0. Whatever definition demands is that we can write 0 as a linear combination of v1, v2 up to vn in a different manner than writing it as a linear combination with coefficients 0's or not all of ai's should be 0, at least one of the ai's should be non-zero and a1 v1 plus a2 v2 up to an vn should be equal to 0. Then we say that the set s or v1, v2 up to vn are linearly dependent or more generally, the set s is linearly dependent.

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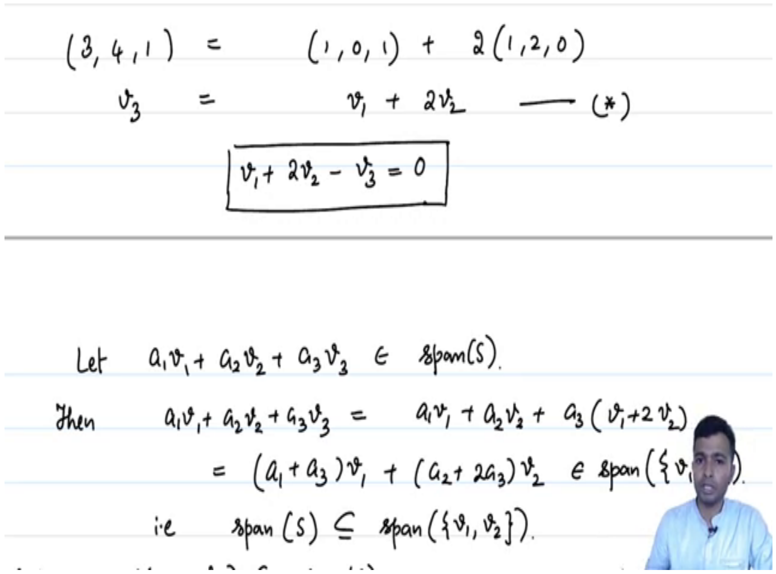

So if you notice carefully, the box here effectively told us that v1, v2 and v3 are linearly dependent. So, when S is finite which many times will be our case.

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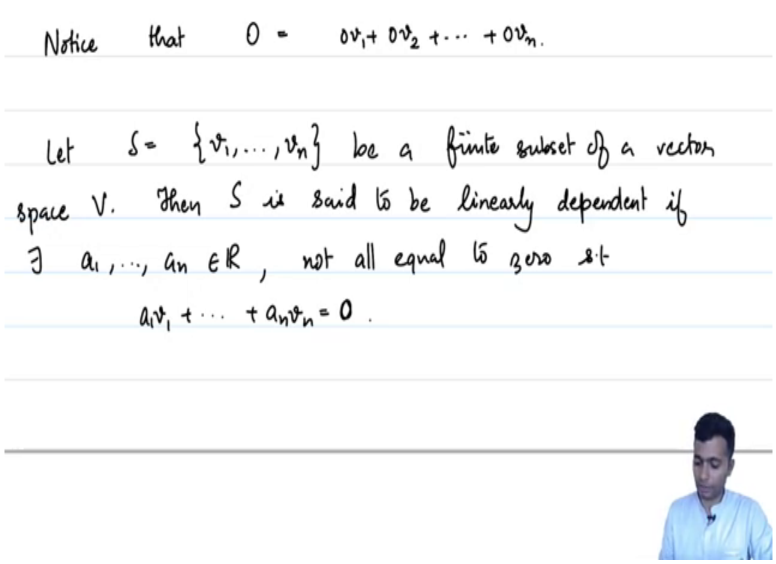

So let S be equal to v1 to vn be a finite subset of v of a vector space v, then S is said to be linearly dependent if there exists a1 to an in the field of scalars, not all 0, not all equal to 0 such that a1 v1 plus an vn is the 0 vector.

So, if you notice, it is not very different from, there is a subtle difference, we just need in the first definition, some sub collection v1 to vn, such that their linear combination is equal to 0. Here we are just demanding that all of the v1, v2, vn up to v1, v2 up to vn get involved in linear combination. It is a subtle difference, but it is the same just by introducing all the vectors if needed, by putting the coefficient as 0 we get back the other definition.

So, this is a useful definition to put it on record, because many times we deal with a finite set and we are asked whether the finite set is linearly dependent. And yes, this will tell us that we do not need to worry about whether there is a subset or sub collection, any linear combination involve in all the vectors need to be considered, that is all. So, we have defined what is meant by linear dependence.

Linear independence is just the negation of this concept. Set S is said to be linearly independent if it is not linearly dependent.

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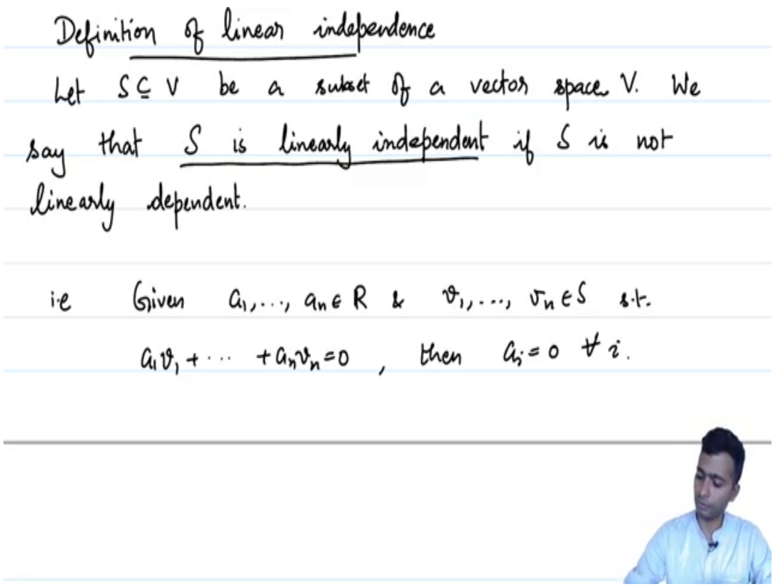

So, definition of linear independence, so let S contained in v be a subset of a vector space v, so notice that we have not demanded that S be finite, we have not at all demanded that. So, of a vector space v, we say that, S is linearly independent if S is not linearly dependent.

So, if you carefully notice, this can be also written as in the following manner i.e. given a1 to an in R and v1 to vn in S, such that a1 v1 plus a2 v2 plus an vn is equal to 0, then ai is necessarily equal to 0 for all i . That is precisely what it means to say that it is not linearly dependent. This, not all 0 would mean that it is linearly dependent. So, this is the definition of linear independence.

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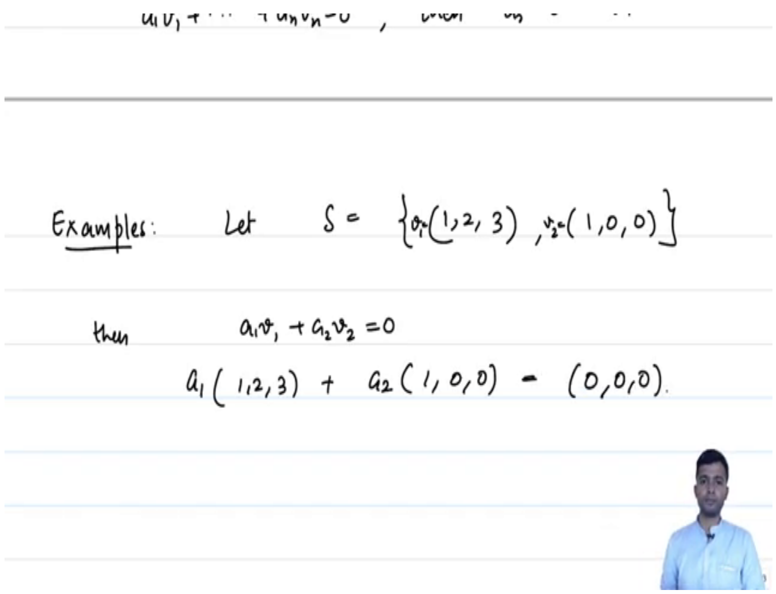

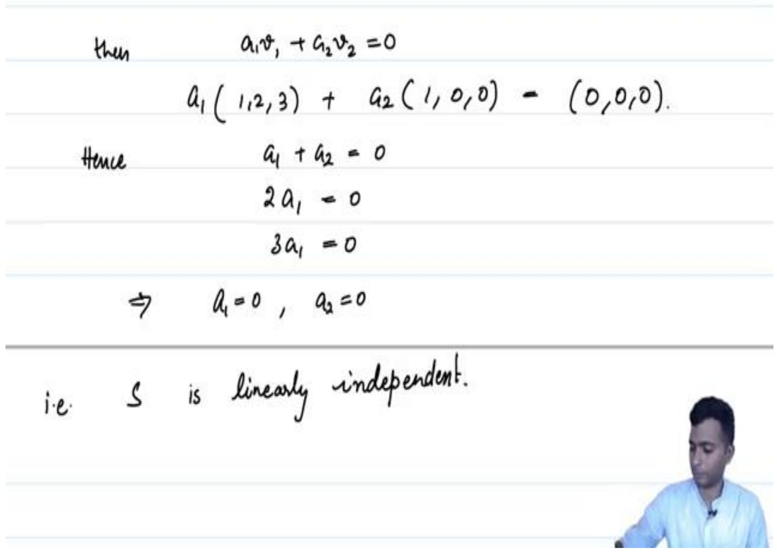

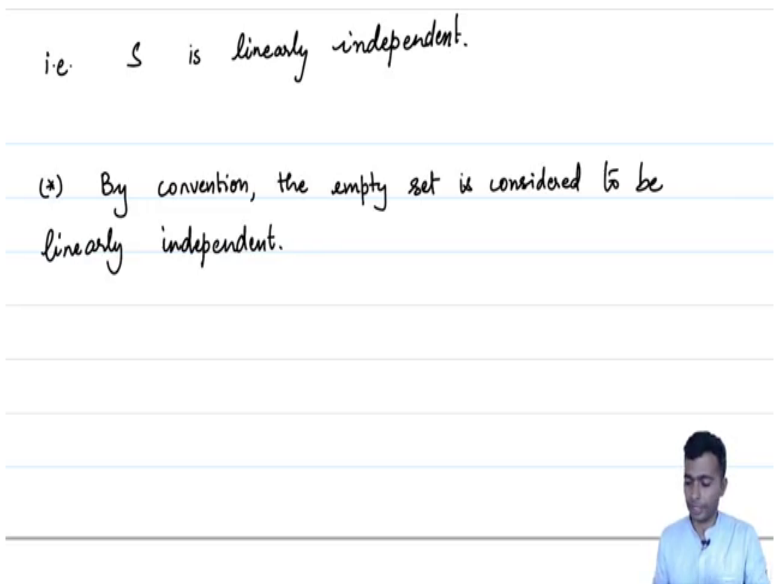

So, let us look at a few examples, so let us consider the following set, let S be equal to 1, 2, 3 and maybe 1, 0, 0. Then let us see if S is linearly dependent or independent. So, here there is a finite set, so consider any linear combination of our vectors v1 and v2, then a1 v1 plus a2 v2 is equal to 0, the 0 vector can be rewritten in the following manner, so this will be a1 times 1, 2, 3 plus a2 times 1, 0, 0 is equal to the 0 vector which is 0, 0, 0.

So, as is clear from the context the right hand side, the 0 vector, the 0 is the 0 vector in R3, which is 0, 0, 0. And this implies, hence we have a1 plus a2 is equal to 0 a2 times a1 is equal to 0 and 3 times a1 is equal to 0. The last two equations imply that a1 is equal to 0, therefore both a1 and with the in conjunction with the first equation, both a1 and a2 is equal to 0. This implies i.e. S is linearly independent.

By convention, the empty set is considered to be linearly independent, let me just note that, by convention, the empty set is considered to be linearly independent. Also, any set which contains the 0 vector will not be linearly independent, it will be linearly dependent.

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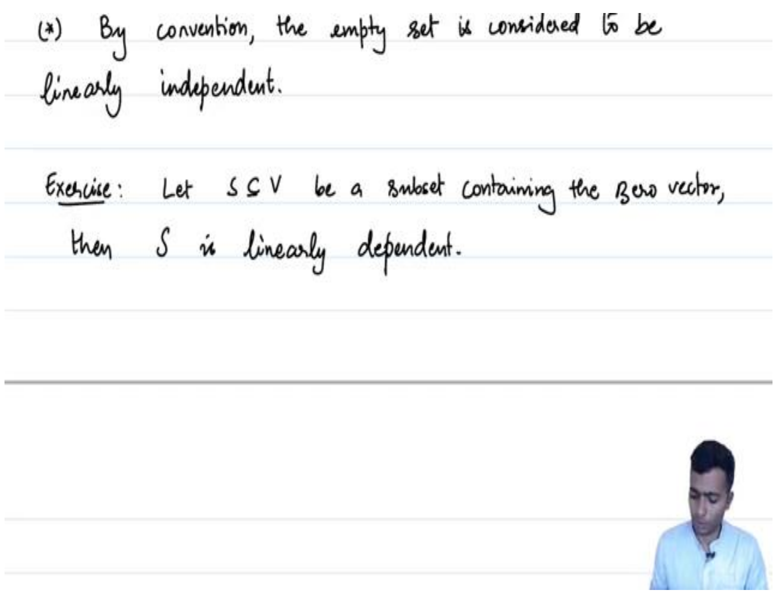

So, maybe I will give it to you as an exercise to check. Let S contained in v be a subset containing the 0 vector, then is S is linearly independent.

So, maybe another exercise might be to check that if S is linearly independent, a subset of S will also be linearly independent.

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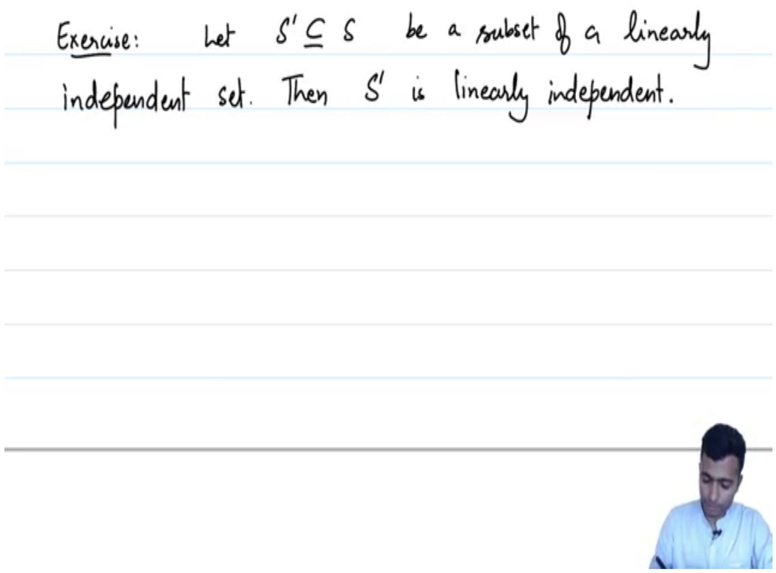

Exercise, let S prime contained in S be a proper subset, does not matter actually proper or not be a subset of a linearly independent set, then S prime is linearly independent. Notice that, we cannot make such a similar statement for linearly dependent sets, it could happen that the subset of a linearly dependent set is linearly independent.

So, maybe we should think about getting hold of an example. Maybe we should think about it now before you listen to the next exercise.

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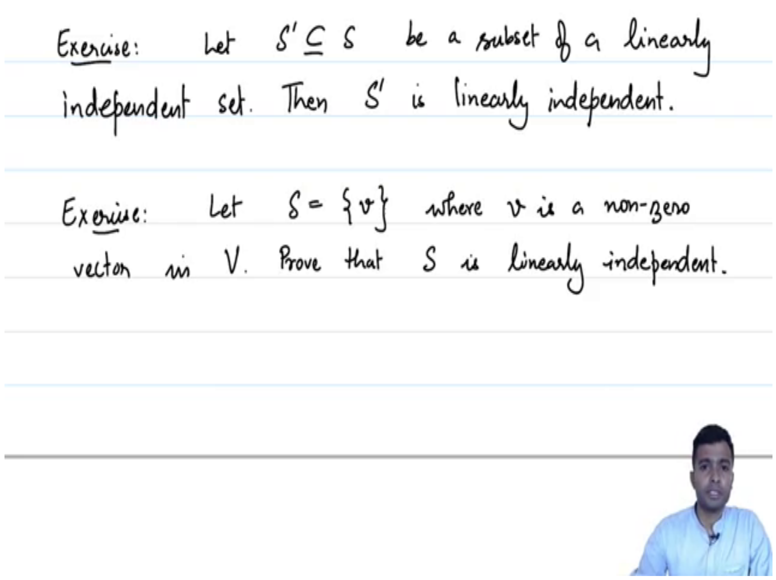

So, let S be a singleton, where v is a non-zero vector in v, a vector space v. Prove that S is linearly independent. So, let us just revisit the example we started of with.

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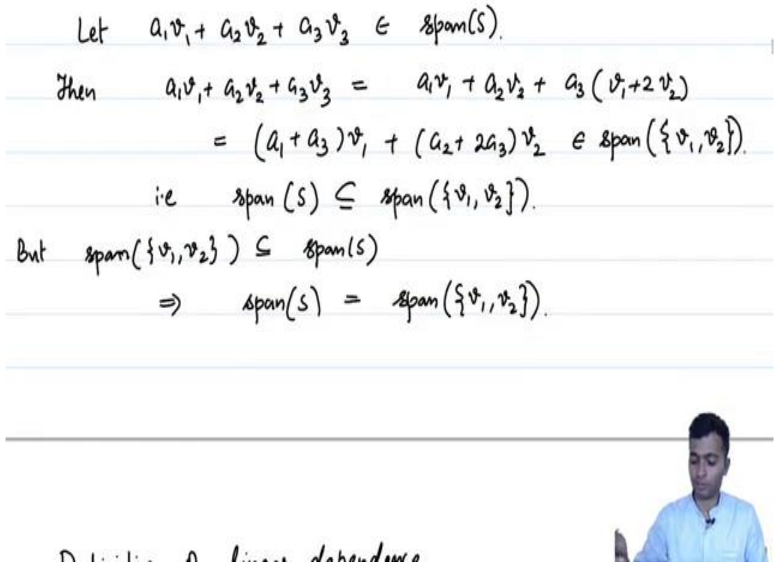

If you notice, linear dependence had some real impact on the span of the set. What it effectively told us is that if you throw out v3 from S, the span remains unchanged.

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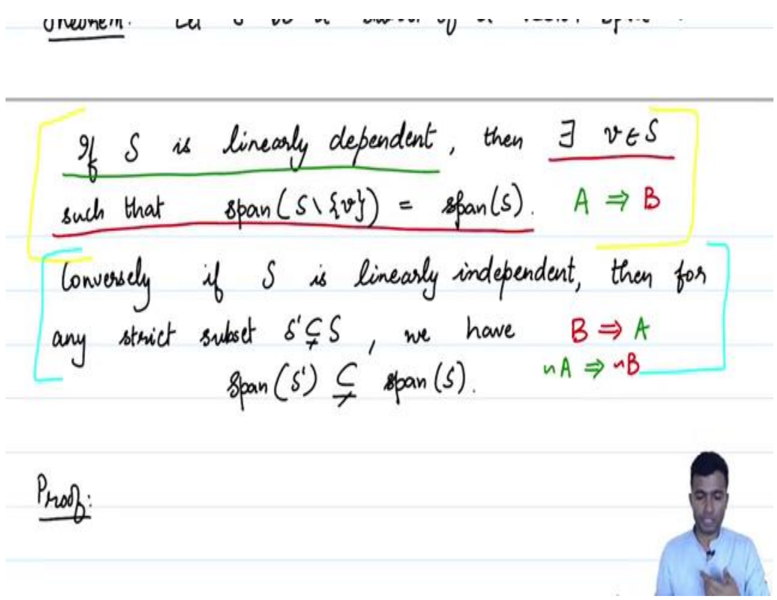

So, let us make a general theorem or let us state a general theorem which will probably capture this idea very precisely. So, next is a theorem, so let S be a subset of a vector space v,then if S is linearly dependent because that in that example, that is the case we were in, v1, v2, v3 were there and v3, v1, v2, v3 had a linear combination equal to 0, non zero linear combination which was equal to non-trivial linear combination which was equal to 0.

So, if s is linearly dependent the conclusion there was then there exists a v in S such that the span of S minus v is equal to the span of S, you throw out one vector from S, the span does not change. That is what this theorem, this part of the theorem says. We will also write a converse. Conversely if S is linearly independent, then for any strict subset S prime contained in S, we have span of S prime is a strict subset of span of S.

Let us maybe spend a couple of minutes trying to look at the theorem. So, there are two parts to this theorem, this is the one in maybe yellow is one statement and the one in blue, which is the converse is the second statement. And to draw your attention here, let me just underline with green, the assumption in the first statement and with red the conclusion in the first statement. So, the converse to this ideally should have been that the thing underlined in red implies the thing underlined in Green.

Or in other words, if there exists some vector v in S, such that span of S minus V is equal to span of S, then S is linearly dependent, that should have been the converse, but if you carefully observe here what the converse we have what the statement of the converse we have written, we have written that, if S is linearly independent, then for any strict subset S prime contained in S, span of S prime is a strict subset of S.

So, it does not really might look like that we are not saying the actual converse, we are trying to really say, but if the thing underlined in green is say A, and what the statement says is that this implies whatever is written down in red. But this is the converse should have been B implies whatever is written down in A. But it is the same as telling that the negation of A implies the negation of B. So, let us see what is the negation of A.

The negation of A tells us that, if S is not linearly dependent, if S is linearly independent, that should mean that should imply that the negation of B happens, that means that does not exist any such vector v such that if you throw it out, the span remains intact, or in other words, if you look at any strict subset, the span will be a strict subset. So, the converse whichever is written here is actually need capturing the converse of the statement written above.

So, having said all that, let me now, maybe I should not drop all these things. Now, let us give a proof of the statement here.

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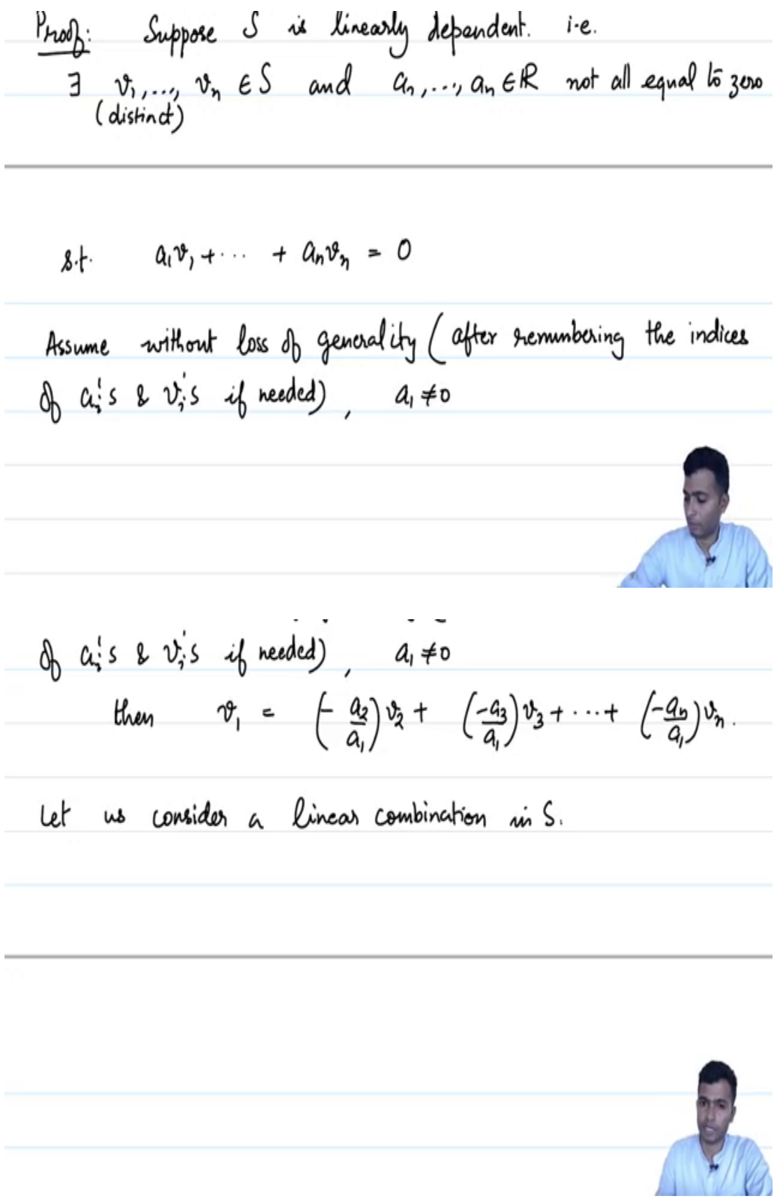

So, a proof, so the first statement here says that if S is linearly dependent, then there exist a vector v which can be thrown out, such that the span of S minus v is the same as span of S. So, suppose S is linearly dependent, so let us give a rigorous proof.

What does it mean to say that the set is linearly dependent, means that there exist finitely many vectors v1, v2 up to vn in S and scalars a1 to an, real numbers not all equal to 0. There is at least one of them not equal to 0. So, let us assume va's are distinct. So, let me assume distinct, third loss of generality that can certainly be assumed, and not all equal to zero such that a1 v1 plus up to an vn is the 0 vector. That is the definition of S being linearly dependent.

So, one of the ai's is not 0, after renumbering of vi's and ai's. Assume the third loss of generality that a1 is not equal to 0, without loss of generality after renumbering ai's, renumbering the indices of ai's of ai's and vi's if needed, it might not be needed at all. We can assume that a1 is not equal to 0. After all, one of them might have been nonzero, suppose aj was not 0, then let us call aj to be a1 and a1 to be aj.

Similarly, v1 to be vj and vj to be v1 after renumbering assume that a1 is not equal to 0. This implies, then by using the various properties involved in the definition of a vector space v1 can be written as minus of a2 by a1 times v2 plus minus of a3 by a1 times v3 plus dot dot dot minus of an by a1 times vn. V1 can be written as a linear combination of v2, v3 upto vn. Now, consider let, let us consider a linear combination in S.

So, what is our goal? Recall that our goal is to show that there exist some vector v such that S minus v has the same span as S. Our candidate for our v is going to be v1. We will show that the span of S minus v1 is equal to the span of S, but to do that, we should take some arbitrary linear combination in S and show that it is in the linear combination of S minus v1. So, let us take some arbitrarily linear combination in S, which will be an element in the span of S.

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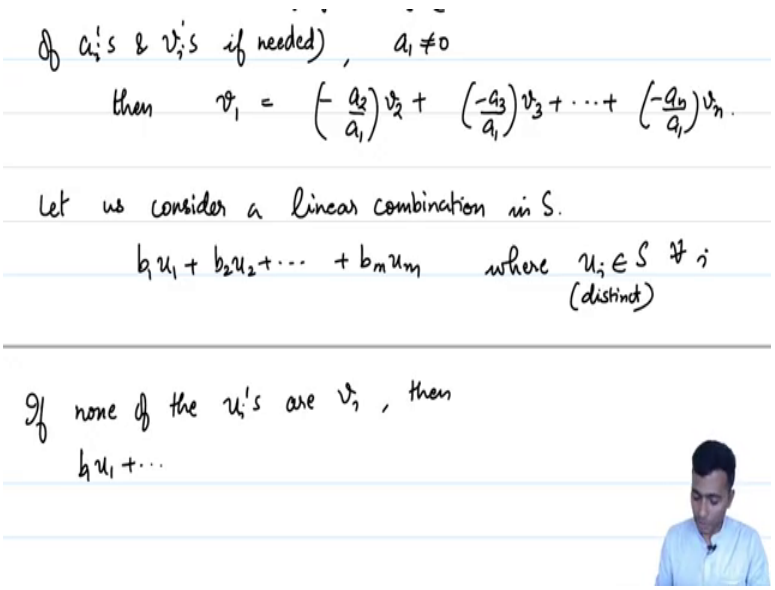

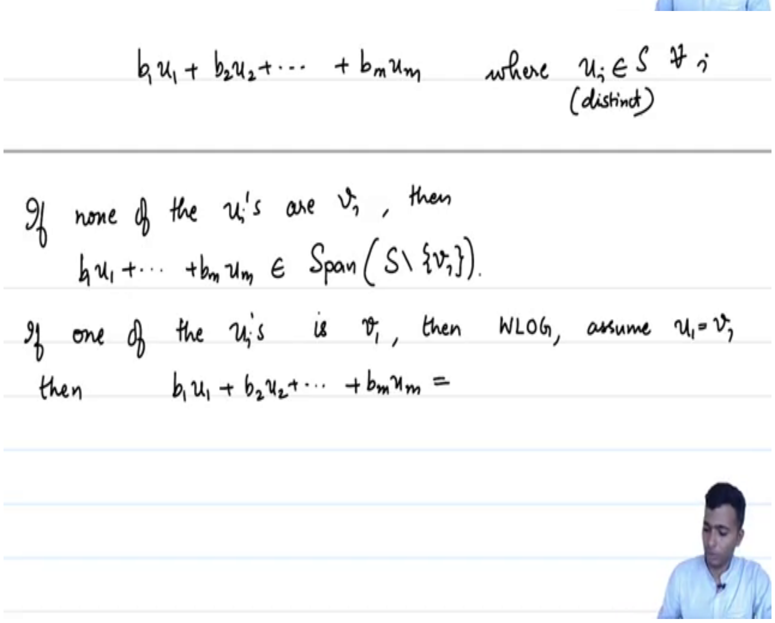

So use the word b1 u1 plus b2 u2 plus up to bm um, where ui's, ui belongs to S for all I, if none of the u1, u2, um is in, is one of, if none of the ui's are v1, let me note it, if none of the ui's are v1, then clearly b1 u1 plus, so let us always assume distinct because otherwise we can just observe and ui's are all distinct. We can observe otherwise into the some involving the other ui and again assume without loss of generality that all of them are distinct.

If none of the ui's are v1, then b1 u1 plus b2 u2 plus up to bm um already belongs to the span of S minus v1. So, the problem comes if one of them is u1, so without loss of the generality again, if one of the ui’s is equal to is v1 then without loss of generality, let me just write down in short, without loss of generality, assume u1 is equal to v1 after renumbering, if needed of the indices, just like in the previous case.

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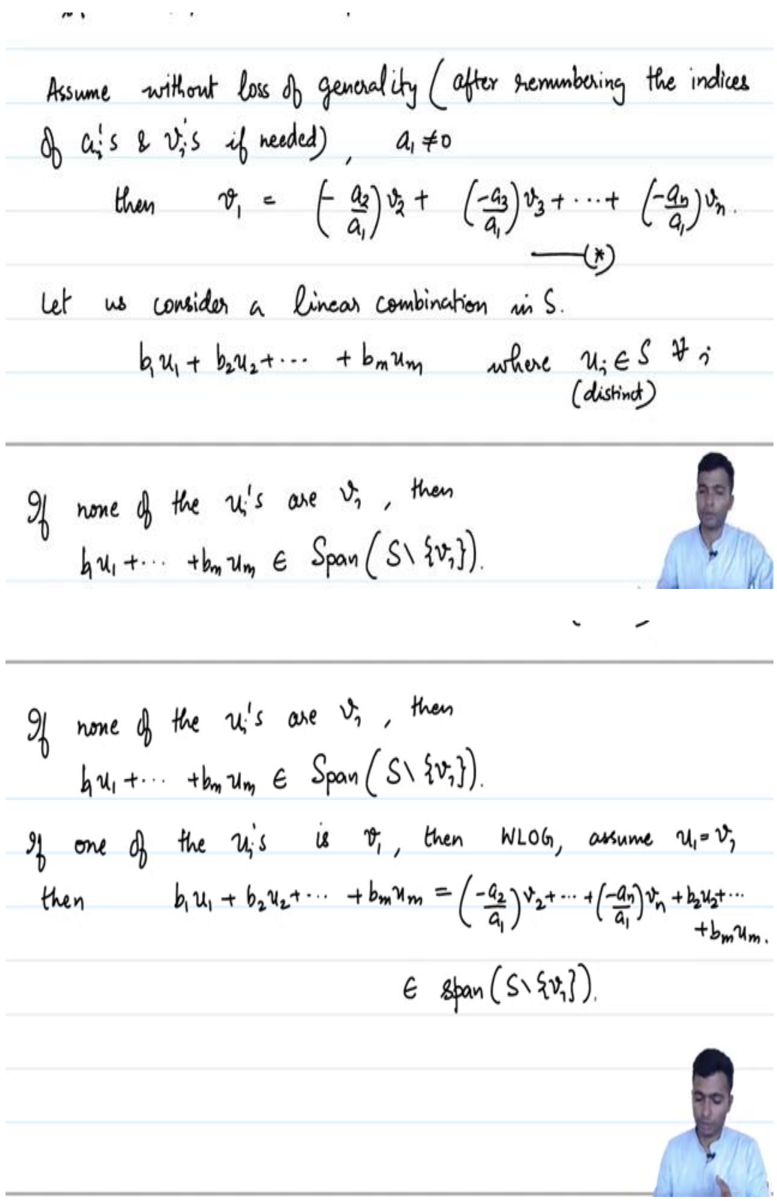

then $b_1u_1 + b_2u_2 + \cdots + b_mu_m = \left(\frac{-a_2}{a_1}\right)v_2 + \cdots + \left(\frac{-a_n}{a_1}\right)v_n + b_2u_2 + \cdots + b_mu_m.$

$\in \text{span}(S \setminus \{v_1\})$.

$\Rightarrow \text{span}(S) \subseteq \text{span}(S \setminus \{v_1\})$.

Since $S \setminus \{v_1\} \subseteq S$, we have $\text{span}(S \setminus \{v_1\}) \subseteq \text{span}(S)$.

Hence $\text{Span}(S \setminus \{v_1\}) = \text{span}(S)$.

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Then b1 u1 plus b2 u2 plus up to bm um is again equal to, let us invoke this equation of star where we wrote v1 as the linear combination of v2, v3 up to vn. This is equal to minus of a2 by a1 times v2 plus dot dot dot minus of an by a1 times vn plus b2 u2 plus dot dot dot plus bm um. Notice that, u2, u3 up to um, v2, v3 up to vn are elements in S minus v1, which hence is an element in the span of S minus v1.

So, either scenario if one of the ui's is v1, then the linear combination is in the span of S minus v1 and if one of the ui's is indeed v1 , the first one is none of them are in v1 and the second is when one of the ui's is v1, even then the linear combination of u1 to um with b1, b2 up to bm is an element in the span of S minus v1. So, this implies that span of S, we took an arbitrarily element and showed that that is in the span of S minus v1.

But span of S minus v1 will always be contained in the span of S, because S minus v1 is a subset of S, since S minus v1 is contained in S, you look at any linear combination of S, any linear combination in S minus v1, it should necessarily be a linear combination in S. We have the reverse inclusion span of S minus v1 is contained in the span of S. Hence, we have proved, span of S minus v1 is equal to the span of S.

That is precisely what it means for two sets to be equal, right? A is equal to B if and only if A is contained in B and B is contained in A, at the same time. We have shown that.

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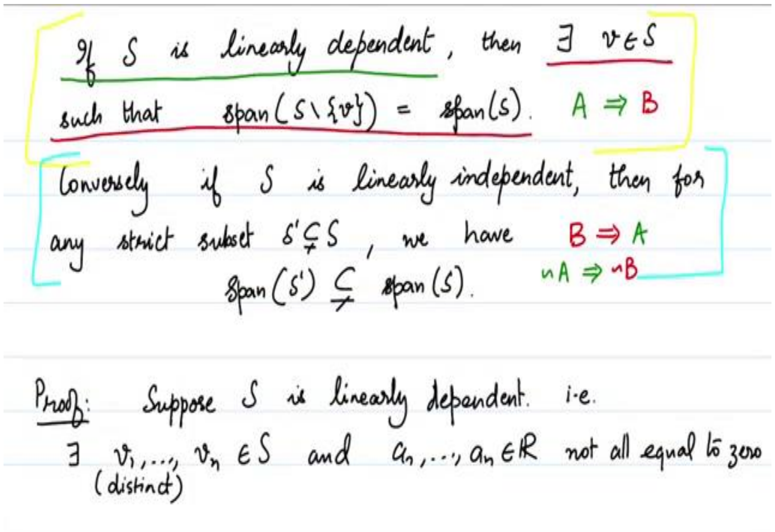

So, essentially we have shown the yellow statement here, whatever is put in the yellow bracket. Let us now show the converse. So, to show the converse, let me recall the statement. Conversely, what was the converse?

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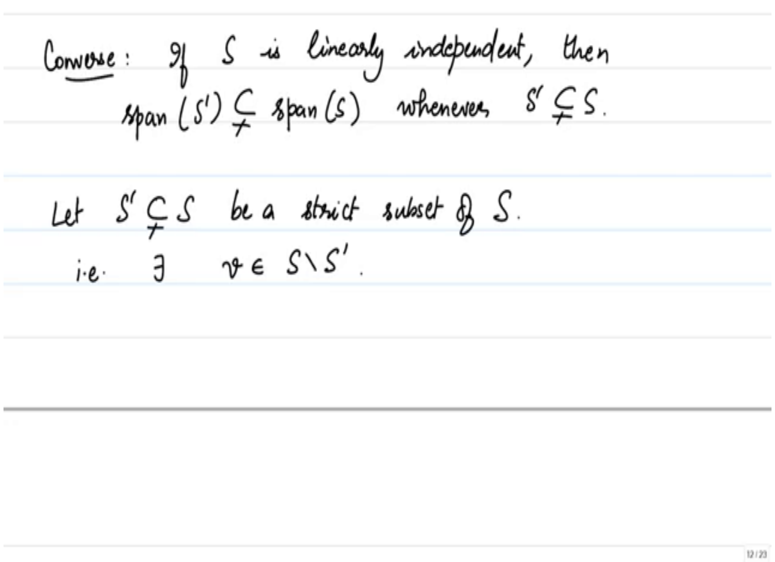

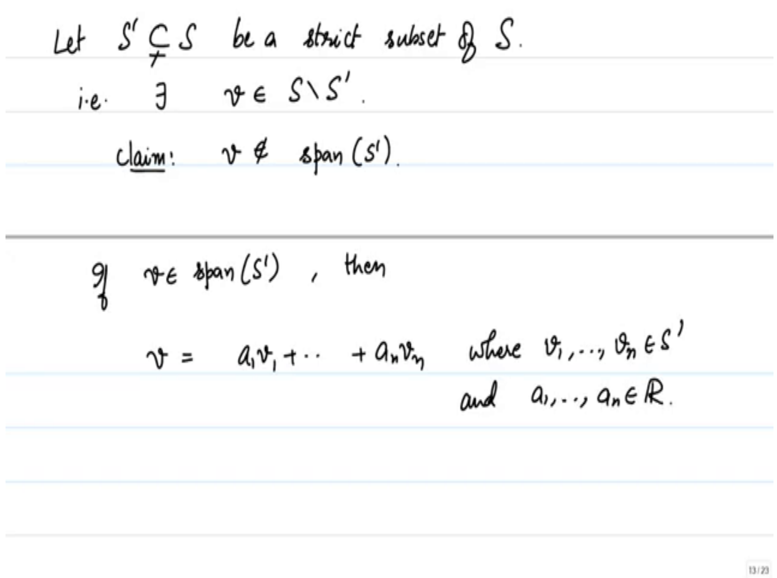

If S is linearly independent, then span of S prime is strictly contained in span of S whenever S prime is strictly contained in S. So, the assumption is that S is linearly independent. So, let us start with some subset S prime, which is strictly contained in S. So, let S prime be a strict subset of S, what does that mean? That means that there is some element v in S, which is not in S prime i.e. there exist v which is S minus S prime.

So, my claim is that this v does not belong to span of S prime. So let me just prove that claim, v that does not belong to span of S prime. Let us prove the statement by contradiction. If v belongs to span of S prime, then v will be equal to something like a1 v1 plus a2 v2 plus an vn which is a typical every element of span of S span should be an element of this type, where v1 to vn belongs to S prime and a1 to an are scalars or real numbers.

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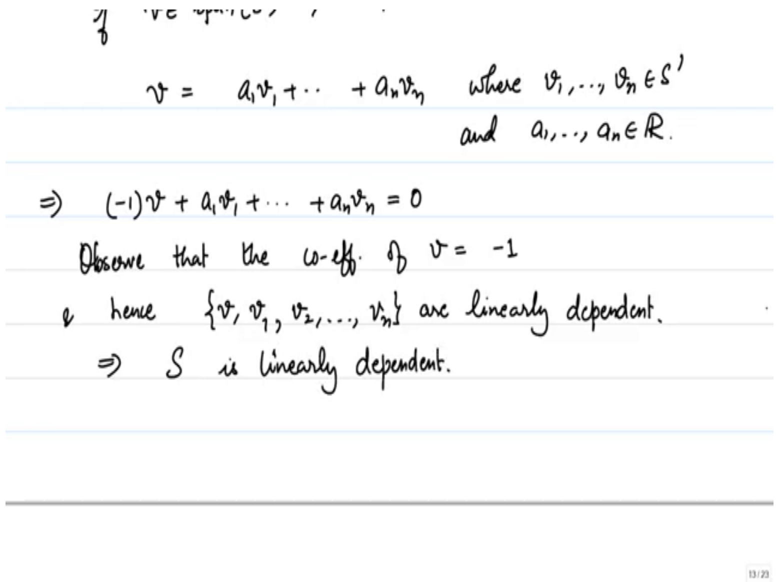

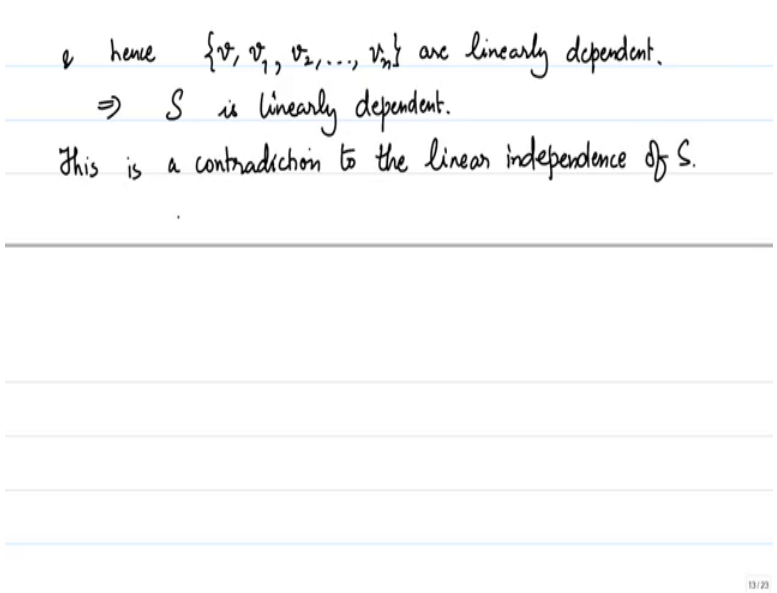

But re-writing this, this implies minus of 1 times v notice that v1, v2 up to vn all are distinct and not equal to v because v is not in S prime and that is a linear combination of S prime. So, this is equal to minus of 1 times v plus a1 v1 plus up to an vn is equal to 0 vector, by just adding the additive inverse of v. But here, there is a linear combination of v, v1, v2 up to vn which is equal to 0, such that not all coefficients are equal to 0 because in particular the coefficient of v is not equal to 0.

Observe that the coefficient of v is equal to minus 1 and hence v, v1, v2 up to vn are linearly dependent which implies S is linearly dependent. These are elements in s, but that is a contradiction because by assumption our S is linearly independent. This is a contradiction, to our assumption that S is linearly independent. Therefore, no it is not this contradicts, this is a contradiction to, let me rephrase it, contradiction to v linear independence of S.

Let me just show you the statement. The converse is exactly reading this, if S is linearly independent then something follows. So, the hypothesis is being contradicted, and therefore something in our assumption is wrong. So, our assumption all started with this, this is our assumption, this implies that this assumption has to be false, because if this assumption is true, then we are arriving at a contradiction to the hypothesis.

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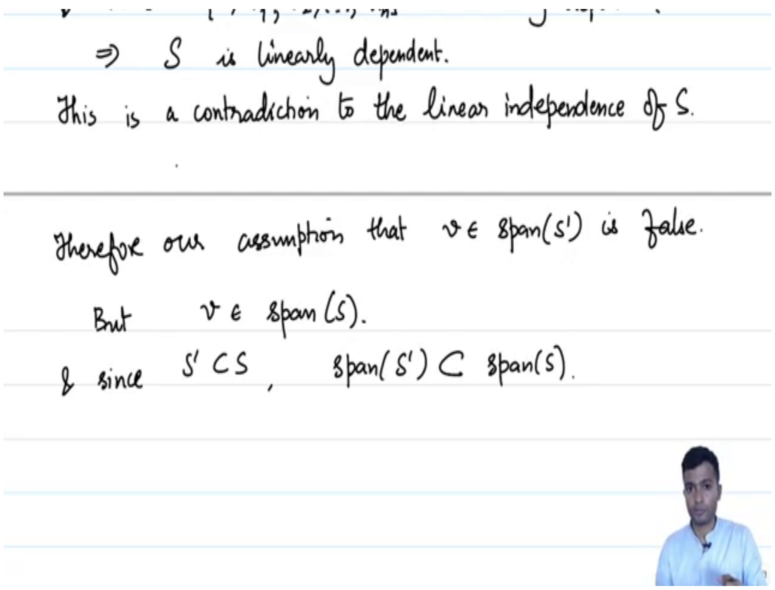

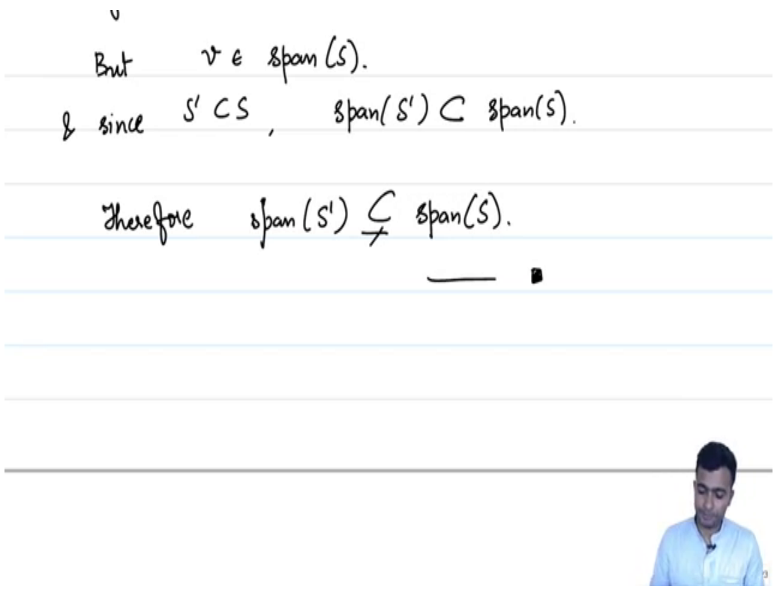

And therefore, our assumption that v belongs to span of S prime is false. That means v does not belong to the span of S prime, but then v is equal to one times v is an element in the span of S and since S prime is a subset of S, span of S prime should be contained in span of S. So we have a subset of, we have realized span of S prime as a subset of span of S.

And we have also found one element in span of s, which is not in span of S prime. Therefore, span of S prime is a strict subset of span of S. And this is precisely what we had set out to prove. So, linear independence is a very important concept, so along with the notion of spanning set and linear independence, we see that a very important notion of a basis can be defined, which we will do in the next video.