Linear Algebra
Professor Pranav Haridas
Kerala School of Mathematics, Kozhikode
Lecture 46
Normal Operator

In the final week of this course, we will be exploring the notion of diagonalizability a bit further. So, if you recall from the material in the eighth week of this course, we had defined a linear operator on a vector space to be on a finite dimensional vector space to be diagonalizable if we can get hold of a basis consisting of eigenvectors of T.

In the setting of an inner product space, we can see a little more about diagonalizability of certain special type of operators. So, in this week, we will explore some of these special type of operators, and also study the property of diagonalizability of these operators. So, let us begin by defining, what is normal operator on a finite dimensional inner product space.

So, let us begin by inserting the case when a linear operator T, when it is given on an inner product space or finite dimensional inner product space, suppose we have an orthonormal basis consisting of eigenvectors of T, let us consider that particular scenario.

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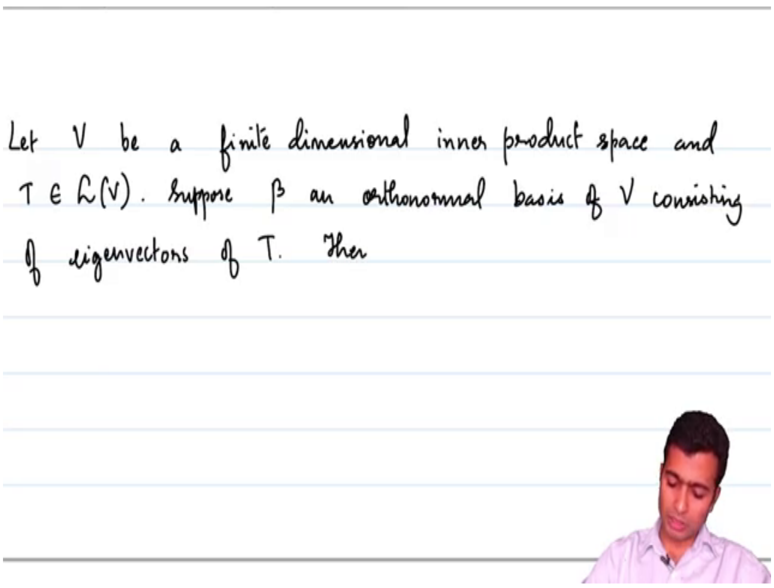

So, the setup is this, let V be a finite dimensional inner product space and we have a linear operator T and let me start using this notation L of V which is basically the space of all linear operators on V linear transformations from V to itself. L of T be in L of V, consider suppose we get hold on an orthonormal basis Beta and orthonormal basis of Beta bases of V consisting of Eigen vectors of T, this is amongst the best possible setup we can think of eigenvectors of T. So, not only is T diagnosable, but we can also get hold of bases which consists of Eigen vectors which are orthonormal rather than orthogonal then, then let us see what happens then, let us consider the matrix of T.

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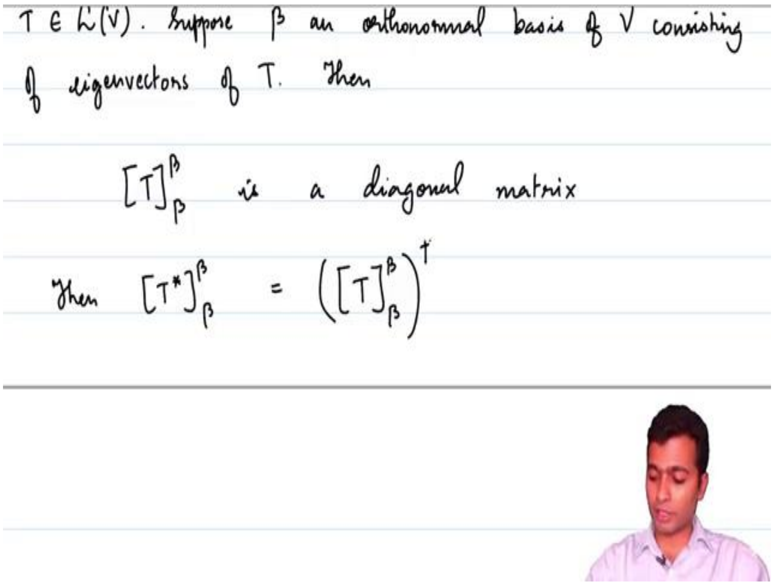

This is a diagonal matrix that is something which we know for sure. It is already a diagonal matrix without it being an orthonormal basis, in particular here it is an orthonormal basis. But the moment we are in an inner product in a finite dimensional inner product space and we have an orthonormal basis and if we look at the matrix of T with respect to this particular orthonormal basis, we also know what is the matrix of T with respect to the matrix of T star with respect to the orthonormal basis.

Then by whatever we have done in the previous week, the matrix of T star with respect to the basis Beta, this is nothing but the advent of the matrix of T with respect to Beta, this is equal to the matrix of T with respect to Beta adjoint, recall that the adjoint of a matrix is the conjugate transpose. But we know that T Beta here in this case is a diagonal matrix. So, what can we say about the conjugate transpose of the matrix we again know that this is a diagonal matrix.

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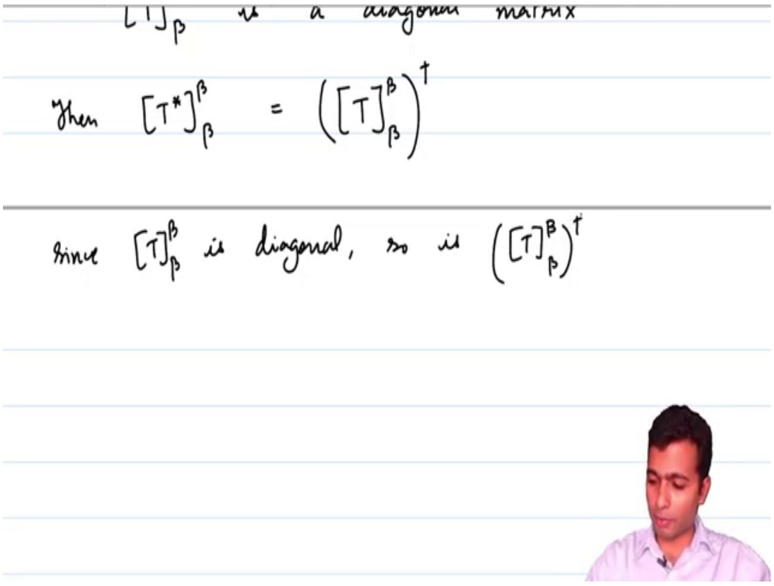

Since T Beta Beta is diagonal so is the adjoint of that and we know that diagonal matrices commute when we multiply.

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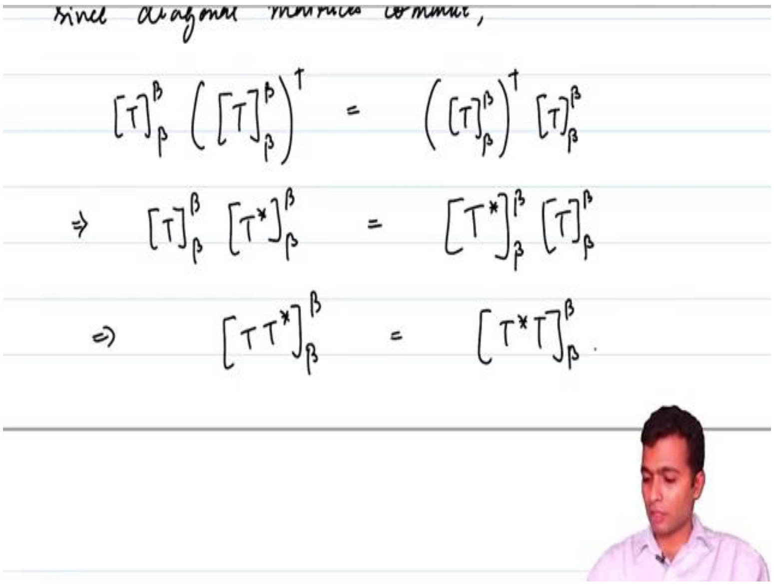

So, since diagonal matrices commute, M be a set to commute if A B is equal to B A commutativity since if the T Beta Beta is diagonal matrix so, is T Beta with a adjoint and hence, T Beta Beta times T Beta Beta adjoint is equal to the adjoint of matrix T Beta Beta times the matrix of T Beta Beta.

Now, we will invoke whatever we have just observed from the previous week, we know that the adjoint of the matrix of T with respect to Beta is the matrix of the adjoint. So, let us use that and conclude here that this is okay. Let me just read one more step this is just telling us that the Beta Beta times T star Beta Beta is equal to T star Beta Beta the adjoint of times the matrix of T itself. But this also implies that the matrix of T T star with respect to Beta is equal to the matrix of T star T with respect to Beta.

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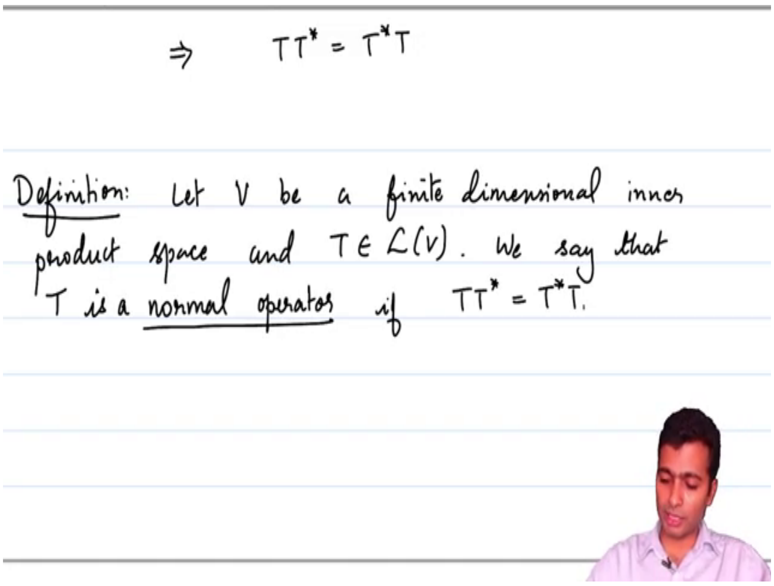

So what it tells us is that with respect to this orthonormal basis, the matrix of T is an diagonal then n star commutes. And as it turns out, this is the right notion to look at, to study the diagonalizability of operators to study thesis. So, that is a notion of the diagonalizability of operators which satisfies this particular condition, namely T star is equal to this, further this also implies that T T star is equal to T star T.

So, this is exactly what is called as a normal operator. So, let me now get give you a definition. So, we just saw that if the linear transformation linear operator on V has a basis orthonormal basis consisting of rather if we have an orthonormal basis consisting of Eigen vectors of a linear operator T, then the matrix of T is going to commute with the matrix of T star and from that we concluded that T T star is equal to T star times T, so, that is what is called as normality normal operators.

So, let V be finite dimensional inner product space and the setup was just as we defined and T be a linear operator on V. So, it is a finite dimensional inner product space and T is a linear operator on V, we say that T is normal. So, let me write the word underline it to stress upon the fact that we are defining this we say that is normal, T is a normal operator so it must be then both this, if T T star is equal to T star T, what is that, is a linear operator. So, it is a linear transformation from V to itself and therefore, the adjoint is also from V to itself. So, T T star makes sense. And we demand further that T T star is equal to T star T.

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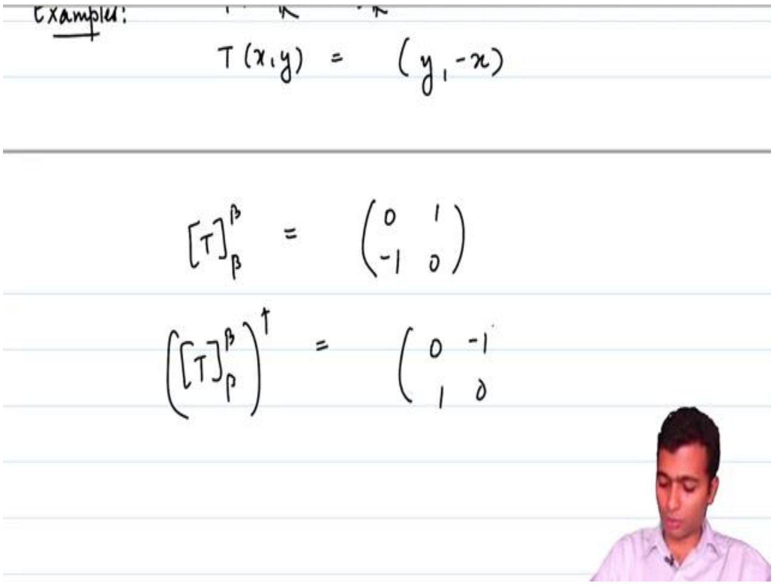

Let us look at a few examples. Let T be from say R 2 to R 2, and let us look at E of say x comma y to be equal to minus of maybe y comma minus of x. Let us look at the map of x comma y 1 2 y comma minus of x, so let me just quickly compute the matrix of T with respect to the orthonormal basis consisting of the standard vectors that will be equal to (())(9:02) 10. T of 1 0 will be 0 minus 1, so this is going to be 0 minus 1. And how about 0 1? T of 0 1 will be 1 comma 0. And how about T Beta, Beta adjoint? That is just meant to be equal to 0, minus 1, 1, 0.

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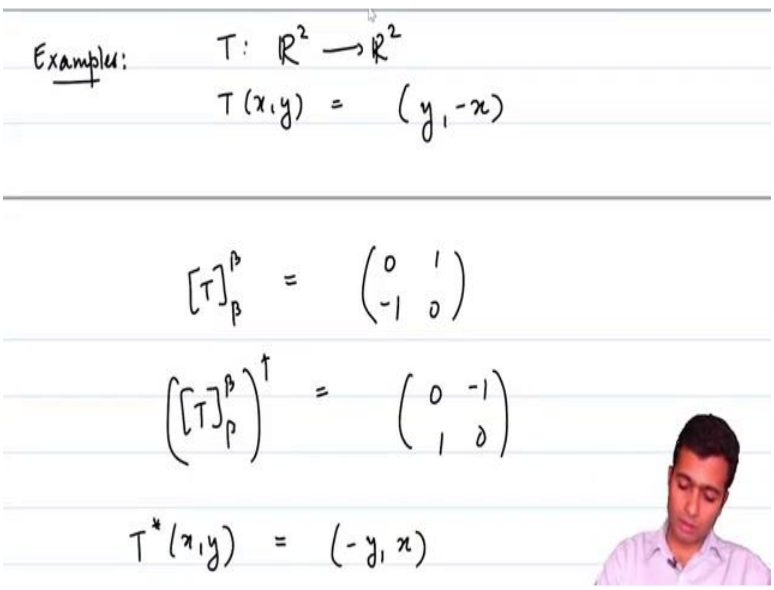

And if you look at what is T star of x comma y is, that will just be multiplication by this matrix. All this is basically things which we are borrowing from the previous week's lectures. So, this is going to be x comma y is minus y comma x. So, if you notice, we started off with T of x comma y being equal to y comma minus x we have ended up with T star of x comma y being equal to minus of y comma x so, let us see what is T start of x comma y.

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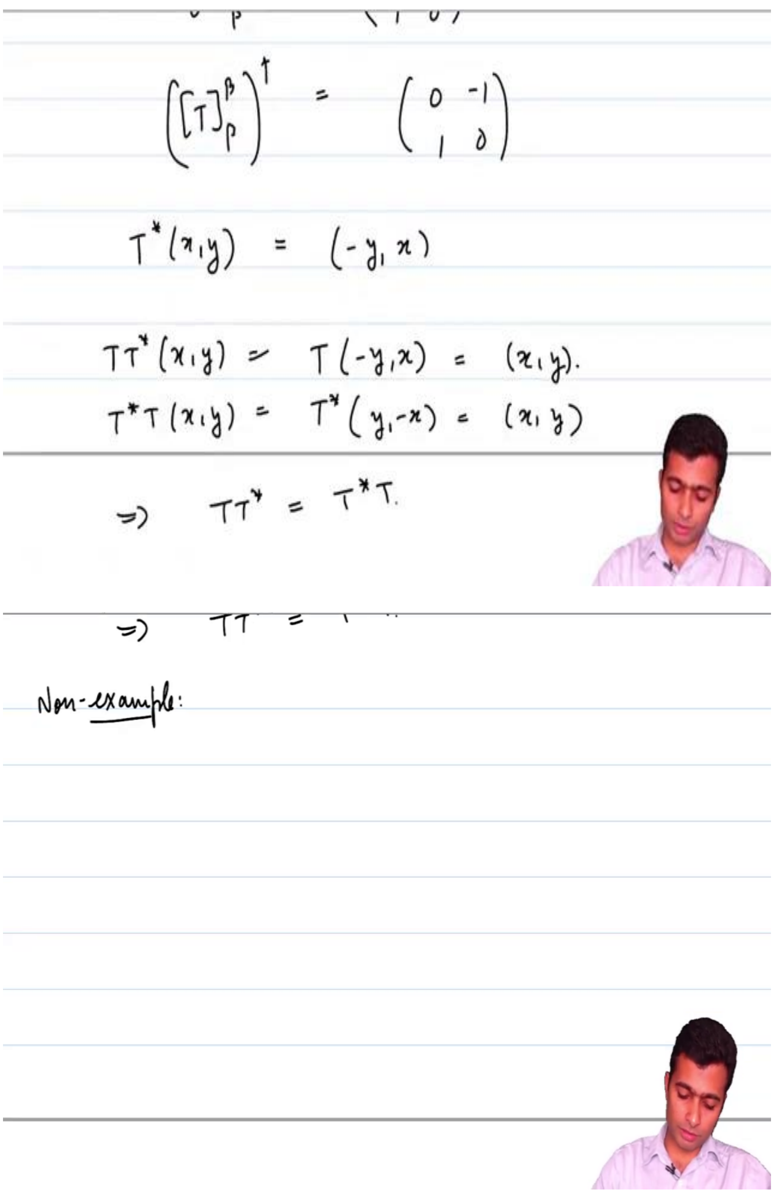

T star of x comma y is equal to T of minus of y comma x, which is equal to x comma y. And what is T star T of x comma y? T star T of x comma y is T star of y comma minus x, if I am not mistaken, that is what our definition of T of x y is yes. And this again by definition is going to be equal to x comma y because T star of x comma y is minus of y comma x. So, this is this implies that T star in this case is equal to T star. So, this is an example of a linear operator which is normal, a non-example that is also an order.

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$$T\,T(x,y) \quad T(y,-x) = (x,y)$$

$$\Rightarrow \quad TT^* = T^*T.$$

Non-example:

$$T(x,y) = (0,x)$$

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left([T]_\beta^\beta\right)^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T^*(x,y) = (y,0)$$

So, let us look at T of x, y being equal to say 0 comma x, instead of minus y comma x let us consider this and let us see what is T Beta Beta with respect to the standard basis, the orthonormal basis consisting of T standard basis vectors. This will just turn out to be equal to T of 1 comma 0 is just 0 comma 1 and then there is this. How about T Beta Beta adjoint?

This is just meant to be called a 0 1 0 0. And what does that mean? That means that T star of x comma y is equal to y comma 0. But let us see what is T star of x comma y. This is equal to T of y comma 0, remember if x comma y is 0 comma x. So, this is just going to be equal to 0 comma y. Yeah that is right. So, let me be a little careful here, y comma 0 is into 0 comma y.

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$$TT^*(x,y) = T(y,0) = (0,y)$$

$$T^*T(x,y) = T^*(0,x) = (x,0)$$

Hence T is not norm

How about T star T of x comma y, this just will be T star of x comma y is 0 comma x, which is equal to now x comma 0. So, observe that for say, 1 comma 1 T star is not equal to T star T, hence T is not normal. In this case, we should actually also look at the fact that there are involved. So, there can be an analogous notion of normality that can be defined for matrices so let me give that definition here as well.

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Hence T is not normal.

Definition: Let A be an $n \times n$ matrix. We say that A is a normal matrix if $AA^\dagger = A^\dagger A$.

$$\boxed{[T]_\beta^\beta = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}}$$

$$\left([T]_\beta^\beta\right)^\dagger = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$T^*(x,y) = (-y,x)$$

$$TT^*(x,y) = T(-y,x) = (x,y).$$

$$T^*T(x,y) = T^*(y,-x) = (x,y)$$

Let A be an n cross n matrix we say that is a normal matrix A adjoint times A. So, we say that this is a normal matrix if A A adjoint is equal to A adjoint times A. So, I will not go back, go and check for you this is this particular matrix which now I am boxing in green is just going to turn out to be a normal matrix. But needless to say that is a strong relation between normal matrices and normal operators, we will capture that in the next proposition.

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dimensional inner product space. Let β be an orthonormal basis of T. Then T is a normal linear operator iff $[T]_\beta^\beta$ is a normal matrix.

Proof: $(\Rightarrow)$ Since T is normal

$$TT^* = T^*T$$

$$[TT^*]_\beta^\beta = [T^*T]_\beta$$

So, the setup is going to be the same let T be a linear operator a finite dimensional inner product space. So, let T be a linear operator and a finite dimensional inner space. And suppose, we have a orthonormal basis some orthonormal basis, so let Beta be an orthonormal basis of T, then T is a normal linear operator if and only if T Beta Beta is a normal matrix.

So, the case of Beta being an orthonormal basis consisting of eigenvectors of T, we had already checked that this is going to be a normal operator but this is true in the general case as well. So, let us give a proof of this and establish this. So, what do we know about T? We know that T is a normal operator since T is normal that is the forward direction since T is normal. Let me just put an arrow here as of now, let T be a normal operator.

We have that T T star is equal to T star T that is the very definition of T being normal. Now, let us look at the matrix of T T star with respect to Beta that will be equal to star T with respect to Beta.

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$$\Rightarrow \quad TT^* = T^*T$$

$$\Leftrightarrow \quad [TT^*]_\beta^\beta = [T^*T]_\beta^\beta$$

$$\Leftrightarrow \quad [T]_\beta^\beta [T^*]_\beta^\beta = [T^*]_\beta^\beta [T]_\beta^\beta$$

$$\Leftrightarrow \quad [T]_\beta^\beta \left([T]_\beta^\beta\right)^\dagger = \left([T]_\beta^\beta\right)^\dagger [T]_\beta^\beta$$

$$\Leftrightarrow \quad [T]_\beta^\beta \text{ is a normal matrix}$$

And we know that so, let me put implications here. And this implies in particular that the matrix of T Beta Beta will be equal to the matrix of times the T star matrix of T star with respect to Beta will be the matrix of star with respect to Beta times the matrix of T with respect to Beta. But Beta is now an orthonormal basis that is where the feature of inner product spaces come into the picture. Because it is an orthonormal basis we know what the definition what will be the matrix of the adjoint, it is just going to be the adjoint of the matrix of T by one of the theorems we have proved in the previous week.

So, this is just going to be equal to T star T or maybe I should be a bit careful, this is T Beta Beta adjoint times the matrix of T with respect to Beta. But it just tells us that T Beta Beta is a normal matrix. So, yes we have proved one direction we have proved that if T is a normal linear operator, then T Beta Beta is a normal matrix so, now let us prove the converse.

To prove the converse I will just run you through the steps and notice for you that it T is a normal matrix this implies that this particular identity is satisfied and if this is satisfied this implies the because it is a orthonormal basis, the adjoint of the ethics of is the matrix of the adjoint.

And this implies that T star of Beta Beta is the same as T star T of with respect to Beta Beta and that implies that T T star is equal to T star T as linear operators which implies that T is normal so, let me just change it a bit. So, let me just say that, T is normal we will imply all this up till here and back that way we have established this.

Normal operators have some nice properties, let us discuss some of them we will first discuss the eigenvectors eigenvalues and eigenvectors of Normal operators. So, in the last week, we have already seen that if lambda is an eigenvalue of T, then lambda bar is an eigenvalue of T star. But we had no clue about what the Eigen vectors of lambda bar would be of T star would be corresponding to lambda bar.

As it turns out there is a good chance that if V is an Eigen vector corresponding to lambda of T, and V need not be an eigenvector of lambda bar corresponding to T star. However, in the case of normal operators that is first so, that is the content of our next theorem. So, let me just write down the theorem for you.

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Theorem: Let T be a linear operator on a finite dimensional inner product space. Then v is an eigenvector of T corresponding to the eigenvalue λ iff v is an eigenvector of T^* corresponding to $\bar{\lambda}$.

So, again let T be a linear operator on a finite dimensional inner product space then V is an eigenvector of T corresponding to the eigenvalue lambda if and only if V is an eigenvector of T star corresponding to lambda bar. So, we already know that lambda has to be an eigenvalue of T star. But this theorem tells us that the same Eigen vector works for lambda bar as well. So then, if and only if V is an eigenvector of T star corresponding to lambda bar.

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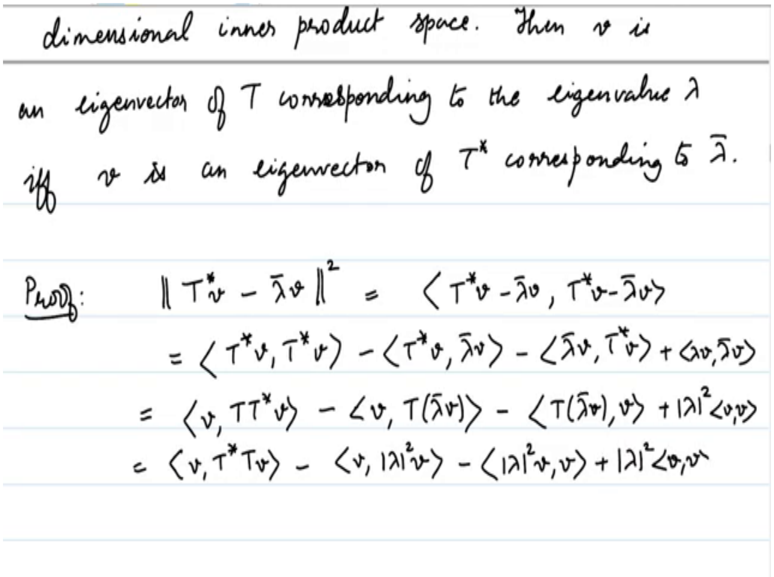

The proof is actually quite straightforward, we are going to use the properties of normal operator quite extensively. So, let us look at T star v minus lambda bar v norm of this, we will prove that this is equal to 0. Rather we will prove that the square is the length of the square of the length of T star v minus lambda bar v.

But that is just equal to the inner product of T star v minus lambda bar v with itself and this is equal to the inner product of T star v with T star v. Let us see minus inner product of T star v comma lambda bar v minus inner product of lambda bar v with T star v. Finally, this is going to be inner product of lambda v lambda bar v, this is the exact expression that we will have. But by the properties of the edge, this is just the inner product of v with T star v.

And how about the second term, the second term will be the inner product of v with T of lambda bar v which is equal to I will come to that. So, this minus similarly, T of lambda bar v comma v plus mod lambda square times inner product of v with itself. Now, T is a normal operator this implies that T star is equal to T star T, this is equal to hence v comma T star T v. How about yes, T of lambda bar V is lambda bar times of T of V which is equal to, v comma mod lambda square times v minus yet again mod lambda square times v comma v plus mod lambda square times v comma v.

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iff v is an eigenvector of T^* corresponding to $\bar{\lambda}$.

Proof: $\| T^*v - \bar{\lambda} v \|^2 = \langle T^*v - \bar{\lambda}v, T^*v - \bar{\lambda}v \rangle$

$$= \langle T^*v, T^*v \rangle - \langle T^*v, \bar{\lambda}v \rangle - \langle \bar{\lambda}v, T^*v \rangle + \langle \bar{\lambda}v, \bar{\lambda}v \rangle$$

$$= \langle v, TT^*v \rangle - \langle v, T(\bar{\lambda}v) \rangle - \langle T(\bar{\lambda}v), v \rangle + |\lambda|^2 \langle v, v \rangle$$

$$= \langle v, T^*Tv \rangle - \langle v, |\lambda|^2 v \rangle - \cancel{\langle |\lambda|^2 v, v \rangle} + \cancel{|\lambda|^2 \langle v, v \rangle}$$

$$\langle Tv, Tv \rangle - |\lambda|^2 \langle v, v \rangle$$

$$= \langle \lambda v, \lambda v \rangle - |\lambda|^2 \langle v, v \rangle$$

$$= |\lambda|^2 \langle v, v \rangle - |\lambda|^2 \langle v, v \rangle = 0$$

Now, let us look at it a little more carefully, till mod lambda square will come out here that will cancel with the last. Finally, what we will be left is v T star T v minus inner product of v mod lambda square v. So, the second term is just going to be equal to mod lambda square of inner product with v with itself.

And how about the first term, the first term this is going to be T v with itself inner product of T v with itself because this is the definition of the adjoint. Recall that the adjoint of T is T itself so that is what we have used here. And we will now use the fact that this is lambda v and again lambda v because v is an eigenvalue of T minus mod lambda square times inner product of v with itself.

But this is equal to lambda times lambda bar which is mod lambda square times the inner product of v with itself minus mod lambda square the inner product of v with itself, and which is equal to 0. So, what had we started off with? We had started off with the length of T star v minus lambda bar V, and we have just established that at object has to be necessarily equal to 0.

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$$= \langle \lambda v, \lambda v \rangle - |\lambda|^2 \langle v, v \rangle$$

$$= |\lambda|^2 \langle v, v \rangle - |\lambda|^2 \langle v, v \rangle = 0$$

Hence $\| T^* v - \bar{\lambda} v \| = 0 \Rightarrow T^* v = \bar{\lambda} v.$

Converse: exercise.

$\blacksquare$

Hence length of T star v minus lambda bar v is equal to 0, which implies that T star v minus lambda bar is equal to 0, the 0 vector, which is equal to hence implies that T star v is equal to lambda bar v that establishes the forward directions, we have assumed that v is an eigenvector corresponding to lambda of T.

And we have established that lambda bar v is an eigenvector of lambda bar corresponding lambda bar of the adjoint of T. The converse is also quite similar, I will not prove that, I will just leave it as an exercise because the proof is exactly the same. You could also use the fact that the adjoint of T is T and then use whatever we have used whatever we have just proved you can apply it to T star and T star star that is also another way of looking at it anyway.

So, the converse also follows so I will just leave that as an exercise, Converse as an exercise for you either way, simple observations. And with that, we have established that the Eigen vector of T will also be an eigenvector of T star. In fact, the Eigen vectors of normal operators behave in a very special manner.

So, in the next proposition we will capture exactly that, we will prove that if you have Eigen vectors corresponding to distinct Eigen values, we have already seen that they are linearly independent right. We will now show that if T is a normal operator then the distinct Eigen value Eigen vectors corresponding to distinct eigenvalues they will also turn out to be orthogonal to each other. So, we can yeah so, let us see that that is not generally the case only in the case of normal operators can we say that. If you look at a normal operator not a normal operator on a finite dimensional product space we cannot conclude this.

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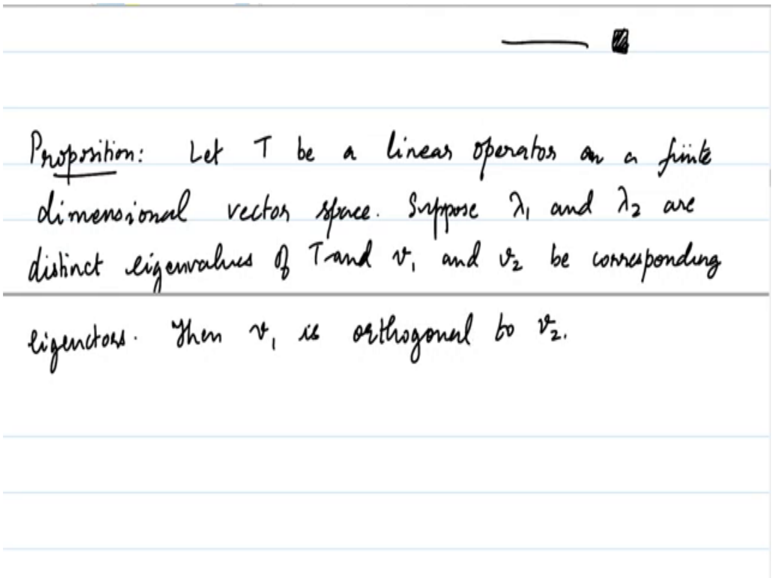

So, proposition which captures what I just said. So, let me just put it up I should slowly maybe I should I just mention that in the entire lecture, T will be linear operator on a finite dimensional vector space then T is… suppose lambda 1 and lambda 2 distinct eigenvalues of T and v 1 comma v 2 be corresponding Eigen vectors, v 1 and v 2 be responding Eigen vectors so they are Eigen vectors corresponding to distinct Eigen values, then v 1 is orthogonal to v 2.

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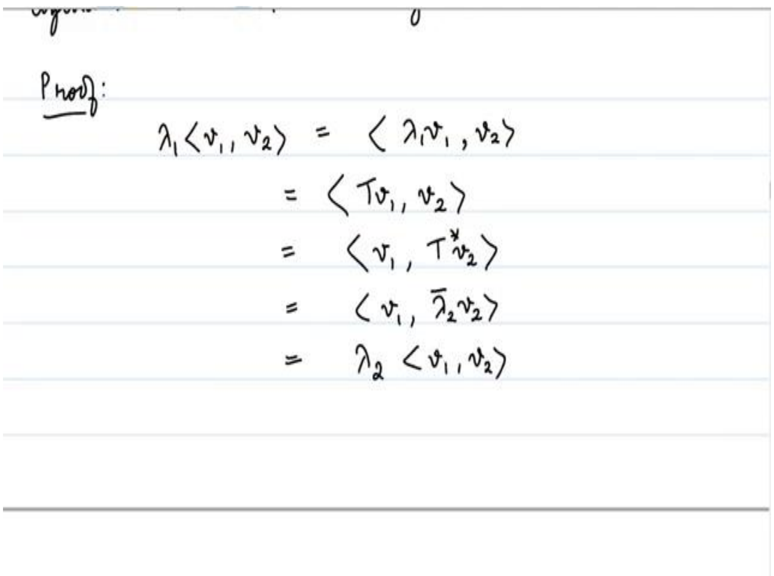

So, let us see, let us look at a proof of this particular proposition. So, what do we know about? Let us see, what is T v 1 comma v 2? We know that this is equal to or maybe I should start like this, lambda 1 times v 1 comma v 2, if you look at this particular number, this is after our scalar right? So this is equal to the inner product of lambda 1 times v 1 comma v 2, by the very basic properties of the inner product.

Now, that is equal to because we want Av 1 is an Eigen value, Eigen vector corresponding to lambda 1, this is just T v 1, inner product with v 2. But now we will invoke the definition of adjoint or the property of adjoint of T to conclude that this v 1 inner product with T star of v 2 the adjoint of v 2, and v 2 is an eigenvector of T corresponding to lambda 2.

So, v 2 is also hence an eigenvector of T star corresponding to lambda 2 bar that is the content of the previous proposition. So, this is just v 1 comma lambda 2 bar times v 2. And by the properties of inner product, this is lambda 2 times the inner product of v 1 with v 2.

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$$= \langle v_1, \bar{\lambda}_2 v_2 \rangle$$
$$= \lambda_2 \langle v_1, v_2 \rangle$$
$$\Rightarrow (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0.$$

$$\Rightarrow \langle v_1, v_2 \rangle = 0$$

Hence v_1 is orthogonal to v_2. —— ∎.

What does that mean? This implies that lambda 1 minus lambda 2 times the inner product of v 1 and v 2, this is just equal to the scalar 0. But we observed that lambda 1 and lambda 2, the hypothesis tells us that lambda 1 is not equal to lambda 2, so lambda 1 minus lambda 2 cannot be 0. So, the product of 2 numbers equal to 0, and one of them is nonzero, forces the other to be equal to 0, so we v 1 comma v 2 in the in the field of scalars, this is just going to be equal to 0 and that is precisely what we had set out so v 1 orthogonal to v 2 right.

So, normal operators have some really nice properties when it comes to eigenvectors so Eigen vectors corresponding to distinct Eigen values are orthogonal to each other. This is going to play a very crucial role in proving that in the right setup normal operators is diagonalizable so, we will come to that later. Let us explore a few more properties of normal operators.

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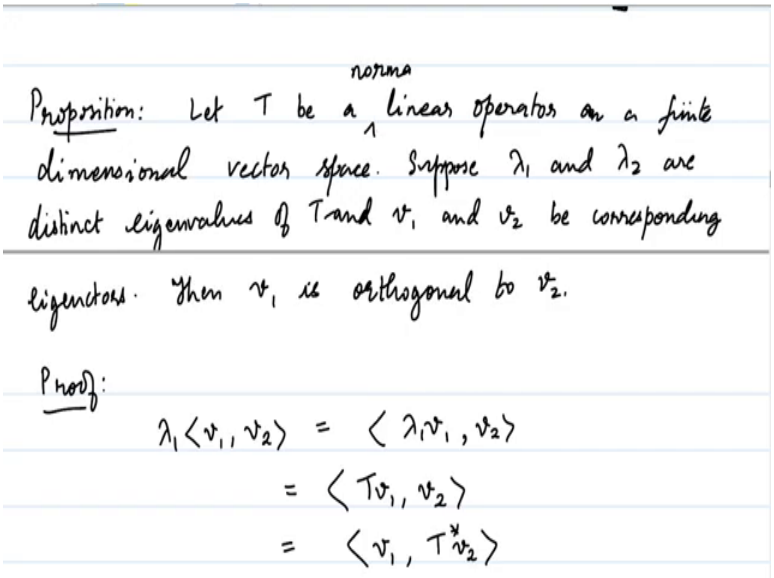

So, capture the next in the following proposition. So, yet again let T did I so that T is normal in this, Oh yeah, T be a normal linear operator, this is true only in the case when T is indeed a normal linear operator, and otherwise we cannot conclude something of this sort. Yeah, I hope I have written the case here as well.

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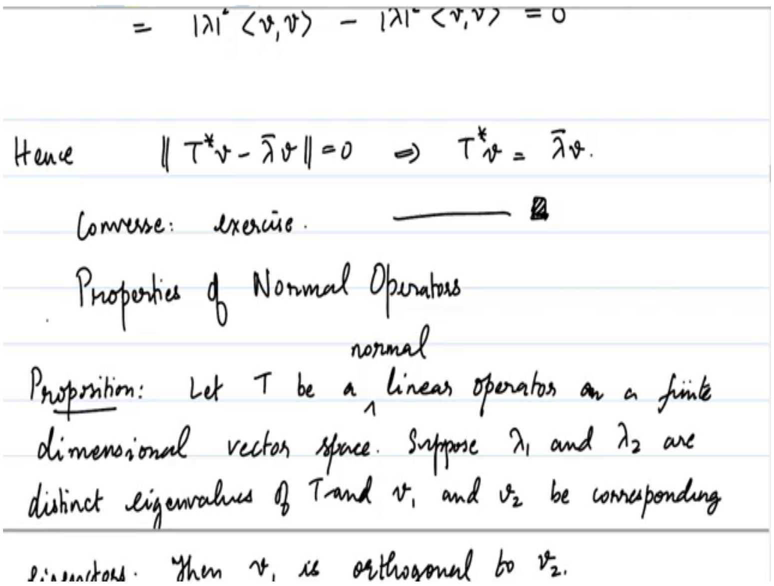

Okay yet again and missed the condition in the so this lecture. All the linear operators that we are considering are normal so this is okay this statement was given as a character, the previous statement was given as a characterization of T being normal. And here we are studying the properties of normal operators. So, this is these are all on properties of normal operators, so what next?

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Proposition: Let T be a normal linear operator on a finite dimensional inner product space. Then $\|T^*v\| = \|Tv\| \ \forall\, v \in V$.

Proof: $\|T^*v\|^2 = \langle T^*v, T^*v \rangle$

$= \langle v, TT^*v \rangle$

$= \langle v, T^*T$

So, let T be normal, let me not miss it this time normal linear operator on a finite dimensional inner product space then the length of T star v is the same as the length of T for all v, then the length of T star v is equal to the length of T v for all v in capital V. This is quite straightforward, actually, let us give a quick proof, so we will rather prove that the square is the same.

These are the standard tricks that we use in inner product spaces, because the length is just the inner product of T star v with itself, the adjoint of the adjoint of T is T. So, this is going to be equal to V T T star v, and T is a normal operator so, T T star is equal to T star T, v comma star T v.

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Hence v_1 is orthogonal to v_2. ■

Proposition: Let T be a normal linear operator on a finite dimensional inner product space. Then $\|T^*v\| = \|Tv\| \ \forall\, v \in V$.

Proof: $\|T^*v\|^2 = \langle T^*v, T^*v \rangle$

$= \langle v, TT^*v \rangle$

$= \langle v, T^*Tv \rangle$

$= \langle Tv, Tv \rangle$

But again, this by the property of the adjoint is just T v, comma T v which is the length of T v square. So, we started off with the length of T star v square, we ended up with the length of T v square, this implies length of T star v is equal to the length of T v. So, T and T star have this T v and T star v have the same length for all v in capital V so that is something which we can say about normal operators.

Okay, let us solve some problems now. So, just like in the previous week, we will try to solve as many problems as possible during the lectures itself, there will be no separate problems session, and problems will be integrated into the lectures so, let us solve the first one.

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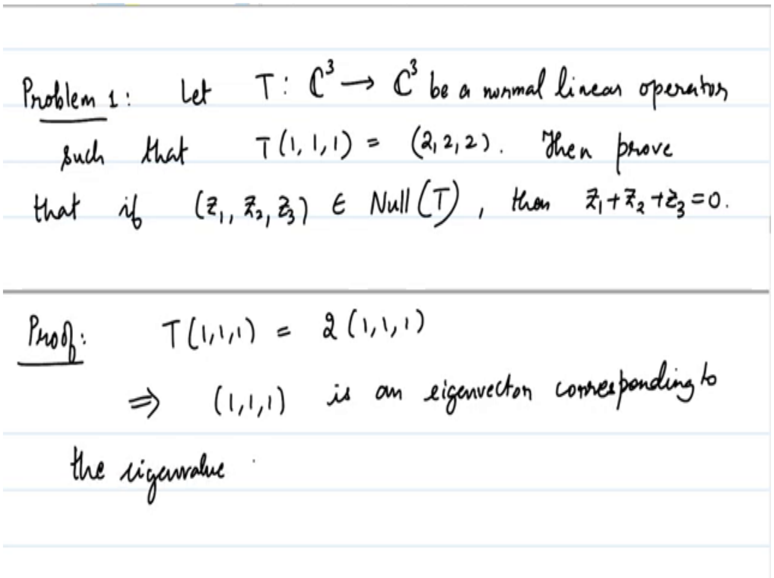

First problem states the following. So, let T be linear operator from C 3 to itself, be a linear operator such that T of say 1 comma 1 comma 1 is equal to 2 comma 2 comma 2 this information is given to us then, prove that if z 1, z 2, z 3 is in the null space belongs to the null space of T, then z 1 plus 2 plus z 3 is equal to 0, this is what we have to prove.

So if z 1, z 2, z 3 is in the null space, then sum of its coordinates is 0. So, yeah, so what is it that we are given? Oh, this is not again yet again. I am sorry that I am missing it again and again, we are considering only normal linear operators in this particular lecture. So yeah, T here is a normal linear operator.

So, let me just restate the problem here, T is a normal linear operator such that T of 1, 1, 1 is 2, 2, 2. And suppose z 1, z 2, z 3 is in the null space of T, then the sum of the coordinates is equal to 0, so let us look at a solution. So, what do we know about T of 1 comma 1 comma 1, that is just 2 comma 2 comma 2 which is 2 times 1 comma 1 comma 1, which implies that 1, 1, 1 is an Eigen vector corresponding to the eigenvalue 2.

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$(z_1, z_2, z_3) \in \text{Null}(T)$

$\Rightarrow \quad T(z_1, z_2, z_3) = 0 = 0(z_1, z_2, z_3)$

$\Rightarrow \quad (z_1, z_2, z_3)$ is an eigenvector of T corresponding to the eigenvalue 0.

And what do we know about z 1, z 2, z 3? z 1, z 2, z 3 is in the null space of T or the kernel of T, this implies that T of z 1, z 2, z 3 this is the 0 vector, but that is just equal to 0 times that z 1, z 2, z 3. So, that means that if the null space of linear operator is nonzero, if it is not an injective linear transformation then 0s in particular and Eigen value of T and any nonzero vector in the null space will turn out to be an eigenvector corresponding to 0.

So, here in this case, z 1, z 2, z 3 is an eigenvector of T corresponding to the eigenvalue 0. So, maybe I should add in the problem here that T be a nonzero vector. Actually, let me not bother about that because if it is, indeed the 0 vector, then there is nothing to prove right. So, let me not unnecessarily, it is there, it is self-evident.

If z 1, z 2, z 3 is 0, then z 1 plus z 2 plus z 3 is also equal to 0. So, in particular, this is true for all vectors in the null space of T be is an eigenvector of T if z 1, z 2, z 3 is nonzero. If z 1, z 2, z 3 is 0, I will not bother writing the argument on because then z 1 plus z 2 plus z 3 is anyways there. Now, what does this mean corresponding to 0 to the eigenvalue 0? Yeah Maybe I should have finished this, what is the outcome which we have proved a bit earlier?

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dimensional vector space. Suppose λ_1 and λ_2 are distinct eigenvalues of T and v_1 and v_2 be corresponding eigenctors. Then v_1 is orthogonal to v_2.

Proof:

$$\begin{aligned}\lambda_1 \langle v_1, v_2\rangle &= \langle \lambda_1 v_1, v_2\rangle \\ &= \langle Tv_1, v_2\rangle \\ &= \langle v_1, T^* v_2\rangle \\ &= \langle v_1, \bar{\lambda}_2 v_2\rangle \\ &= \lambda_2 \langle v_1, v_2\rangle\end{aligned}$$

We have proved that this one, if you have a normal linear operator if T is a normal linear operator on a finite dimensional vector space and lambda 1 and lambda 2 are distinct eigenvalues of T, then its Eigen vectors are going to be orthogonal to each other.

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the eigenvalue 2.

$(z_1, z_2, z_3) \in \text{Null}(T)$

$\Rightarrow T(z_1, z_2, z_3) = 0 = 0(z_1, z_2, z_3)$

$\Rightarrow (z_1, z_2, z_3)$ is an eigenvector of T corresponding to the eigenvalue 0.

$\Rightarrow \langle (z_1, z_2, z_3), (1,1,1)\rangle = 0$

$\Rightarrow z_1 + z_2 + z_3 = 0$ —— ■

So, here in particular, what this tells us is that 0 and 1 or 2 are Eigen values here, distinct eigenvalues, the inner product of z 1, z 2, z 3 and 1 comma 1 comma 1 which are the Eigen vectors correspond to 0 and 2 respectively, this is just going to be equal to 0. But by the inner product definition of the inner product we know that z 1 plus z 2 plus z 3 is exactly the inner product of z 1, z 2, z 3 and 1 comma 1 comma 1 and hence we have concluded that this is equal to 0, we have solved the problem.

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Problem: Let $T: V \longrightarrow V$ be a normal operator on a finite dimensional inner product space. Suppose $\|v\| = 2 = \|w\|$ and $Tv = 3v$, $Tw = 4w$. Then calculate $\|Tv + Tw\|$

Solution: Since v and w are eigenvectors of T corresponding to distinct eigenvalues.

Solution: Since v and w are eigenvectors of T corresponding to distinct eigenvalues, then

$$\langle v, w \rangle = 0 \quad \Rightarrow \quad \langle 3v, 4w \rangle = 0$$

$$\|Tv + Tw\|^2 = \|3v + 4w\|^2$$

$$= 9\|v\|^2 + 16\|w\|^2$$

Another problem along the same line, so let us consider in an abstract inner product, so select T from V to itself be normal operator on a finite dimensional inner product space. Suppose norm of v is equal to norm of w and T of v be equal to T v comma T of w be equal to 4 W, then calculate T v plus T w.

Let us see, T is equal to three times V, and T w is equal to four times W, this indicates that v and w are orthogonal to each other because they are the Eigen vectors corresponding to distinct Eigen values so, let me first note that down solution. Since, v and w Eigen vectors of T corresponding to distinct Eigen values, I mean by that theorem. Then inner product of v with w is 0, they are orthogonal to each other.

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$$\|Tv + Tw\|^2 = \|3v + 4w\|^2$$

$$= 9\|v\|^2 + 16\|w\|^2$$

$$= 25 \times 4 = 100.$$

$$\Rightarrow \quad \|Tv + Tw\| = 10$$

So, let us look at what is the length of T v plus T w. In fact, we will look at the square of the length of this, T v you notice this is just equal to two three times v, and the vector T w is four times w, the square of this, but we know that by Pythagoras theorem if v and w are orthogonal to each other, so the vectors 3 v and 4 w they are after all scalars, right?

So, check that this implies 3 v with 4 w, which is just four times v comma w is equal to 0. And therefore, these are orthogonal vectors and you are looking at the square of the length of the sum of two orthogonal vectors. By Pythagoras theorem, this is just equal to the length of nine times v length of v square plus 16 times the length of w square.

But length of v and length of w both are equal to 2 and therefore this is just 25 times 4, which is equal to 100. This implies that the length of T v plus T w is equal to 10, that is the number we are looking for.

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Problem 3: Let $v_0, w_0 \in V$ and define $T: V \to V$ given by $Tv = \langle v, v_0 \rangle w_0$. Then prove that T is normal iff v_0 and w_0 are linearly dependent.

Yeah, maybe we should have proved this problem first, this is problem 3, let us call it problem 3. So, let T from V to itself be a linear operator or before that let is fix 2 vectors. So, let v 0 comma w 0 be in capital V. So, we have seen this linear operator before and define T from V to itself, given by T v is equal to the inner product of v with v 0 times w 0.

So, here the question as to prove that then prove that T is normal, notice that in this case, T is from V to itself so T is normal, it is a special case of the linear operator we were considering last week. If and only if, v 0 and w 0 are linearly dependent, let us look at that.

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dependent.

Proof: Suppose T is normal.

Recall that $T^* w = \langle w, w_0 \rangle v_0$

$$TT^* w = \langle w, w_0 \rangle T v_0$$

$$= \langle w, w_0 \rangle \langle v_0, v_0 \rangle w_0$$

Let us prove the forward direction first, suppose T is normal that means T T star is equal to T star T. So, we have calculated what T star is, recall that T star of let me just use w here for consistency, even though w is also some vector in v, this is going to be equal to w inner product with w naught times v naught.

And T T star w is just equal to the inner product of w with w naught times T w times T v naught, why is this because T is a linear operator then this is equal to the inner product of let me be a little bit careful this is going to be w comma w naught times what is T v naught? T v naught by definition is v naught with itself inner product of this times w naught. So, that is the definition of if you recall here.

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$$= \langle w, w_0 \rangle \|v_0\|^2 w_0$$

$$T^*Tw = T^*(\langle w, v_0 \rangle w_0)$$

$$= \langle w, v_0 \rangle \langle w_0, w_0 \rangle v_0$$

$$= \langle w, v_0 \rangle \|w_0\|^2 v_0$$

$$\underline{T^*Tw = TT^*w}$$

$$\Rightarrow \langle w, v_0 \rangle \|w_0\|^2 v_0 - \langle w, w_0 \rangle \|v_0\|^2 w_0 = 0$$

This the scalar w comma w 0 times the length of v naught square times w 0. Now, let us look at what is T star T of w, that will just be equal to T star of what is T of w that is inner product of T of w will just be inner product of w with v naught times w naught. So, this is equal to inner product of w with v naught times T star w 0, What is T star w naught we have just said what T star is, T star w is w w naught.

So, this is just going to be inner product of w naught w naught times v naught, which is equal to w comma v naught times the length of w naught square times v naught is normal. T star T w is equal to T T star w implies that the first term here is w inner product of this with itself times the length of w naught square times v naught.

And how about the other one let me just rewrite this in this following manner, this is just going to be the inner product of w with w naught times the length of v naught square times w naught equal to 0. So, this is just written in this manner.

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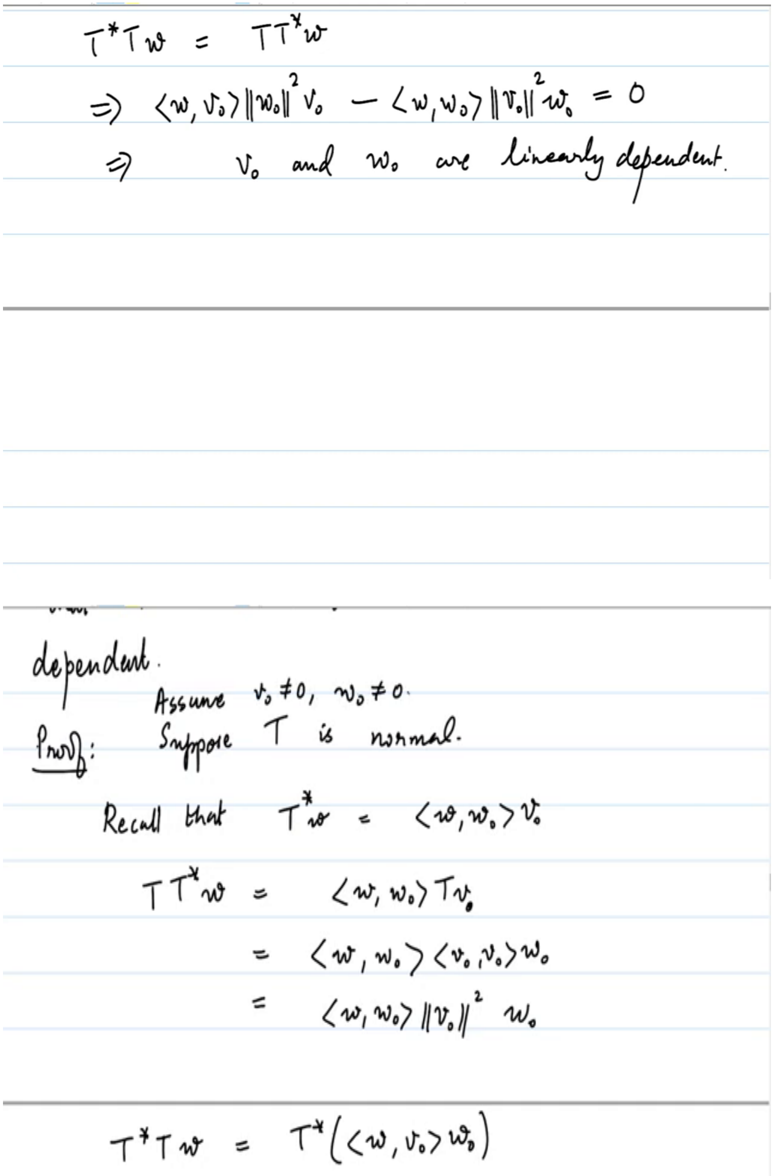

That implies that v naught and w naught are linearly dependent, that is true because we have assumed so, let us assume our nonzero vectors. Actually, we do not need to assume that if v 0 and if even one of them is 0, then T will be 0, star will be 0 so, hence T is a normal linear operator and naturally if even one of them is 0 they are linearly dependent.

So, this is true in the case when w 0 and v 0 are 0 as well. So, assume without loss of generality, so, assume v 0 is not the 0 vector and w 0 is also not the 0 vector, the case when one of them is 0 itself is quite, quite straightforward. And because it is not 0, this term will not be 0, this term will not be 0, and this will not be 0 for example, this is not 0 for when w is equal to v 0.

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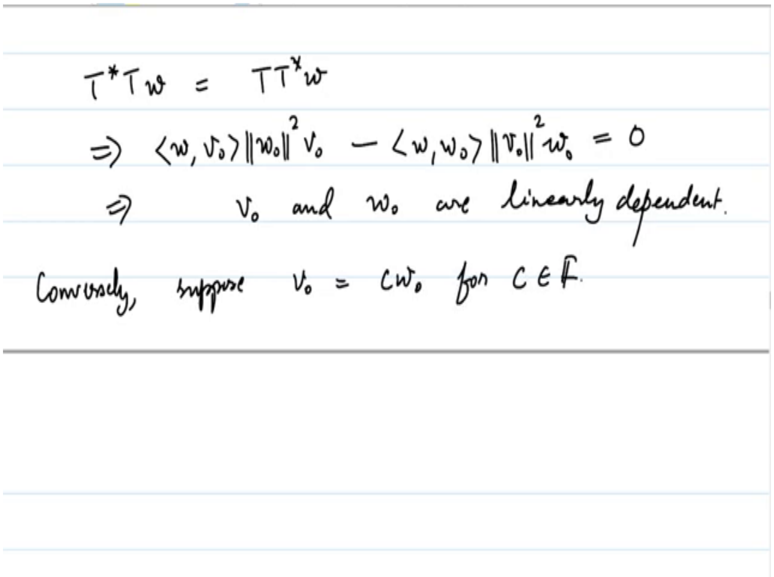

So yeah, so this is a linear combination non trivial linear combination which is equal to 0 and hence v 0 and w 0 are linearly dependent. Now, let us prove the converse, conversely suppose v naught and w naught, v naught is equal to Alpha times w 0 maybe C times w 0 for C in the field of scalar. So, here we are not assuming that this is real or complex inner product space, so let me denote it by F. We know what is T star w, and T star T w, so let us use both to see what happens when v 0 is C times w 0.

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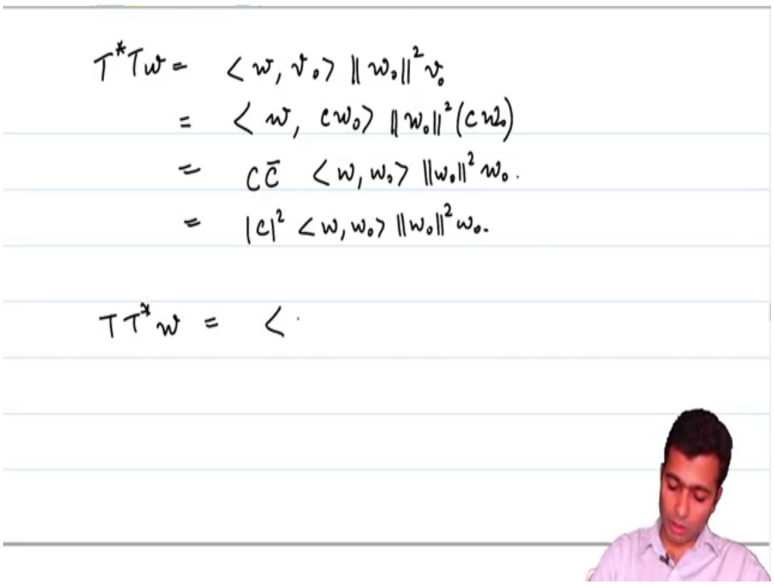

T star T, w let us first pick that, T star T w is inner product of w with w naught times the length of w naught square times v naught that is what our T star T w is, but we know what v naught is, v naught is C times w naught. So, this okay, there is a slight mistake, I guess. Yeah, that is so this is not w 0, this is v naught.

So, this from here, we will turn out to be equal to the inner product of w with C times w naught times the length of w naught square times C times w naught, the inner product, the properties of the inner product tells us that the C bar times inner product of w with w naught times the length of w naught square and the C will be taken out here times the w naught here.

So, this is mod c square times the inner product of w, w 0 so let me just write it down, times the length of w naught square times w naught. So, f star let me put so that c is not equal to 0 again it does not matter, but still. Now, what is T T star w, that will just turn out to be equal to well, let me go up.

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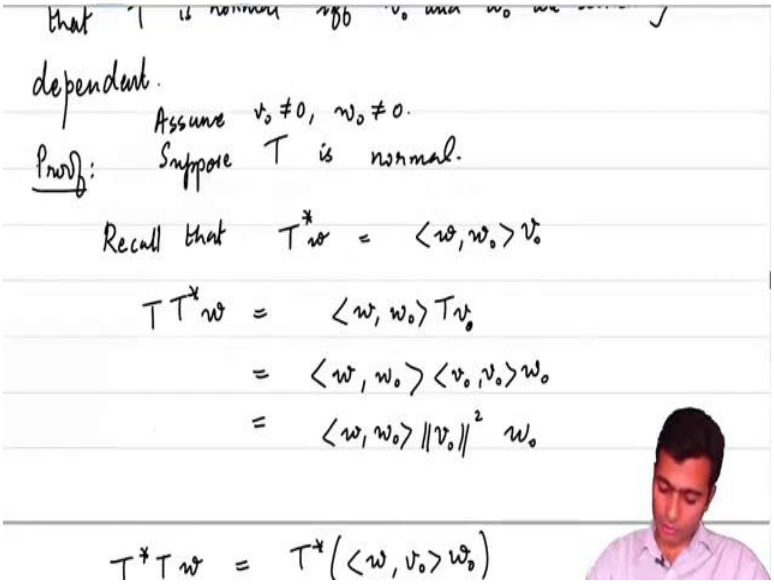

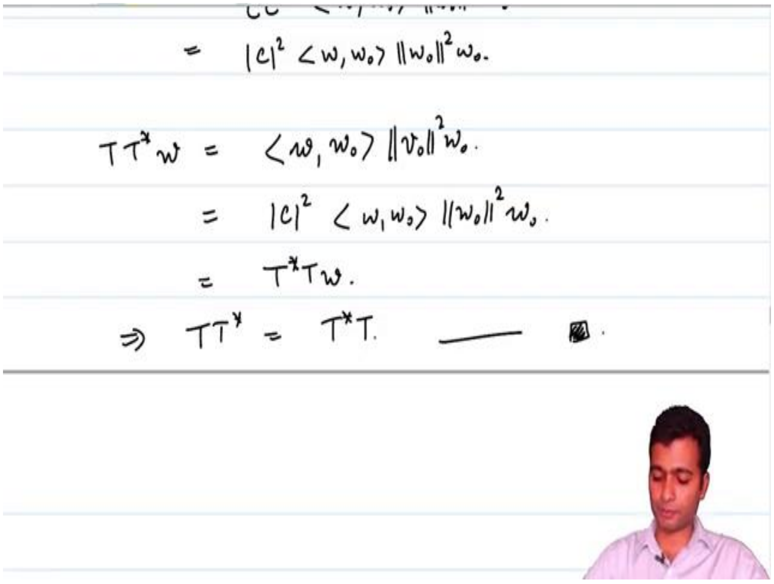

Let me not make mistakes, this is w, w naught length v naught square times w naught. W, w 0 length of v 0 square times w 0, but v 0 is C times w 0 so, this is just directly equal to mod C square times w, w 0 times w, w 0 length squared times, but that is precisely equal to T star T w, which implies that T T star is equal to T star T. So, in that case when v 0 and w 0 are linearly independent, then T is necessarily a normal linear operator. That completes the proof, let me stop here.