Linear Algebra.
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Lecture 47
Self Adjoint Operators.

We have already discussed the notion of a normal operator on an inner product, finite dimensional inner product space. A normal operator's linear transformation from V to itself, whose adjoint commutes with the given operator. So, in this lecture we will discuss a special subclass of normal operators. They are called as self adjoint operators. As the name suggests self adjoint linear operators are those operators whose adjoint is the operator itself. So, let us begin by defining what are adjoint operators.

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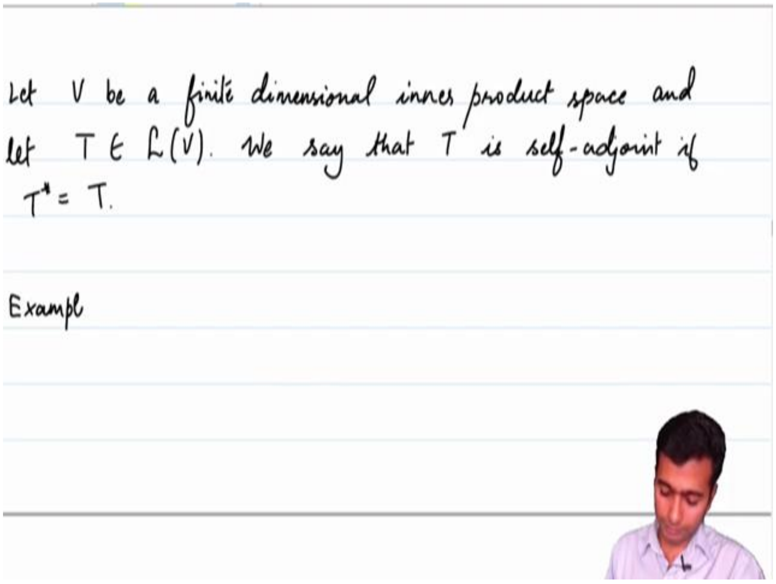

So, let V be an inner product space, be a finite dimensional inner product space. And let T from V to itself be a linear operator and let T be in L of V, so linear transformation from V to itself. We say that T is self adjoint if T star is equal to the operator T itself. So, let us look at an example.

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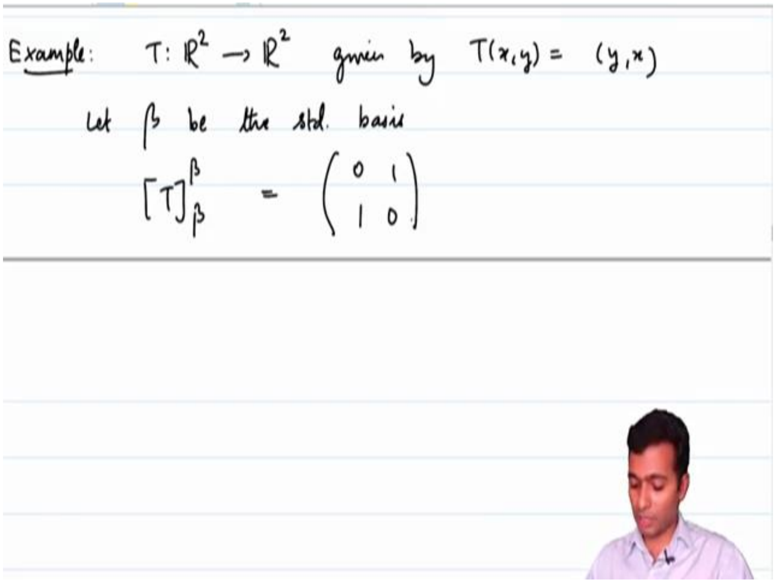

So define E to be from say, R 2 itself given by T of x comma y is equal to y comma x. So, let us fix the standard basis which is an orthonormal basis with respect to the standard inner product. So, let beta be the standard basis and what is going to be T beta beta, already the matrix of T with respect to beta. This is just going to be let us see, what is T of 1 0 T of 1 0 is going to be 0 comma 1, and T of 0 comma 1 is 1 comma 0. So, this is precisely the matrix of T with respect to beta.

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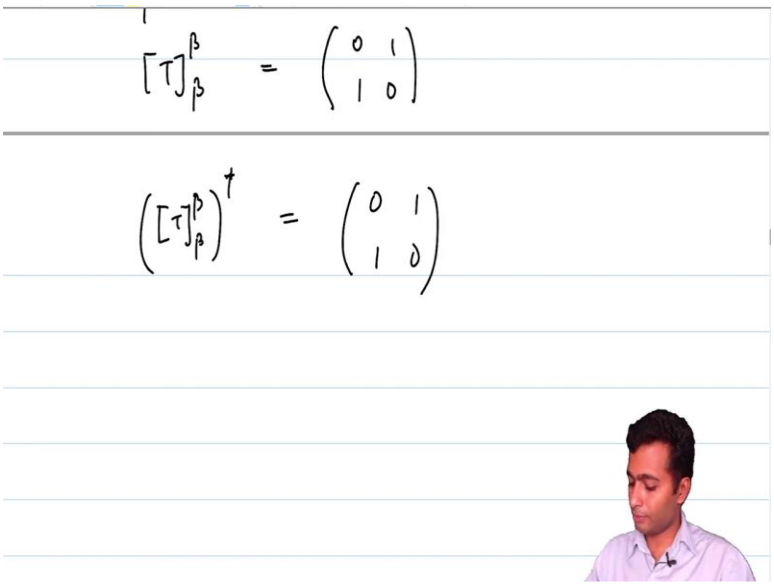

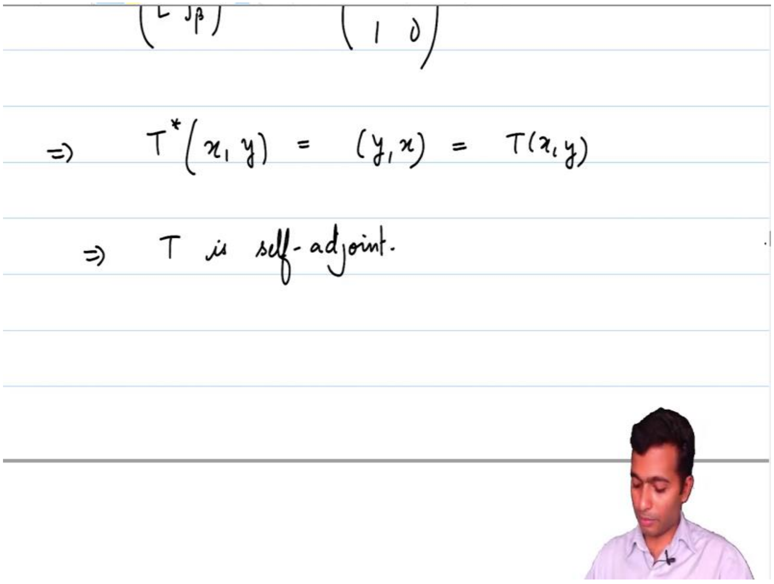

And it has real entries, what is the adjoint, so T star matrix of. So let us see T beta beta adjoint, which is the matrix of the adjoint of T is again the same matrix. And therefore, the matrix of T star is going to be 0, 1, 1, 0. And therefore T star of x comma y is just going to be equal to y comma x, which is equal to T of x comma y. So, this implies that T is indeed a self adjoint linear operator. By the very definition, T is self adjoint.

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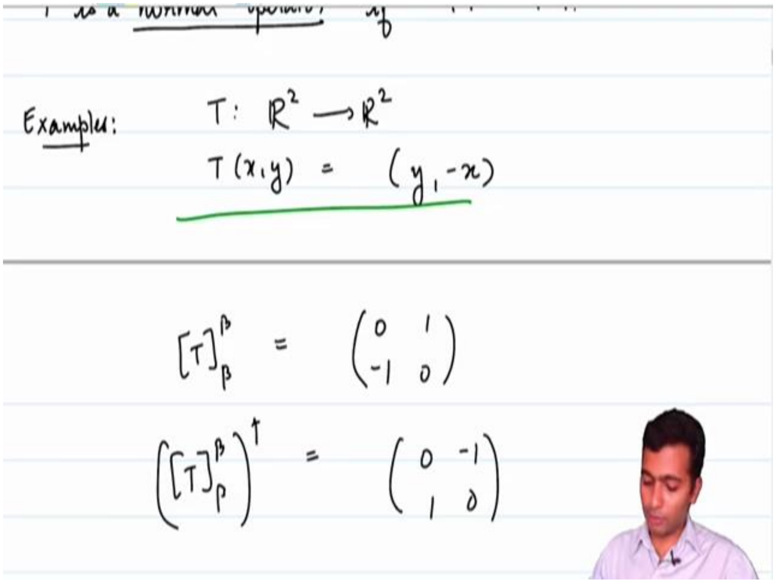

Okay, now let us look at 1 more example, rather a non example. In fact, let us just scroll up go to the example of normal linear operators and look at maybe let us focus on this particular operator. T of x, y being equal to T of T of x, y being equal to y comma minus x.

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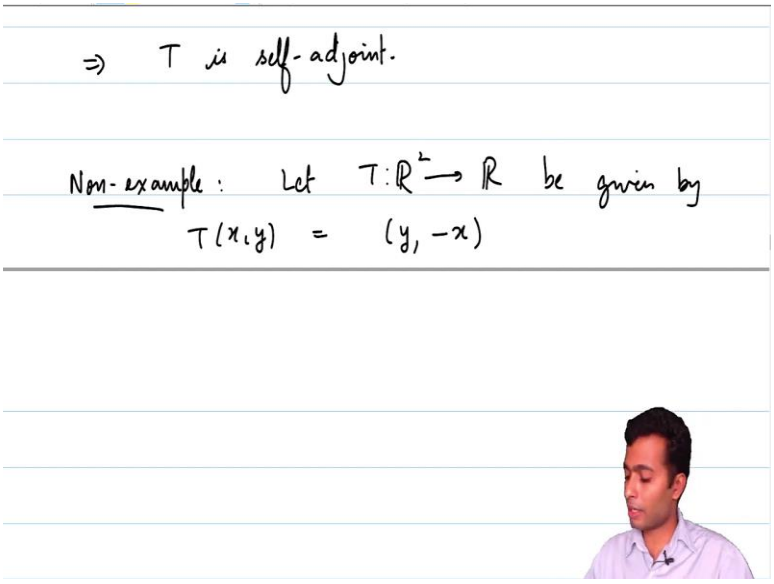

So, example from about continued, so, let me write it as an example, because this is going to be a linear operator which is not self adjoint. So, let T again from R2 to R2 be given by T of x comma y is equal to y comma minus x.

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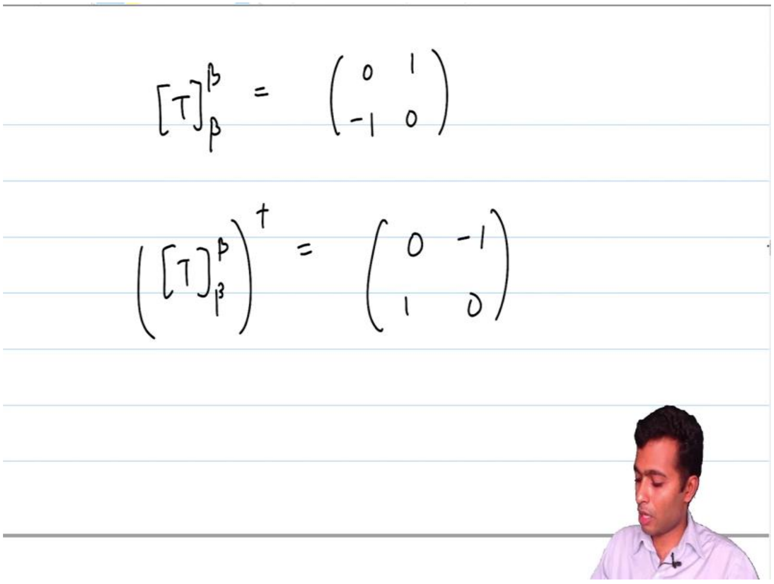

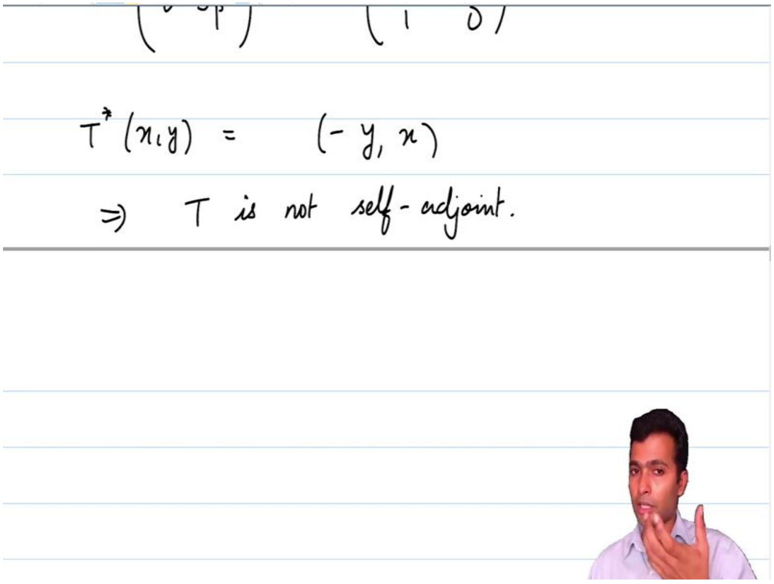

And if you go back and check the calculations involved, we actually had this with respect to the standard basis, this is just going to be equal to 0 minus 1, 1, 0. And therefore, if you calculate the adjoint, the adjoint is just 0 minus 1 here in the first row, 1 0 in the second row. And that would give us that T star of x comma y is equal to minus of y comma x, which is not equal to t of x comma y because T of x comma y was y comma minus x.

This implies that T is not self adjoint. So, this immediately tells us that not all normal operators will be self adjoint, T is not self adjoint because this is an example of a normal operator. Normal operator the condition is weaker. Okay, so maybe I should make a small lemma here.

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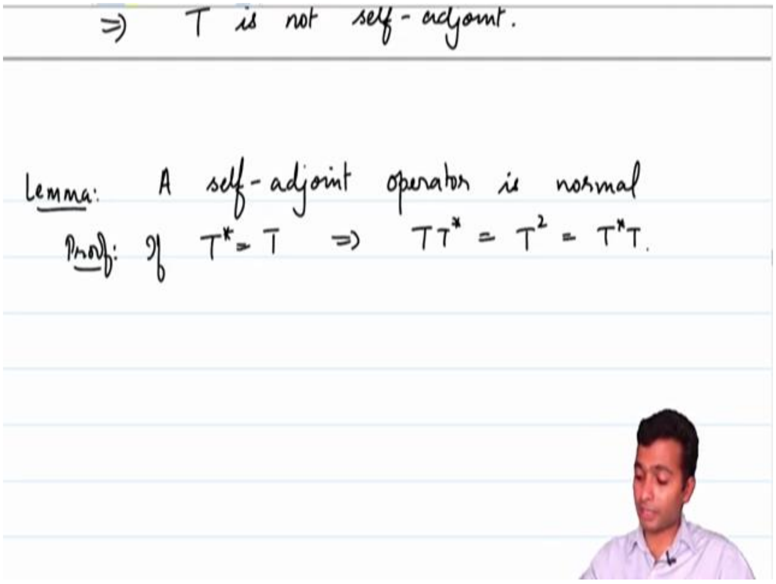

It is a very straightforward observation, a self adjoint operator is normal, self adjoint operator is normal. And why is that the case? Well, it is a one line proof, if T star is equal to T, this would imply T T star is equal to T square which is the same as T star T, and hence it is normal. So, it is a straightforward observation to see that every self adjoint operator is a normal operator. And in an analogous manner, when we dealt with normal operators, we will also define what a self adjoint matrix is.

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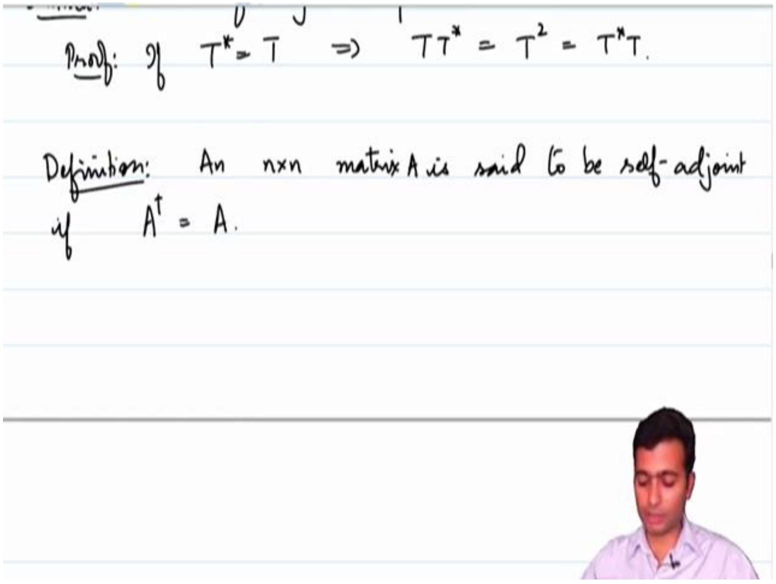

So, as n cross n matrix, so, let me just call it a definition here. An n cross n matrix is said to be self adjoint, if the adjoint, the conjugate transpose of A, let us call it At, if this is equal to the matrix A itself. So, that is a straightforward definition one would expect. And in this case the first matrix here, this is self adjoint. However, this is not self a joint. Just like in the case of normal operators, we can find a relationship between self adjoint transformations, linear time linear operators and Self adjoint matrices which will be the content of our next proposition.

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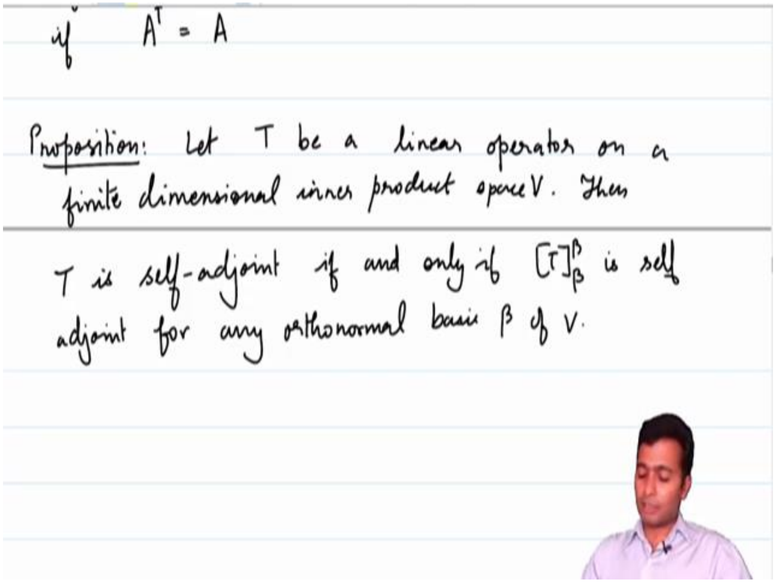

The proof of this proposition is going to be very similar to how we would have put it in the case of normal operator, so I will not venture into giving a proof of this. Nevertheless, it is a very important statement, let me write it down. So, let T be a linear operator on an inner product space, on a finite dimensional inner product space and beta be some orthonormal basis, let me not phrase it that way.

Then T is self adjoint if and only if, the matrix of T with respect to beta is self adjoint for any orthonormal basis, that is important, the orthonormality of basis is important orthonormal basis beta of V. So, let us call the vector space V. We will leave this as an exercise for you to take this is already done, a similar proposition was done in the case of normal operators and that is going to go through, I similar proof is going to go through in the case of normal operators. Let us look at a few more examples of self adjoint linear operators, or rather now I will focus on matrices. In the backdrop of this particular proposition, we just need to focus on matrices and prove results there which we will carry forward to self adjoint operators as well.

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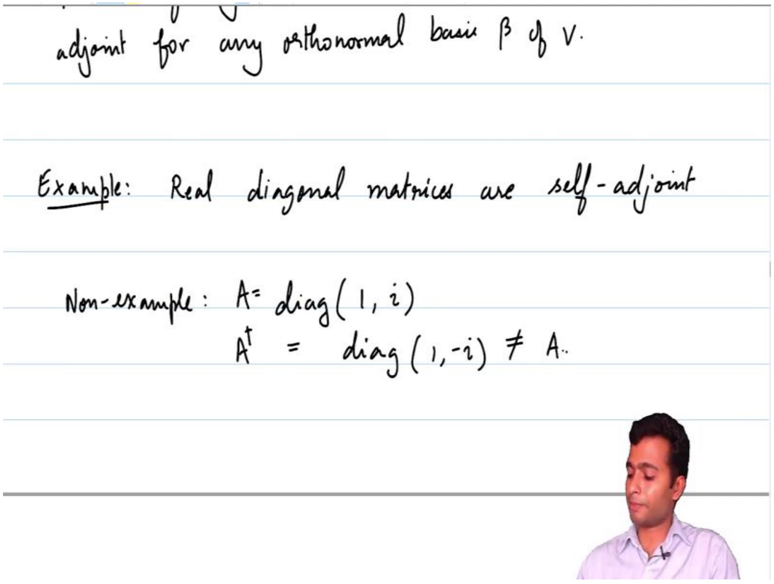

So, the next example, so what was one standard example of a normal operator? Diagonal matrices turned out to be standard examples of normal operators, because if you look at the conjugate transpose of it, it will again turn out to be a diagonal matrix and the product of diagonal matrix will turn out to be a commutative operation. However, we have to be a bit careful when it comes to dealing with diagonal matrices, which will be self adjoint.

So, I will just write it this way, real diagonal matrices are self adjoint. The reason being that if a matrix has real entries, if you look at the conjugate, the entries are going to be preserved. There will be no change in the entries and then if you look at the transpose, again it will not change because it is a diagonal matrix. So, real diagonal matrices are always self adjoint, diagonal matrices with complex entries, where the non real entries are not going to be Self adjoint.

For example, non example again, you look at a diagonal of 1 comma i. And what is going to be the adjoint of A, adjoint of A which will represent the matrix of the linear transformation LA star that is just going to be equal to diagonal allow 1 comma minus i because if you look at the conjugate of i, that is going to be minus i, which is not equal to A. So, in the case of matrices with complex entries, we have to be a bit more careful, the diagonal matrices will not turn out to be self adjoint.

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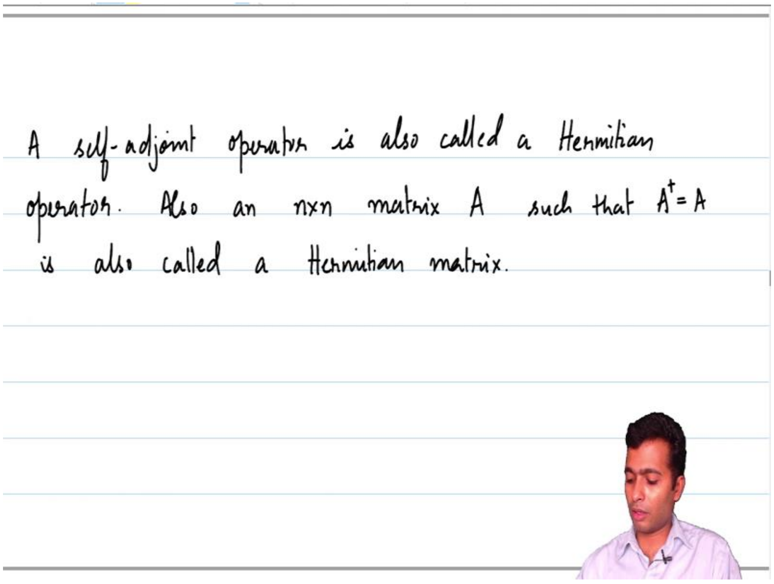

So, there is an alternate term that is used to describe a self adjoint operator. So, let me just write that down. A self adjoint operator is also sometimes called as a Hermitian operator. Self adjoint, this is a term which you can find very, very extensively in literature, a self adjoint operator is also called an Hermitian operator.

So, a matrix whose, which is equal to its adjoint is called a Hermitian matrix, is also matrix, an n cross n matrix A whose adjoint, let me write it like this such that A adjoint is equal to A is also called, it is called a self adjoint matrix, it is also called a Hermitian matrix.

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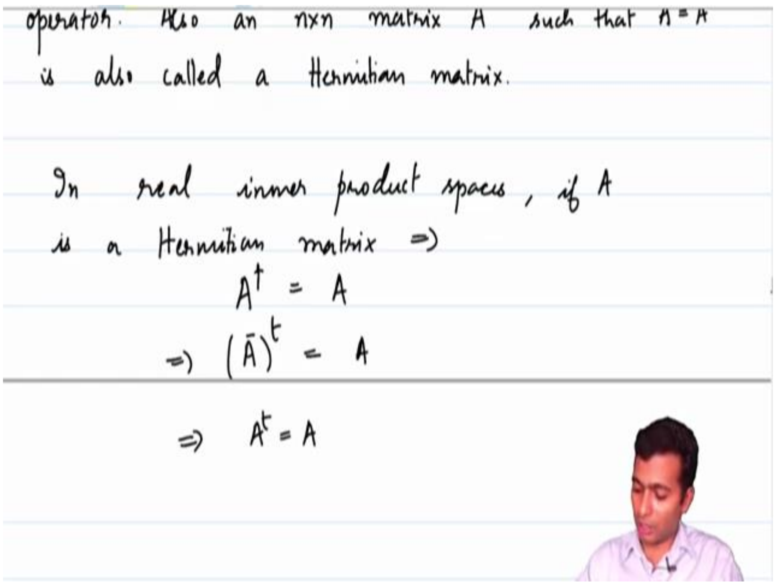

If we are focusing our attention on some real inner product space, then Hermitian matrices are going to boil down to something which we are quite familiar with. So, in real inner product spaces, if you notice what is going to be the conjugate transpose, inner product spaces. If A is Hermitian, A is a Hermitian matrix, that would imply that A adjoint is equal to A. But what is the meaning of A adjoint, A adjoint is the conjugate transpose, this is equal to the matrix A itself. But what is the conjugate in a real inner product space of a matrix.

So, the matrix of the, so the inner real inner product. So, let me just put it, matrix of a linear operator on a real inner product space or a matrix A, a Hermitian matrix A with real entries. That is what this statement means. Should I rewrite it, it is its context is clear I hope it is not confusing. What this statement means is that if the matrix A is an n cross n matrix with real entries or in real inner product space, you consider the consider a linear operator T which is Hermitian or self adjoint and look at the matrix of T with respect to an orthonormal basis.

Then A transpose will be equal to A implies that A, A adjoint is equal to A implies that A bar transpose is equal to A which would imply A transpose is equal to A, because the conjugate of a matrix is going to be the matrix itself.

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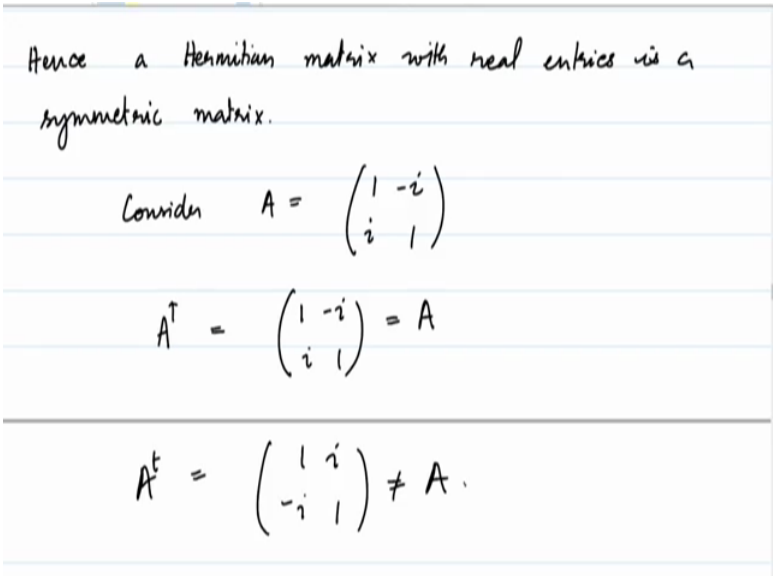

Hence, a Hermitian matrix with real entries is a symmetric matrix. However, when we are dealing with matrices with complex entries, we have to be careful because the 2 notions do not coincide then. So, for example consider A to be equal to, let us say, 1 minus i, i, 1. And let us look at the adjoint of A. The adjoint of A will be the conjugate transpose which is going to be 1 minus i, i, 1, which is equal to, therefore this matrix is actually Hermitian. But if you look at the transpose of A, A transpose is just going to be equal to 1, i minus i 1, which is not equal to A. So, this is a Hermitian matrix which is not symmetric.

So, the distinction of Hermitian matrix and the symmetric matrix is quite stark in the case of matrices with complex entries. However, in the case of real entries, we just noted that a Hermitian matrix is certainly a symmetric matrix as well. Let us now explore some properties of self adjoint operator.

So, we have already seen that the matrix, the linear operator T, the Eigen value of the linear operator T, and the Eigen value of the argument of T are related lambda being an Eigen value of T is true if and only if lambda is an eigenvalue of T star. In the case of self adjoint operators we can say more which will be captured in the next theorem.

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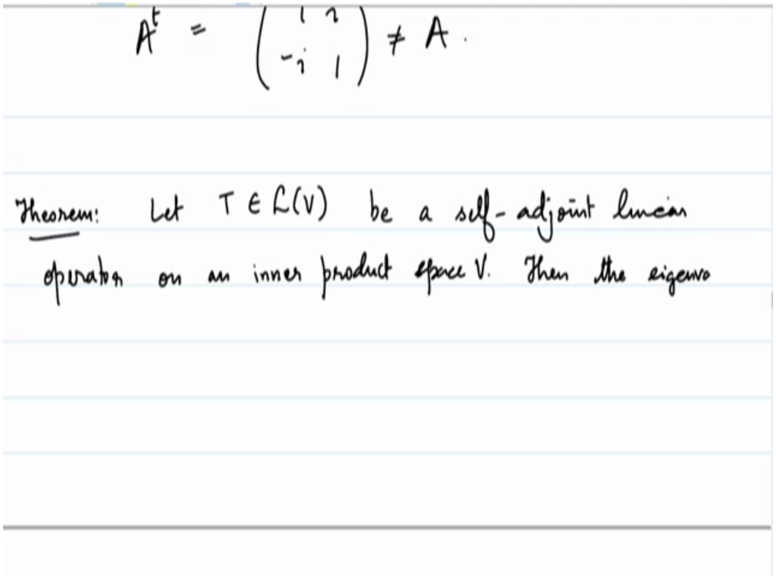

So, let me write down the theorem. Let T be linear operator on V be self adjoint, self adjoint linear operator on an inner product space, then the eigenvalues of T are real. Let us spend a few minutes, let us spend a couple of minutes trying to see what this exactly means. Eigen values of T are real. So, notice that if V is a real inner product space, then this theorem does not say much because the eigenvalues of T are necessarily real in that case.

However, if you look at a complex inner product space, even then, this theorem tells us that even in the case when the field of scalars is complex numbers, if you consider self or adjoint linear operator on V, its eigenvalues should necessarily be real numbers. Of course real numbers are complex numbers. So, there is no contradiction there. It says that it has to be necessarily real.

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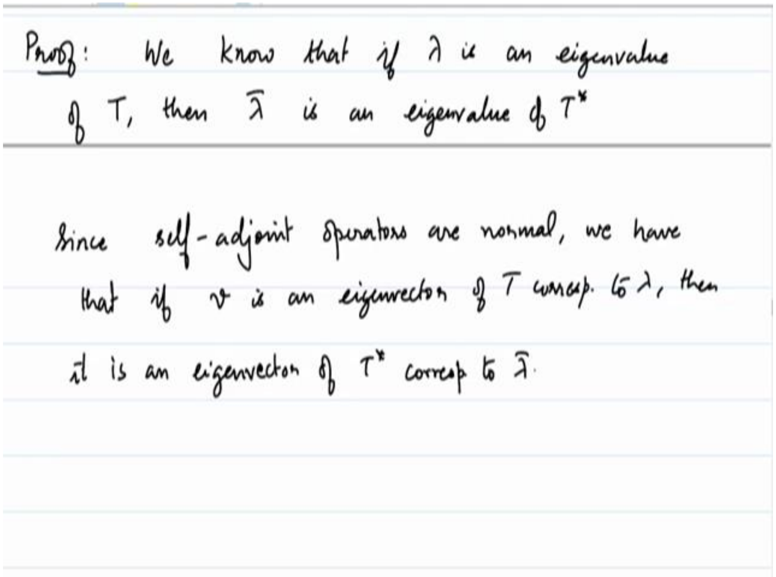

Let us look at the proof of this, the proof is actually quite straightforward and elegant. So, we know that lambda and lambda bar are the if lambda is an eigenvalue. So, we know that if lambda is an Eigen value of T, this is true in the case of every linear operator on a finite dimensional inner product space. Lambda is an eigenvalue of T, then lambda bar is an eigenvalue T. But we also know when t is a normal operator of T star, I said T, but it is T star.

In the case of normal operator, we know more we know that the Eigen value, Eigen vector corresponding to lambda of T will be the same as the say the Eigen vector as an Eigen vector corresponding to lambda of T star. So, since self adjoint operators are normal operators, by one of the theorems proved in the previous lecture, we have that if V is an Eigen vector of T corresponding to lambda then it is an Eigen vector of the adjoint of T corresponding to lambda.

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Hence

$$
\begin{aligned}
\lambda\langle v, v\rangle &= \langle \lambda v, v\rangle \\
&= \langle Tv, v\rangle \\
&= \langle v, T^{*}v\rangle \\
&= \langle v, Tv\rangle \\
&= \langle v, \lambda v\rangle \\
&= \bar{\lambda}\langle v, v\rangle
\end{aligned}
$$

So, let us see, so that means, hence lambda times v comma v, let us look at what lambda comma v comma v is going to be. That is going to be equal to lambda v comma v, which by the definition of an Eigen vector is equal to the inner product of Tv with v. But this is also the same as the inner product of v with the adjoint of T acting on v by the very definition of T adjoint. But since T is self adjoint T star is equal to T.

And that implies that this is just the inner product of v with Tv because T star is equal to T, but this is equal to v comma lambda v. And by the properties of the inner product, this lambda bar times the inner product of v with itself. So, what do we have? We have lambda times the Length of v squared as lambda bar times the length of v square.

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$$
\begin{aligned}
&= \langle v, Tv\rangle \\
&= \langle v, \lambda v\rangle \\
&= \bar{\lambda}\langle v, v\rangle
\end{aligned}
$$

$$\Rightarrow \quad (\lambda - \bar{\lambda})\|v\|^2 = 0$$

$$\Rightarrow \quad (\lambda - \bar{\lambda}) = 0 \quad \Rightarrow \quad \lambda = \bar{\lambda}$$

$\Rightarrow$ λ is a real number.

So, this implies lambda minus lambda times the length of v square is equal to 0. But V is an Eigen vector, hence a nonzero vector and therefore the length of V has to be necessarily nonzero and therefore, the length of v square is also nonzero. This implies that one of the two has to be 0 here because it is a field and therefore, lambda minus lambda bar is equal to 0 which implies that lambda is equal to lambda bar. But what is the meaning of a complex number being equal to its conjugate?

That means that the imaginary part has to be necessarily 0 because if a plus ib is equal to a minus ib, that implies that 2 ib is equal to 0 and for b is equal to 0, this implies that lambda is a real number. That is good, at this time, at this juncture, it might be a good idea to introduce you hyper tool called the fundamental theorem of algebra.

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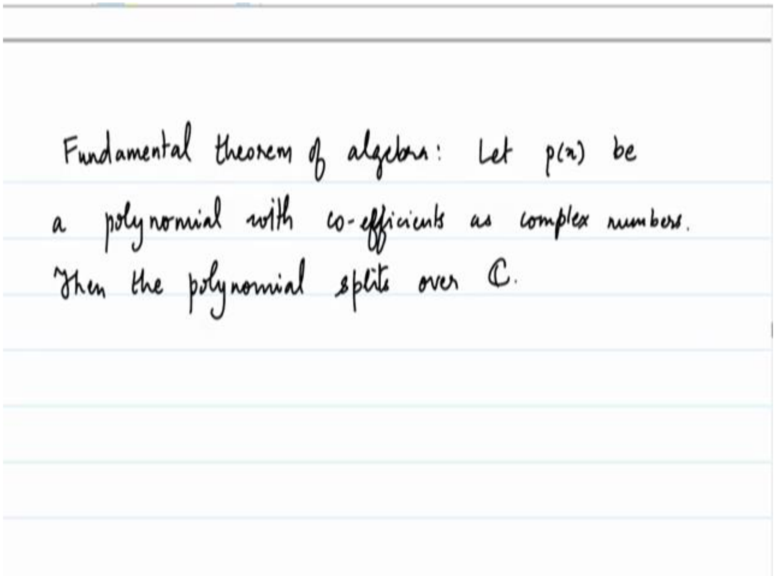

So, let me just state that theorem here fundamental theorem of algebra states that if you look at a polynomial, which has complex coefficients, then it splits in the field of scalars being complex numbers. So, let me just write it down, fundamental thing. That is one version of fundamental theorem of algebra stated differently, but this is more suitable for our purposes. So, let p of x be a polynomial over the field of scalars being complex numbers or to write it differently, be a polynomial with complex coefficients with coefficients in complex numbers, as complex numbers.

So, this is a polynomial over C, then the polynomial, then p of x splits over C. So, notice that if p of x is some a 0 plus a 1 x plus up to a n x to the power n, what this tells us is that p of x will be, it will be possible to write p of x as x minus lambda 1 times x minus lambda 2 upto x minus lambda. Always, any polynomial you take, you will be able to write it as a product of linear factors. That is quite remarkable in our case, because the applications of this is of course vast.

Let us see how it is, how the fundamental theorem of algebra impacts the study of self adjoint operators. So, we already know that okay, I just immediately state down the theorem before elaborating on that. As a consequence of the fundamental theorem of algebra, we will be able to say that the characteristic polynomial of self adjoint operator always splits. So, let me write it down.

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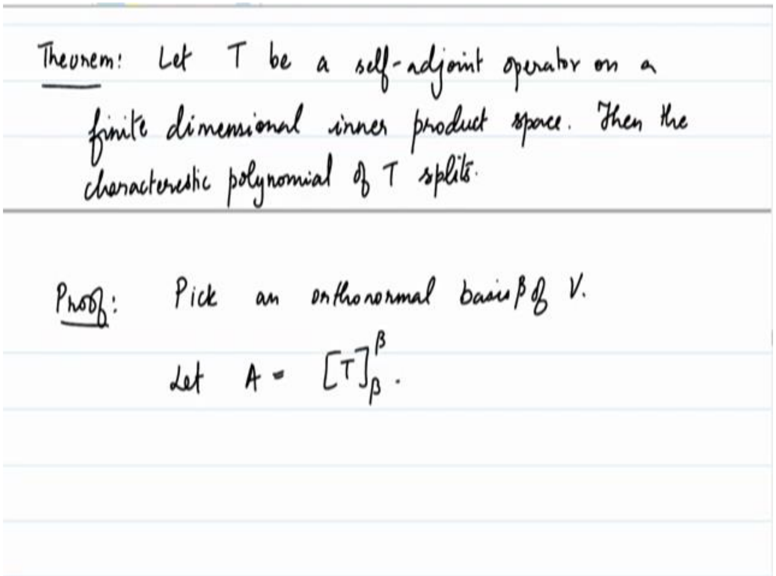

So, let T be a self adjoint operator on an inner product space, self adjoint operator on a finite dimensional inner product space, then the characteristic polynomial of T. So, recall that the characteristic polynomial of linear operator is basically the characteristic polynomial of one of the matrices corresponding to T with respect to some basis. Of course, this is well defined because the characteristic polynomial of similar matrices are going to be the same and therefore, a change of basis matrix will not alter the characteristic polynomial.

And this says, this theorem says that the characteristic polynomial of T splits irrespective of whether it is a real inner product space or a complex non product is true. In the case of a complex inner product space there is not much that the theorem is telling because if you look at a complex inner product space, and if you look at the matrix of T, that will be a matrix which has complex entries. And if you look at the characteristic polynomial by the fundamental theorem of algebra that will be a polynomial.

If you look at the characteristic polynomial it will be a polynomial with coefficients from the field of complex numbers, and by the fundamental theorem of algebra, it splits. This theorem is really telling something substantial when you consider real inner product spaces, because there the characteristic polynomial will be a polynomial with real entries. Of course, it might, you can think of it as a complex polynomial as well, and it might split, but there is no guarantee that it will split over the field of real numbers, it will be splitting over the field of complex numbers.

This theorem tells us that it should necessarily split over the field of real numbers as well in the case of real inner product spaces. So, let us look into it.

So, proof so what we will do is we will immediately pick an orthonormal basis, pick an orthonormal basis of V and let A be equal to the inner product, sorry the matrix of T with respect to, so let us call this orthonormal basis beta and AB, the matrix of T with respect to beta.

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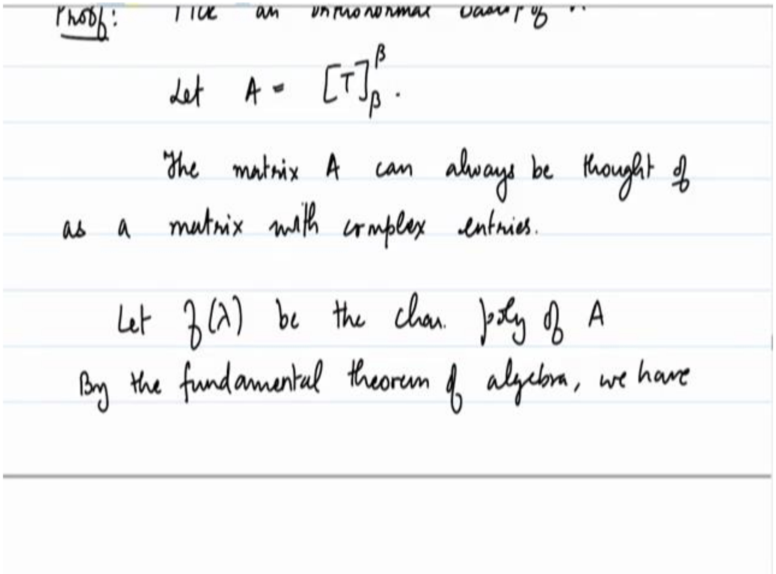

So, we will now forget about T and the characteristic polynomial of T is just the characteristic polynomial of A, we will focus on the characteristic polynomial of A. Now, the matrix A, A is a matrix in our case with either real entries or complex entries, but the matrix A can also be thought of as, if it is a complex inner product space A will be having complex entries, if it is a real inner product space a will have real entries, but nevertheless we will think about it as a matrix in the complex over the complex numbers.

So, the matrix A can always be thought of as a matrix with complex entries, with complex entries. Let us, let f of lambda be the characteristic polynomial of A. And then f of lambda will be by the fundamental theorem of algebra, this is a polynomial of degree n by the fundamental theorem of algebra.

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Since A is a self-adjoint matrix.

$\Rightarrow \lambda_i$ is real $\forall i$.

Hence $f(\lambda)$ splits. ——— $\blacksquare$.

We have f of lambda splits into linear factors. We know what is the degree of f, degree of f is equal to the dimension of V, let us say it is n. That means, it splits into lambda minus lambda 1 into lambda minus lambda 2, up to lambda minus lambda n, where lambda i are the roots of f of lambda. But what do we know about the roots of the characteristic polynomial where lambda i, which are the roots of the characteristic polynomial are the same as the Eigen values of A, lambda i are the eigenvalues of A.

But what do we know about A, we know that is the matrix of t, which is a self adjoint linear operator and by one of the propositions that we wrote earlier, we know that A is self adjoint matrix. Because A is self adjoint, what do we know about the eigenvalues of A. We just proved that the eigenvalues of A are necessarily real. This implies that lambda i is real for all i. But what does that mean? That means that f of lambda would be just lambda minus, it is going to split the real numbers.

Hence irrespective of whether it is over complex numbers or not f of lambda splits. So, the key thing to note is that the characteristic polynomial f always splits even in the case when it is over real numbers by the fundamental theorem of algebra, the fact that lambda 1, lambda 2, up to lambda n are all reals in the case of self adjoint operators tells us that even if it is a real inner product space, this splits.

Okay, that completes the proof. Well, this is an important theorem in the sense that when we were studying diagonalizability, many times we not made as a theorem that we proved wrong it began this way. If you start off with a linear operator, and if it is characteristic polynomial splits, then we say, then we had a characterization of the diagonalizability.

Yeah, so this tells us that in the case of self adjoint operator, it always splits. So, that is maybe one step towards diagonalization. But we will come to that in due course. Let us now focus on some more properties of self adjoint operator. So, let us now look into self adjoint operators on complex inner product spaces. But before that, let us prove one statement of linear operators on complex inner product spaces.

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Proposition: Let V be a complex inner product space & $T \in \mathcal{L}(V)$ such that $\langle Tv, v \rangle = 0 \ \forall v \in V$.
Then $T \equiv 0$.

Note: Consider $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by
$$T(x, y) = (-y, x).$$
Then $\langle Tv, v \rangle = 0 \ \forall v \in \mathbb{R}^2$.

So, proposition so let V be a complex inner product space. And T be a linear operator on V then T be a linear operator on V such that inner product of Tv with V is 0 for all V in capital V. Then this forces T to be equal to identically equal to the 0 linear operator. So, notice that we have imposed a very specific condition of V being a complex inner product space. Because in the case of real inner product spaces, this proposition is not true. So, I will leave it to you to check that, let me just note it note.

In the case of let me just look at this. Consider T to be a linear map from R2 to itself given by T of x comma y being equal to minus of y comma x or x y comma minus x, one of the 2. This linear transformation that satisfies the condition that T of then inner product of Tv with itself with V is always plays 0 for all V in capital in R2. However, T is not the 0 linear transformation here. So, notice that in the case of real inner product spaces, this proposition that we just wrote is not true. This is true for complex inner product spaces.

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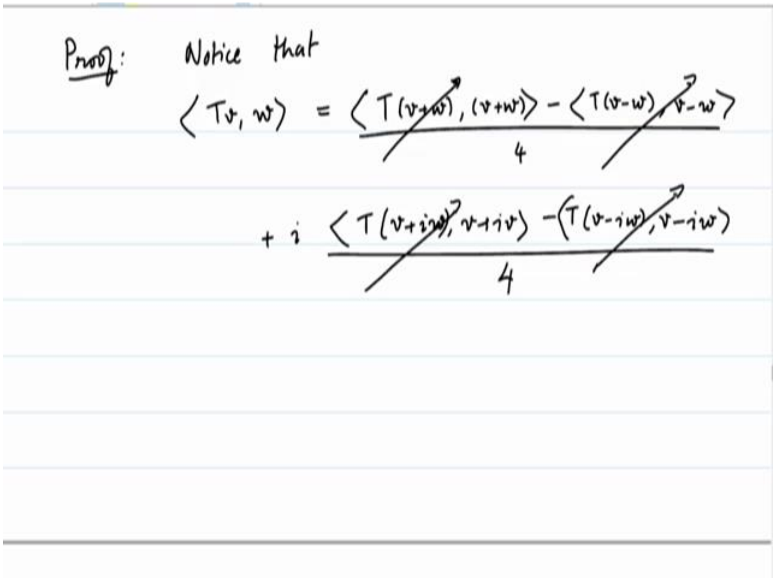

So, the proof of this is going to be an identity which I will not prove, it is very similar to the idea identity which we had proved in one of the problems sessions last week, I will just write down the formula. Notice that it is your job to sit down and check the inner product of Tv with w, this is equal to the inner product of T v plus w comma v plus w minus the inner product of T v minus w, v minus w.

The whole divided by 4 plus i times in our product of Tv plus i w, v plus i w minus b of inner product of T of v minus iw, v minus iw, the whole divided by 4. So, I will suggest that you sit down and work out this identity just like it was done in the problem session earlier. And if you have indeed worked it out, the next step is to notice that every element here this will vanish, this will vanish, this will vanish and this will vanish, because all of them are of the type t of u comma u, the inner product of t of u comma u. And we have just put that as the condition in our hypothesis that T v comma v 0 for all V in capital V.

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$$\Rightarrow \langle Tv, w\rangle = 0 \quad \forall\, v, w \in V.$$

$$\text{For } w = Tv \Rightarrow \langle Tv, Tv\rangle = 0$$

$$\Rightarrow Tv = 0 \quad \forall\, v \in V.$$

$$\Rightarrow T \equiv 0.$$

But that would imply that the inner product of Tv with w is equal to 0 for all v comma w in capital V. But what do we know about the case when w is equal to Tv. For w is equal to Tv this would imply that the inner product of Tv with itself is 0, which would imply that T v is equal to 0 for v. that means E is identically equal to 0. Why did we make this observation this observation is to give you a characterization of self adjoint operators on complex inner product spaces.

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Proposition: Let V be a complex inner product space. Then T is a self-adjoint operator iff $\langle Tv, v\rangle \in \mathbb{R} \quad \forall\, v \in V$.

Proof: T - self-adjoint

So, let me write down that as the next proposition. Again this is true only in the case when V is a complex inner product space let V be a complex inner product space then T is a self adjoint operator if and only if this is a characterization Tv, v, inner product of T v, v is a real number 4v in capital V. Maybe I should remark at this point of time that the self adjoint operators behave like the real numbers behave in the case of complex numbers. So, notice that the conjugate is being captured, the idea of conjugate is being captured by the adjoint.

And what happens when the complex number z is equal to the conjugate of z? z is equal to z bar forces it to be real? So, in some sense, the self adjoint operator captures the idea of real numbers in the world of complex numbers. That is the idea that is being captured by self adjoint operators in the world of operators itself.

So, let us prove this proposition. The proposition here is going to use the previous theorem. The previous theorem let us see. So, what is going to be T, so if is T is suppose T is self adjoint that means that T is equal to T star.

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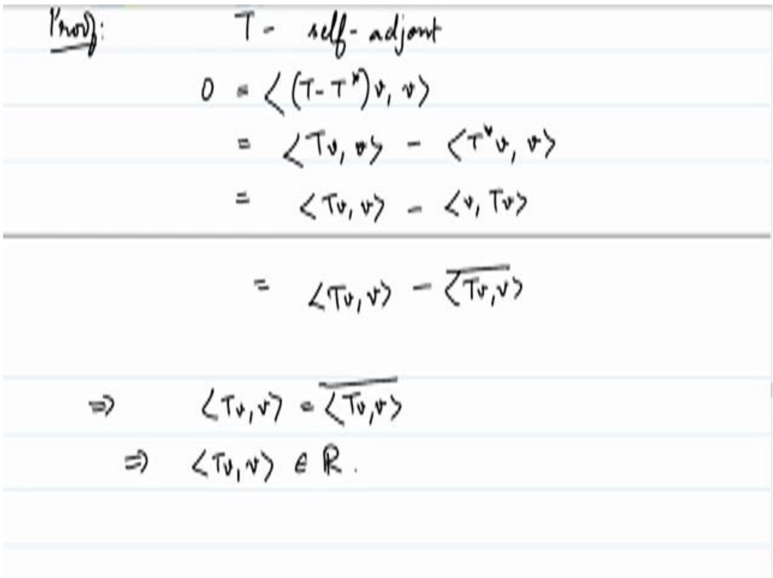

T is equal to T star would imply that T minus T star is 0 and therefore, this is equal to 0. But what is this, this is equal to T v comma v minus T star v comma v and by definition this is equal to T v comma v minus v comma T v. This is equal to T v comma v minus the conjugate of T v comma v. Well I skipped 1 step here, no I did not skip any steps.

So, they are all in place. So, this implies that inner product of Tv with v is equal to the conjugate of inner product of Tv with v. But that implies that this is a real number, right? That is precisely what it means for a complex number to be equal to its conjugate. So, we have pulled one direction.

(Refer Slide Time: 38:34)

$\Rightarrow \quad \langle Tv, v\rangle = \overline{\langle Tv, v\rangle} \quad \forall\ v \in V.$

$\Rightarrow \quad \langle Tv, v\rangle \in \mathbb{R}. \quad \forall\ v \in V.$

Now suppose $\langle Tv, v\rangle \in \mathbb{R} \quad \forall\ v \in V.$

Then by going in opposite direction above,

we get $\langle (T - T^*)v, v\rangle = 0 \quad \forall\ v \in V$

By

What happens if we want to go in the other direction suppose t v v is equal to R for all v, so this is for all v in capital V, so this is also for all v in capital V. And if you want to go in the other direction, suppose Tv, inner product of Tv v is in R, and that would imply that the inner product of t v v is equal to the conjugate, that would imply that this is equal to 0. And that would imply that T minus T star v is equal to okay. So let me just not use the colors. I will just write here that.

Now, suppose Tv v is an element of R for all v in capital V, then by going in the opposite direction above, we get that T minus T star of v comma v is equal to 0 for all v in capital V. But we are in the setup of a complex inner product space.

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we get $\langle (T - T^*)v, v\rangle = 0 \quad \forall\ v \in V$

By by the proposition, $T - T^* = 0$

$\Rightarrow$ T is self-adjoint.

By the previous proposition if any linear operator satisfies the condition that you know t comma V is equal to 0, then the linear operator should be 0, by the previous position, in this case, our T is just replaced by T minus T star, T minus T star is equal to 0, which implies that T is self adjoint.

All right. So, that is one characterization of self adjoint linear operators in complex inner product spaces. Well that is an analogue of this proposition in the case of real inner product spaces. So, we saw that in general this is not true. Let us look at what could be a analog in the case of rare inner product spaces which I will write down in the next proposition.

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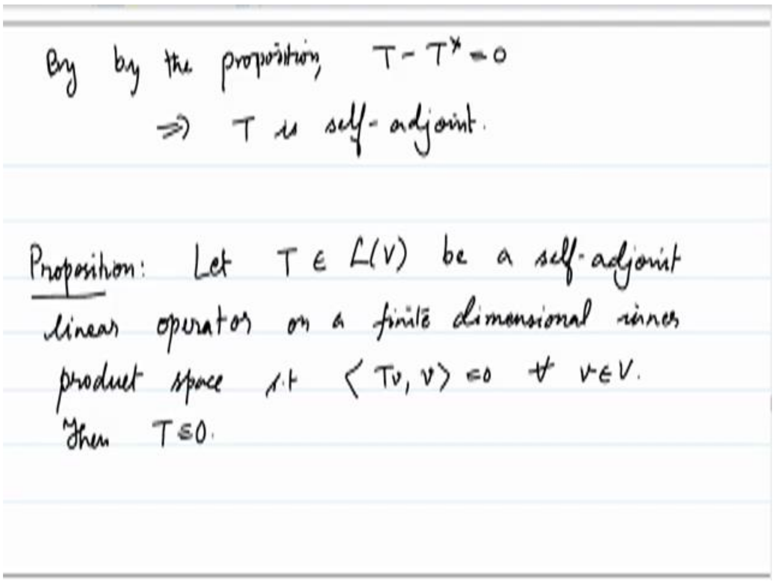

So, let T be a self adjoint linear operator on a finite dimension inner product space, such that T v, v in a product is 0 for all v in capital V, then T is identically equal to 0. So, notice that this proposition has already been proven in the most general setting in the case of complex inner product spaces. We do not need to impose extra condition that T is self adjoint. In the real inner product space, in the case of real inner product spaces, T being a self adjoint is a necessary condition to, it is a sufficient condition to say that T v, v is equal to 0 for all v for this T to be equal to 0.

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product space s.t. $\langle Tv, v\rangle = 0 \;\; \forall\, v \in V$.
Then $T \equiv 0$.

Proof: Assume that V is a Real inner product space. Then

$$\langle Tv, w\rangle = \frac{\cancelto{0}{\langle T(v+w), v+w\rangle} - \cancelto{0}{\langle T(v-w), v-w\rangle}}{4}$$

$$\Rightarrow \langle Tv, w\rangle = 0 \quad \forall\, v, w \in V$$

$$\Rightarrow T \equiv 0. \qquad \blacksquare$$

Let us give a proof of this, we will assume it without loss of generality, that V is a real inner product space because in the case of complex inner product spaces, we have already done this. So, assume that V is a real inner product space, then I will leave it as an exercise for you to check that, then you can check that, Tv comma w, they are just going to be equal to the product of T of v plus w, v plus w minus T of v minus w, v minus w by able 4. By using this self adjointness of T, this will follow.

And the 2 quantities on top, both are going to be 0 because that is of the type T v, v. And that would imply that Tv inner product with w is 0 for all v, w in capital V. And by have very similar argument as earlier, this concludes that T is identically equal to be 0.

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$$\Rightarrow T \equiv 0. \qquad \blacksquare$$

Problem 1: Let S, T be self-adjoint linear operators. Then ST is self-adjoint iff $ST = TS$.

Proof:

So, let us prove maybe 1 or 2 problems. So, let S, T be self adjoint linear operators. Then the composition ST is self adjoint, okay, where on a finite dimension inner product space, let me not write it down again. ST is self adjoint if and only if S and T commute, so let us see. So, this is a proof, so suppose ST is self adjoint.

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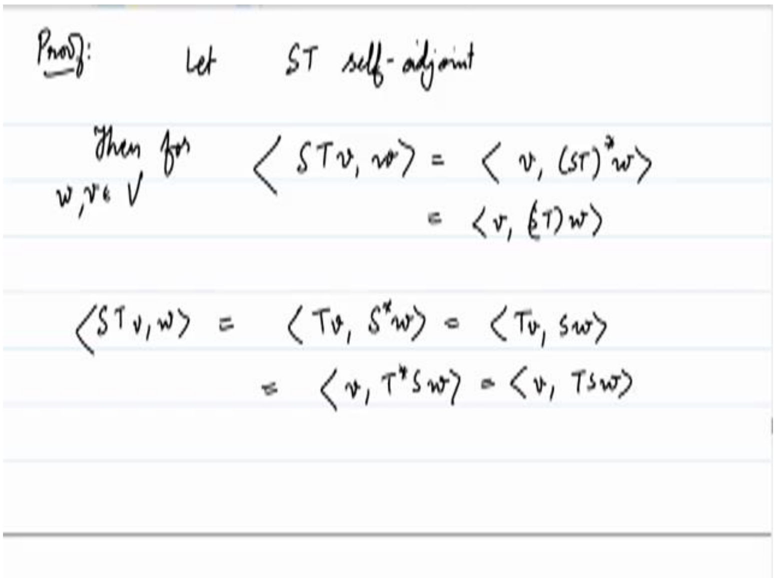

Okay, let us start with ST be self adjoint. Then what is going to be my inner product of ST v with w and for v comma w in capital V that struck to figure out what this is. By definition this is going to be equal to v comma ST star of w. We just assumed that ST is self adjoint, and therefore this is going to be equal to v, ST of w. But what is ST v, w, if you consider it one by one, a map of, a map of T acting on v and there is a map S acting on v. This is just going to be an inner product of Tv with a joint of S acting on w. Which is equal to Tv, Sw, since S is self adjoint. But this is also the same as v, T star Sw, which is the same as v, TS of w.

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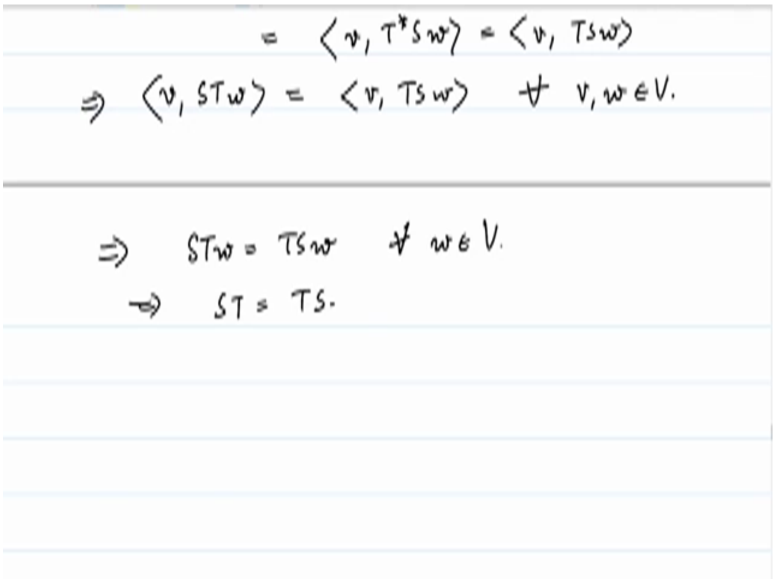

That would imply that v, ST w is equal to v comma TS w for all v comma w in capital V. I am not talking about how we are able to conclude that ST w is equal to TS w for all w, it follows by the uniqueness that this argument have been done many times. And therefore from this we will be able to conclude that this is the case. Which implies that ST is equal to TS.

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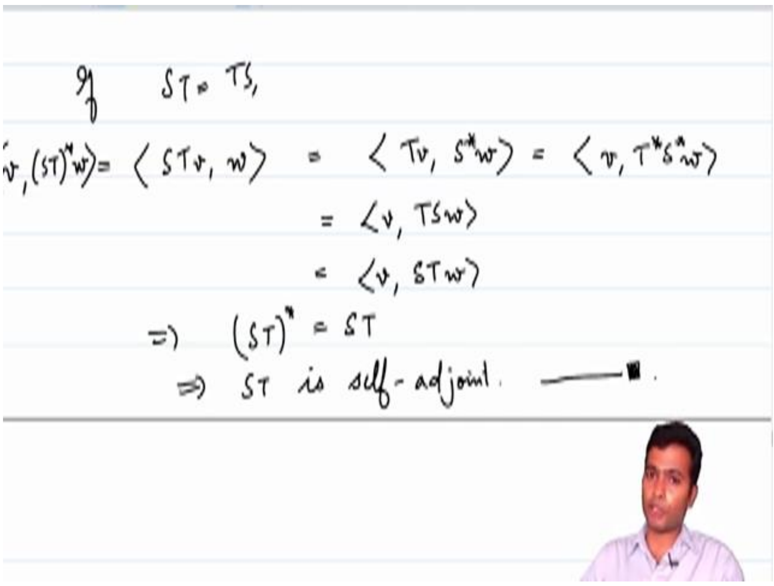

So, what is going to be ST v, w, this is just going to be equal to Tv, S star w, which is equal to v, T star S star w, which is equal to v, TS w because both T and S are self adjoint. But TS is equal to ST will imply that this is a inner product of v with ST w and what is this, this is just by definition inner product of v with ST star w.

This would imply that ST star is equal to ST, which implies that ST is self adjoint. That completes above proof. Okay, so let me stop here, in the next video we will discuss what is called as the spectral theorem of normal and self adjoint operators.