# Linear Algebra
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**Lecture 48**
**Spectral Theorem**

In this lecture, we will be exploring the notion of diagonalizability of normal operators and self adjoint operators in the right setting. From the lectures in week 8, if you recall, linear operator is said to be diagonalizable if we can get hold of a basis of the vector space consisting of eigenvectors of T. We will show that in the right setting normal operators and self adjoint operators are diagonalizable. This is what is classically known as the spectral theorem. And we will be concluding this course by giving a proof of the spectral theorem for these operators. Okay, so, let us begin.

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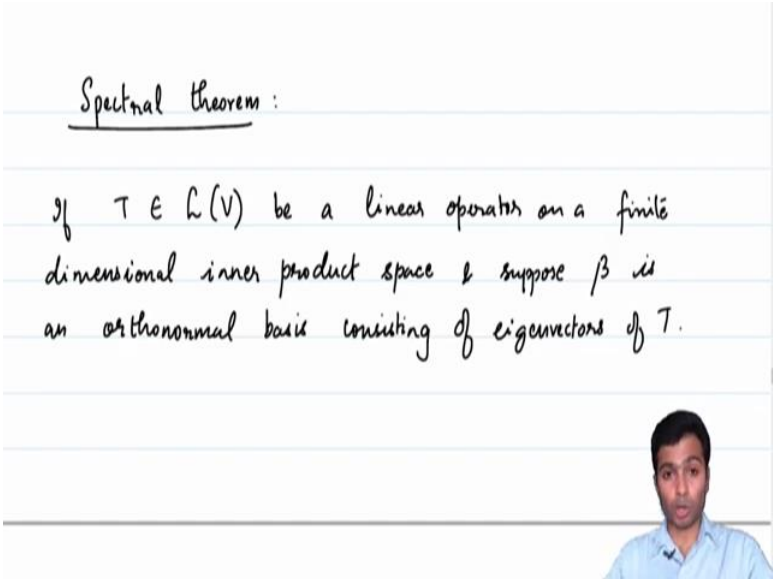


So, let me just write the heading spectral theorem. Suppose, before we enter into the statement of the spectral theorem, so if T be a linear operator on an inner product space be a linear operator on a finite dimensional inner product space. Suppose we have an orthonormal basis and suppose beta is an orthonormal basis consisting of Eigen vectors of T, suppose we are in this setup. So, observe that what we are demanding is that what is given to us is that T is a diagonalizable linear operator, not only that the Eigen vectors are orthonormal to each other.

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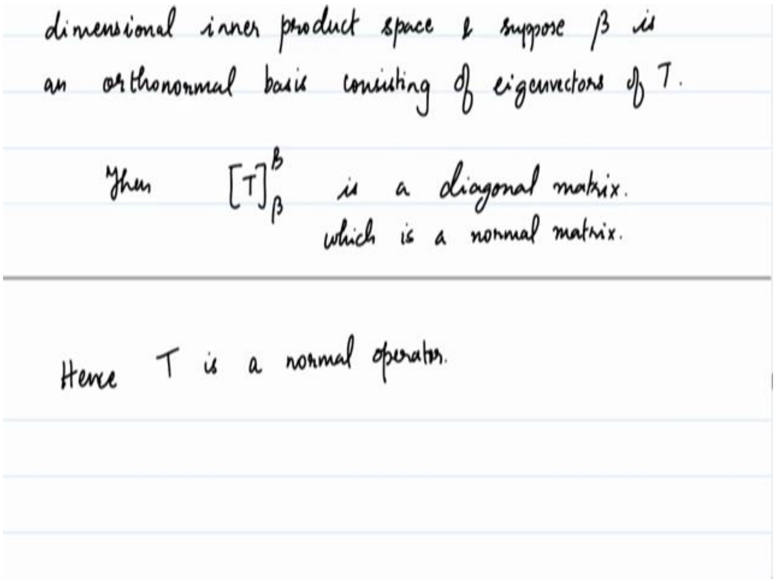


Then recall that the matrix of T with respect to be will be a diagonal matrix. What have we talked about diagonal matrices? We have seen that diagonal matrices are normal, the product of a diagonal matrix and its conjugate transpose will be equal to the product of the conjugate transpose of the matrix and the matrix itself, so which is a normal matrix. And by 1 of the terms we have proved earlier if a linear operator has a matrix representation with respect to some orthonormal basis which will give us a normal matrix, then the linear operator itself is normal.

This concludes, this helps us conclude that T is itself a normal operator. So, what we have observed is that if there is a diagonalizable matrix with a basis consisting of orthonormal vectors of Eigen vectors, then the linear operator T is normal. Our spectral theorem is in some sense a converse to this. What our spectral theorem says is that, this is true in general, spectral theorem says the following.

If we start off with complex inner product space, finite dimensional complex inner product space V, and if T is a linear operator on V which is normal, then there exists an orthonormal basis of V consistent Eigen vectors of T, in particular linear operator T is diagonalizable.

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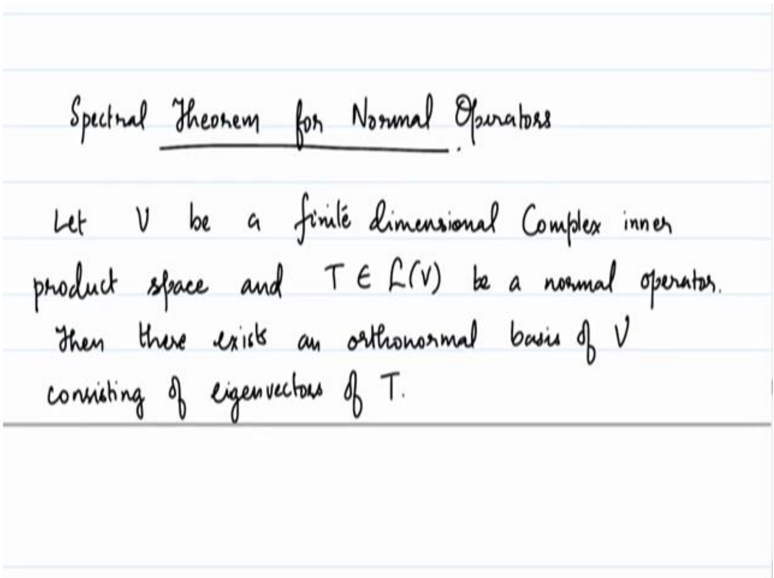


Spectral theorem for normal operators. So, the setup has already been laid out for you, we are in a complex inner product space, finite dimensional complex inner product space and T is a normal operator on V. So, let V be finite dimensional complex inner product space and T L of v be a normal operator, linear transformation from V to itself be a normal operator.

Then the spectral theorem tells us that then there exists an orthonormal basis of V consisting of Eigen vectors of T. The spectral theorem tells us that not only is diagonalizable, but the basis of Eigen vectors of t, they are also orthonormal. All right, so, let us give a proof of this, before giving a proof of the spectral theorem, let us look at what we have actually observed right now.

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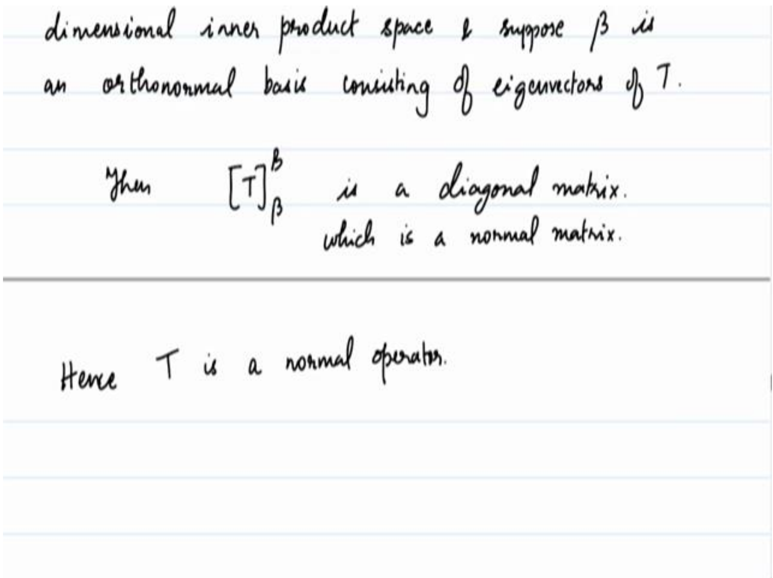

What we have just seen is, in the previous observation, the previous observation here was telling us that if you have a linear operator which has, which satisfies the kind addition that V has an orthonormal basis consisting of eigenvectors of T, then T is necessarily a normal operator. And the spectral theorem tells us that if T is normal then V consists of, V contains an orthonormal basis consisting of Eigen vectors. So, in a complex inner product space normal operators are precisely those operators which has, which satisfies the condition that V has a orthonormal basis consisting of eigenvectors of T.

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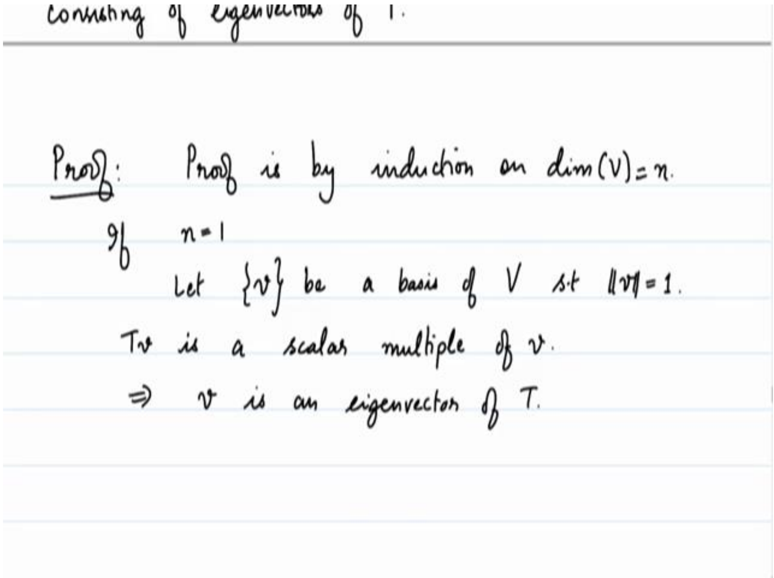


Okay, so let us give a proof of the spectral theorem. The proof is by induction. Induction on what, induction on dimension of V. So, this is a very finite dimensional inner product based proof that we are going to give. So, when which, let us call it to be called it to be equal to n. So, if n is equal to 1, what does that mean? That means that let v be a basis of capital V, which has a norm 1, such that length of V is equal to 1.

So, since capital V has dimension 1, this particular vector will form a basis and because there are no other vectors it is by default an orthonormal basis. We know that T v is a scalar multiple of v since v is one-dimensional vector space and that implies that v is an eigenvector of T.

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$\Rightarrow$ $v$ is an eigenvector of $T$.

Thus $\exists$ an orthonormal basis consisting of eigenvectors of $T$.

By induction hypothesis assume the theorem to be proved for upto $n-1$.

And thus, there exists an orthonormal basis consisting of eigenvectors of T. So that when n is equal to 1, there is nothing to prove. Let us now assume that n is greater than 1. So by induction hypothesis let us assume that the theorem has been proved for up to n minus 1. By induction hypothesis, assume that theorem can be proved for up to n minus 1. So, we will prove that when dimension of v is equal to n, the spectral theorem is satisfied or spectral theorem is true.

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By induction hypothesis assume the theorem to be proved for upto $n-1$.

Let $f(\lambda)$ be the characteristic polynomial.

By the fundemental theorem of algebra

$$f(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

Okay, so the induction hypothesis tells us that for any linear operator, which is a normal operator on a vector space, which is a complex inner product space of dimension less than n, up to n minus 1 there exists an orthonormal basis consisting of Eigen vectors. So, to prove for n, let us consider the characteristic polynomial of our given operator.

So, let f of lambda be the characteristic polynomial. So, recall that if you pick any basis and look at the matrix of the operator T corresponding to that basis, let us call that matrix A then compute the characteristic polynomial of this A. Then that is, we can call that the characteristic polynomial of a given operator.

Because you change the basis, any change of basis will give you a similar matrix to a, to this matrix A and therefore, the characteristic polynomial will not change. So, the characteristic polynomial of a given operator is a well defined polynomial with coefficients from the field of scalars. In this case, it is a complex inner product space and therefore, f of lambda is a polynomial coefficients in the complex numbers.

Now, by the fundamental theorem of algebra, our polynomial splits into linear factors. And we can write f of lambda to be equal to lambda minus say lambda 1 times lambda minus lambda 2 up to say a lambda minus lambda n, where n is the dimension of our given vector space. These lambda 1, lambda 2 up to lambda n are complex numbers. So, the characteristic polynomials splits over these field of complex numbers. What do we know about the roots of the characteristic polynomial?

We know that the roots of the characteristic polynomials are precisely the Eigen values of our given linear operator T. Notice that there is no demand on these lambda i's to be distinct, that has never been a demand even in the case as we were discussing earlier, but then certainly it splits, that the only thing we know and the roots turn out to be the eigenvalues. So, in particular lambda 1 is an eigenvalue.

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Let $v_1$ be an eigenvector of $T$ corresponding to $\lambda_1$
Moreover we may assume that $\|v_1\| = 1$.
i.e $\quad Tv_1 = \lambda_1 v_1$
$T$- normal $\Rightarrow \quad T^* v_1 = \bar{\lambda}_1 v_1$

So, let v1 be an eigenvector of lambda 1 eigenvector of lambda 1, that of T corresponding to lambda 1. That means, what does it mean, it means that corresponding to lambda 1 i.e Tv1 is equal to, moreover we may assume that the length v1 is 1, we assume that length of v 1 is equal to 1. Why can we do that? Because suppose we started off with a v 1 which did not have length v1 equal to 1, we will look at this scalar multiple of 1 by length of v 1 times v 1.

Now, the good thing about the vector v 1 is that it is nonzero, because it is an eigenvector therefore, length of v 1 to begin with was not 0 and therefore, 1 by length of v 1 make sense. And now, if you look at the length of v 1 by the length of v 1 that should have unit length and that is why we may assume without loss of generality that the length of v 1 is equal to 1 to begin with. And Tv1 is equal to lambda 1 v 1 will be the identity, which states that v 1 is an Eigen vector of T corresponding to lambda 1.

But T is a normal operator, T is normal implies, we have already proved this that if lambda 1 is an Eigen value of T then lambda 1 bar will be an eigenvalue of T star in the case of finite dimensional inner product spaces. And in the case of, in the case when T is a normal operator, we have proved more, we have proved that the Eigen vector corresponding to lambda 1, corresponding to lambda 1 of T will be the same as the eigenvector of T star corresponding to lambda 1 bar. So, we will, because T is normal we also have that the adjoint of T v 1 is equal to lambda 1 bar times v 1.

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i.e $Tv_1 = \lambda_1 v_1$

$T$- normal $\Rightarrow$ $T^* v_1 = \bar{\lambda}_1 v_1$

Define $W = span\{v_1\}$.

Notice that $W$ is invariant under $T$ and $T^*$

Notice that $W$ is invariant under $T$ and $T^*$

Consider the orthogonal complement of $W$ given by $W^\perp$.

Let $w \in W^\perp$

$\Rightarrow \langle w, v_1 \rangle = 0$

$$\langle Tw, v_1 \rangle = \langle w, T^* v_1 \rangle = \langle w, \bar{\lambda}_1 v_1 \rangle$$
$$= \lambda_1 \langle w, v_1 \rangle = 0$$

So, this is good, because now what we will do is we will define W to be the vector subspace of V which is defined as the span of v1. Notice that W, notice that W is invariant under both T and T star, under T and T star. Why because T v 1 is lambda v 1 and T star v 1 is lambda 1 bar v 1 which both of which belong to capital W. So, this is in vector subspace which is invariant under both T and T star. Now, consider the orthogonal complement of W. So, consider the orthogonal complement of W, given by W orthogonal.

So, the first observation which we have already seen in one sense is to notice that w orthogonal is invariant under both T and T star. We have already seen a proof of this but nevertheless it is quite quiet a useful thing to see even once more. So, let w be a vector in the orthogonal complement of capital W. What does this mean?

This means that inner product of v with v 1 is equal to 0. Notice that the moment we have this condition, inner product of v1 with w is 0, that means that w is in the orthogonal complement of capital W, because v vector in capital W will be a scalar multiple of v 1, so it will be c times v 1.

And w inner product with v1 is 0 would imply that w inner product with C times v 1 is also 0. So, this is exactly the characterization of w being in the orthogonal complement of capital W. Okay, but this is, this is right. So, let us consider T w and we would like to see that this is also is in the, this is also in the orthogonal complement of capital W.

And in order to do that, we should check that the inner product of Tw with v 1 is 0. But what is this, this is the inner product of w with T star v 1. And v 1 is an Eigen vector of T star, which is the inner product of w with lambda 1 bar times v 1, which is equal to lambda 1 times the inner product of w with v1. Which we just observed is equal to lambda 1 times 0 which is equal to 0.

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let $w \in W^{\perp}$

$$\Rightarrow \quad \langle w, v_1 \rangle = 0$$

$$\langle Tw, v_1 \rangle = \langle w, T^* v_1 \rangle = \langle w, \bar{\lambda}_1 v_1 \rangle$$

$$= \lambda_1 \langle w, v_1 \rangle = 0$$

$$\Rightarrow \quad Tw \in W^{\perp}$$

|||ly $\langle T^* w, v_1 \rangle = \langle w, T v_1 \rangle = \bar{\lambda}_1 \langle w, v_1 \rangle = 0$

$$\Rightarrow \quad T^* w \in W^{\perp}.$$

So, this implies that T w is in the orthogonal complement of capital W. Similarly, or maybe it is just 1 line T star W, inner product with v 1 is equal to the inner product of w with Tv 1, which is equal to lambda bar lambda 1 bar times the inner product of w with v 1 which is equal to 0, which implies that T star w is also in the orthogonal complement of W.

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Hence $T|_{W^\perp} : W^\perp \to W^\perp$ & $\|^{ly}$ $T^*|_{W^\perp} : W^\perp \to W^\perp$

Also notice $\langle T|_{W^\perp} w_1, w_2 \rangle = \langle Tw, w_2 \rangle$

$$= \langle w_1, T^* w_2 \rangle = \langle w_1, T^*|_{W^\perp} w_2 \rangle$$

$$\forall\, w_1, w_2 \in W^\perp$$

So, what does this say this means that T restricted to orthogonal complement of W is a linear map from the orthogonal complement of W to itself. Similarly, T star restricted to the orthogonal complement of W is also a linear map from the orthogonal complement of W to itself.

Also notice that inner product of T restricted to W orthogonal of w comma w 1 comma w 2, this is equal to the inner product of T w 1, w 2, which is equal to the inner product of w 1, T star w 2, which is the same as w 1, T star restricted to W orthogonal w 2 for all w 1, w 2 in the orthogonal complement of W. So, what this says is that the restriction map of T star is the adjoint of the restriction map of T.

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Also notice $\langle T|_{W^\perp} w_1, w_2 \rangle = \langle Tw, w_2 \rangle$

$$= \langle w_1, T^* w_2 \rangle = \langle w_1, T^*|_{W^\perp} w_2 \rangle$$

$$\forall\, w_1, w_2 \in W^\perp$$

$$\Rightarrow \left(T|_{W^\perp}\right)^* = T^*|_{W^\perp}$$

Hence $T|_{W^\perp} T^*|_{W^\perp}(w) = TT^*(w) = T^*T(w)$

$$= T^*|_{W^\perp} T|_{W}(w).$$

So, that is good because this tells us, okay, let me just note that for you T restricted to the orthogonal complement of W star is precisely going to be equal to T star restricted to the orthogonal complement of W. Hence, T restricted to W orthogonal times T star restricted to orthogonal which is equal to T, T star restricted to the other orthogonal complement of W is the same as T star T. Yeah or maybe I should just say that this of w is equal to this of w, which is the same as this of w which is the same as T star restricted to W orthogonal of T star T restricted to W orthogonal of w.

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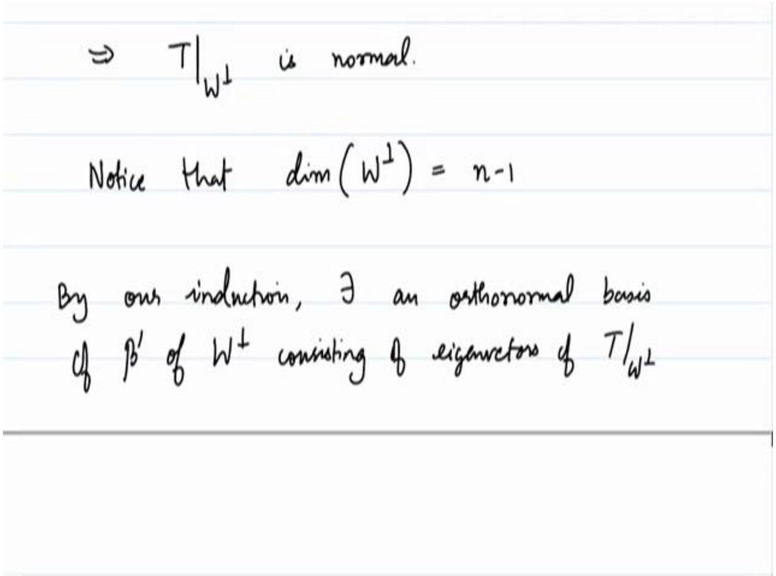


Which tells us that T restricted to the orthogonal complement of W is normal. Now, we are in good shape because w orthogonal is a vector subspace of V, which has dimension n minus 1. Notice that the orthogonal complement of one of the, the was a version of the dimension theorem for the orthogonal compliment you should go back and check. This is just going to be equal to n minus 1 because the dimension of W was equal to 1. And by our induction hypothesis.

By our induction hypothesis, notice that T restricted to W is a normal linear operator on a vector complex inner product space of dimension n minus 1. The inner product is just the one borrowed from V. So, by induction, there exists an orthonormal basis, let us call it beta prime, of W orthogonal such that consisting of Eigen vectors of T restricted to orthogonal complement of w. So, the first thing to notice is that beta prime is a subset of W orthogonal and in particular beta prime is a subset of V as well.

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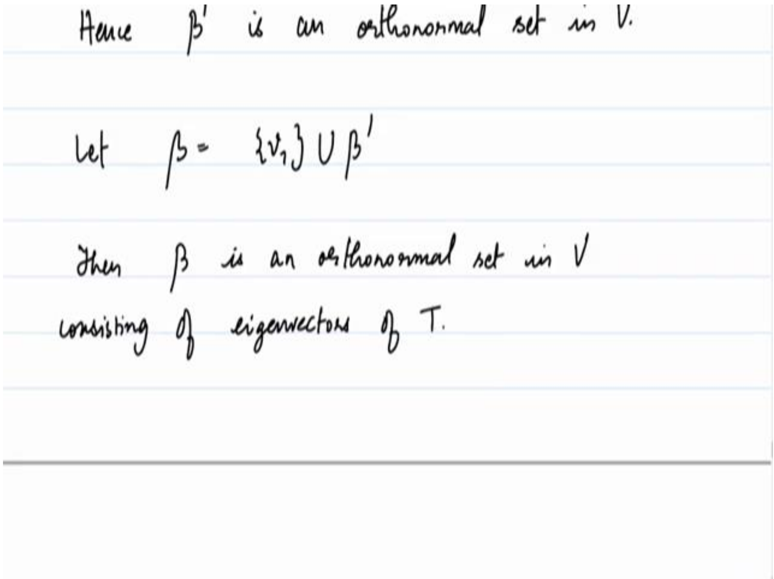


Hence, beta prime is an orthonormal set in capital V in the inner product space V, this is an orthonormal set, they are orthogonal and has length 1. Now, let us look at beta to be equal to v 1 union beta prime. So, if beta prime was a v 2, v 3 up to v n, beta is just now going to be v1, v2 up to vn. And notice that beta is also an orthonormal set.

The reason being that each of the beta diverging in beta prime are vectors in w orthogonal and in particular, they are orthogonal to our vector v1. And therefore, this is a collection of mutually pairwise orthogonal vectors. And we already started off with length of v 1 being equal to 1. So, beta in particular is an orthonormal set in capital V. And notice that each of them are Eigen vectors, consisting of eigenvectors of T. Because v1 to begin with was an Eigen vector corresponding to lambda 1, Eigen vectors of T.

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Notice that $\dim(W^{\perp}) = n-1$

By our induction, $\exists$ an orthonormal basis of $\beta'$ of $W^{\perp}$ consisting of eigenvectors of $T|_{W^{\perp}}$

$\Rightarrow$ $\beta'$ consists of eigenvectors of $T$

$\left( \because \text{ for } w \in \beta' \quad Tw = T|_{W^{\perp}} w = \lambda_j w \right)$

Hence $\beta'$ is an orthonormal set in $V$.

Let $\beta = \{v_1\} \cup \beta'$

One observation here is that an Eigen vector of, consisting of Eigen vectors of T restricted to W orthogonal, this implies that beta prime consists of Eigen vectors of T. Why is that the case, because, if T restricted to W orthogonal of w is equal to some lambda j times, let me just write it down. Since T restricted to W or other for W in beta prime T w is just equal to T restricted to the orthogonal complement of W acting on W and this is just going to be some lambda j times W and therefore W is an Eigen vector, that I just noted.

That would imply that each of the beta prime consists of Eigen vectors of T as well. Well, Eigen vectors of T restricted to W orthogonal is just going to be Eigen vectors of T as well. So, that is the simple observation here.

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Let $\beta = \{v_1\} \cup \beta'$

Then $\beta$ is an orthonormal set in $V$ consisting of eigenvectors of $T$.

$\# \beta = n$

Hence $\beta$ is a set of size $n$ which is linearly independent.

$\Rightarrow$ $\beta$ is a basis of $V$.

independent.

$\Rightarrow \beta$ is a basis of V. —— ∎

And therefore, this beta will have an orthonormal set consisting of Eigen vectors of T. But what is the size of beta? The size of beta is equal to n minus 1 plus 1 which is equal to n, because beta prime has n minus 1 vectors side, after all w orthogonal had dimension n minus 1. And they are all, that is an orthonormal set, hence beta is a set of size n, which is linearly independent. And what do we know about a linearly independent set of size n in a dimension n vector space? We know that it is a basis, this gives that beta is a basis of V and we are done with the proof. So, we have just proved that in a complex inner product space a normal operator is always diagnosable. Not only is it diagnosable, we also have a basis of Eigen vectors which is also normal. Okay, that is pretty nice.

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Remark: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(x, y) = (-y, x)$$

$$[T]_\beta^\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$f(\lambda) = \lambda^2 + 1$. does not split over $\mathbb{R}$.

Hence T is an example of a normal operator on a Real inner product space which is not diagonalizable.

Proposition: Let $V$ be a complex inner product space & $T \in \mathcal{L}(V)$ such that $\langle Tv, v \rangle = 0 \ \forall v \in V$.

Then $T \equiv 0$.

Note: Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x, y) = (-y, x).$$

Then $\langle Tv, v \rangle = 0 \ \forall v \in \mathbb{R}^2$.

So, there are a couple of observations, couple of remarks here. The first one is to observe that this is a statement, which has complex inner product space put into the very hypothesis of the statement. Because if you consider, so remark, consider T from R2 to R2, this is a real inner product space given by T of x comma y to be equal to.

Let us look at the example we had given in 1 of the previous lectures. We will just use the same example here. Was it x minus y? Yes, T minus y comma x. So, let us look at this particular example, minus of y comma x. So, what was the characteristic polynomial? To do that, let us look at what was T beta beta, where beta was the standard basis. If you recall, this is just going to be equal to 0, 1, minus 1 0. And hence f of lambda is just going to be equal to lambda square plus 1.

And if you notice, this is a polynomial which does not split over real numbers, does not split over R. So, the fundamental theorem of algebra is, it just tells us that it splits over the complex numbers, this particular polynomial, however, does not split over R. And hence this is a linear operator which does not even have Eigen values, T from R 2 to R 2. So, it is a normal operator which does not even have Eigen value. So, there is no question of getting hold of a basis consisting of Eigen vectors.

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$$f(\lambda) = \lambda^2 + 1. \quad \text{does not split over } \mathbb{R}.$$

Hence T is an example of a normal operator on a Real inner product space which is not diagonalizable.

So, hence, T is an example of a normal operator on a real inner product space which is not diagnosable. Let us look at however, is when, let us say call it S.

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Example: $S : \mathbb{C}^2 \to \mathbb{C}^2$ Let $\beta = ((1,0),(0,1))$

$$S(z, w) = (-w, z)$$

$$[S]_\beta^\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$f(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

So, let us look at example now S be from C 2 to C 2, a very similar map, let us define. S of instead of x comma y, let us use some z comma w and this will be what minus of w comma z. Just like the map we have defined here. And let us see the what will be the matrix of s with respect to beta. So, let beta be equal to the ordered basis consisting of 1, 0 and 0, 1. Notice that this is in C2. So, all complex numbers you look at the span with respect to all the field of scalar is being been complex numbers.

So, what is S beta beta, as was observed earlier, this is going to be equal to 0 comma 1 in the first column and minus 1 comma 0 and therefore, f of lambda is equal to lambda square plus 1, which is lambda minus i times lambda plus i, where i is the square root of minus 1.

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$$f(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

$\Rightarrow$ $\lambda_1 = i$ & $\lambda_2 = -i$ are the eigenvalues of $S$.

eigenvectors of $i$

$$S(z, w) = i(z, w) = (-w, z)$$

$$\Rightarrow iz = -w$$

$\Rightarrow$ $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector of $i$

Therefore, there are 2 Eigen values here. Hence, lambda 1 is equal to i and lambda 2 is equal to minus i are the eigenvalues. So, notice that the split is to be expected by the fundamental theorem of algebra, here in this case it was easy, hence, I could write it down very easily. Lambda 1, the roots are going to be the eigenvalues of the linear operator.

So, i and minus i are the eigenvalues of T of S rather. That means that that S of, let us look at what the Eigen vectors will be for eigenvector of z of i. Eigenvector, one of the eigenvectors of i will be something of the this type. i times z, w but this we know is equal to minus of w comma z. Yes, that means this is equal that means this gives us that i, z is equal to minus of w. So, this would imply that and z times the first one can be put as 1, if z this 1, w would be minus i. This is an eigenvector of T corresponding to I, eigenvalue lambda 1 equal to i.

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$$\Rightarrow \quad iz = -w$$

$$\Rightarrow \begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ is an eigenvector of } i$$

If $S(z, w) = -i(z, w)$

$$\Rightarrow \quad iz = w$$

$$\Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ is an eigenvector of } -i.$$

How about minus i, if s of z comma w is equal to minus of i times z comma w, this would imply that i times z is equal to w. And minus of i times w is equal to z, makes sense. And therefore, z is equal to 1 comma i is an eigenvector of minus i. So, what would be an orthonormal basis.

So, we already seen that in the case of a normal operator, the Eigen values corresponding to distinct Eigen vectors corresponding to distinct Eigen values should be orthonormal. So, in particular this is orthonormal, you can check it straight forward in a straightforward manner using the inner product space as well.

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$$\beta' = \left( \frac{1}{\sqrt{2}}(1, -i), \ \frac{1}{\sqrt{2}}(1, i) \right) \text{ is}$$

an orthonormal basis consisting of eigenvectors of $S$.

$$[S]_{\beta'}^{\beta'} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So, beta being equal to 1 comma minus 1, after ortho normalizing, it will be 1 by root 2 times 1 comma minus i and 1 by root 2 times 1 comma i is an orthonormal set consisting of orthonormal basis rather, consisting of Eigen vectors of S. Let us call it beta prime, we already called beta as the standard basis of S. And what is going to be S beta prime beta prime, this is just going to be equal to i comma 0, 0 comma minus i all right so, that is, that is an example of yeah, that is.

This example illustrates that in the case of complex inner product spaces, normal operators are diagonalizable, whereas in the case of real inner product spaces, normal operators need not be diagonalizable, the example we just gave illustrates that. However, if you notice our first example was an example of a normal operator which was not a self adjoint operator.

So, of course, all self adjoint operators are normal operators and therefore, if you consider self adjoint operators over complex inner product spaces by the spectral theorem for normal operators, we know that that is also diagonalizable linear operator. But we can ask more. What can we, can we at all say anything about self adjoint operators on real inner product spaces? The answer turns out to be yes, we can say something more. And that is going to be the content of the spectral theorem for this self adjoint linear operator.

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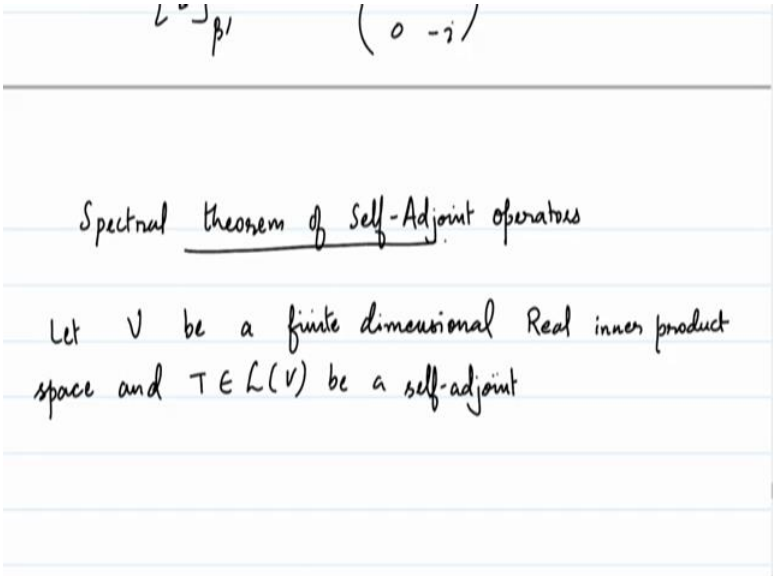

Let $V$ be a finite dimensional Real inner product space and $T \in L(V)$ be a self-adjoint operator. Then $\exists$ an orthonormal basis of $V$ consisting of eigenvectors of $T$.

So, this is going to be the spectral theorem of self adjoint operators. So, now let us look at real inner product space. We have already addressed the case of a complex inner product space right. Self adjoint operators are normal and as I was just mentioning, because of the spectral theorem for normal operators we also have a spectral theorem for self adjoint operators in the case of complex inner product space. Let us now look at a finite dimensional inner product space. The field of scalars in this case is the real numbers. Inner product space.

Then, and T in L of V be a self adjoint operator. Did I mention in the statement of the spectral theorem above that T is normal, yes it is it is mentioned. So, in this case we are going to focus on T in a L of V be a self adjoint operator. Then there exists an orthonormal basis of V consisting of Eigen vectors of T. So, in the case of a real inner product specific given a self adjoint operator, we do have that it is diagonalizable over R. So, we have already done all the major work.

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$\exists$ an orthonormal basis of $V$ consisting of eigenvectors of $T$.

<u>Proof:</u> Proof is done by induction on $\dim(V) = n$

$n = 1$. The same proof works.

$n = 1$. The same proof works.

Assume the theorem to be proved upto $n-1$.

Let $f(\lambda)$ be the characteristic polynomial of $T$.

So, let us give a proof of this, the proof has already been done to a large extent already. The question would be we will mimic the proof that we were trying to do in the case of normal operators. Where would be the hurdle that that would come? So, recall that again the proof can be done by induction on the dimension of V, which is equal to n, n is equal to 1, it is straightforward is the same proof works and let us assume that the theorem has been proved up to n minus 1.

Now, if you are to mimic the power of the spectral theorem for normal operators, we will be considering the characteristic polynomial be the characteristic polynomial of T. However, now we are in a real inner product space and we will not be able to say that the characteristic polynomial splits over real numbers in the general case.

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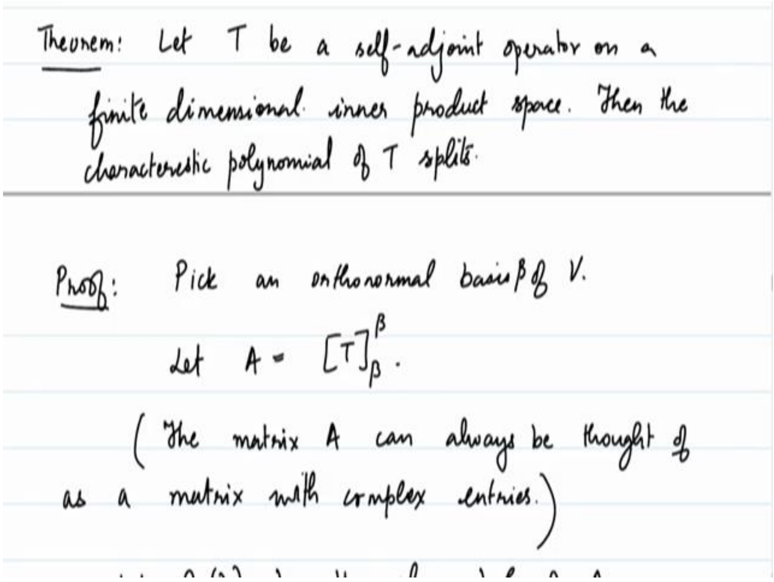


However, let me go back to a result from your, from the previous lecture regarding the characteristic polynomial of self adjoint operators, yes this is precisely the statement. So, notice that if T is a self adjoint operator on a finite dimension inner product space, whether complex inner product space or real inner product space. In the case of complex inner product space, this theorem is redundant because anyway fundamental theorem of calculus, fundamental theorem of algebra tells us that the characteristic polynomial space.

In the case of real inner product is what this is telling us substantial amount of information. This tells us that it splits, I did not actually mention over R, but that is understood here. I did not write over here because the statement was more general, it splits over the field of scalar. So, if it was in the case of a complex inner product space, it splits over C, if it was in the case of real inner product space, it splits over R, the proof did capture that. So, I do not want to go through the roof again. But what it tells us is that splits over R because T is self adjoint. So, we will use that information.

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Let $f(\lambda)$ be the characteristic polynomial of T.

Since T is self-adjoint, the char. poly of T splits over $\mathbb{R}$.
Let $\lambda_1$ be a root of $f(\lambda)$.

splits over $\mathbb{R}$.
Let $\lambda_1$ be a root of $f(\lambda)$.

Let $v_1$ be an eigenvector corresp. to $\lambda_1$

$W =$ span$\{v_1\}$.

Consider $W^{\perp}$

By mimicking the proof for normal operators, we get an orthonormal basis of V consisting of eigenvectors of T.

Since T is self adjoint, this is very important, because it is self adjoint, the characteristic polynomial of T splits over R. And let lambda 1 be a root of f of lambda. After this the proof is extremely similar to what was done in the case of normal operator. So, let v1 be an Eigen vector corresponding to lambda 1, let W be the span of v 1 and consider W orthogonal. We noticed that R and star both are invariant, sorry W orthogonal is invariant under both T and T star.

And by mimicking the proof for normal operators using the induction hypothesis, we get an orthonormal basis of V consisting of Eigen vectors of T. Let us say that it should go back to the proof of the spectral theorem for normal operators from the case when the induction hypothesis came into force, the proof is going to be exactly the same, finally getting hold of it, finally we will be getting hold of an orthonormal basis of V, which consists of eigenvectors of the self original operator T.

With that, that will be the proof, completion of the proof of the spectral theorem for the self adjoint operator. So, even though I did mention that linear, the normal operators and self adjoint operators are quite special and not, not many operators are, most operators do not turn out to be normal and self adjoint. However, I should mention here that in real life, there are many cases when the operators that we are studying does turn out to be normal and self adjoint, normal and or self adjoint.

This is in fact true in physics many times, especially when you do say for example, quantum mechanics. So, the study of these operators is very, very useful and important. And the spectral theorem goes a long way in that front. So, that is more or less everything that I would like to speak about spectral theorem and with this we have come to the conclusion of this particular course. I hope that you have enjoyed this course, I wish you all the luck and all the best for all your future endeavors. Thank you.