Linear Algebra
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Lecture 12 1
Problem session

So, this is a problem session which is based on the material that was covered in week 10 of this course. Let us begin by considering a problem where we deal with the matrix of a given linear operator with respect to an orthonormal basis, so that is the problem 1.

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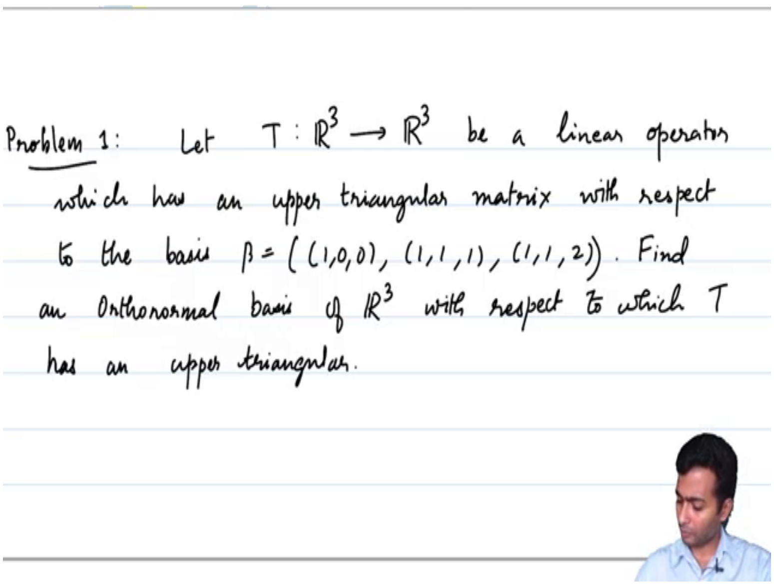

Let T from R 3 to itself be a linear operator which has an upper triangular matrix with respect to a given basis to the basis let us call it beta which is given by 1 0 0, 1 1 1 and 1 1 2. So, we know that T has an upper triangular matrix with respect to this basis. The problem is to find an orthonormal basis of R 3 with respect to which T has an upper triangular matrix, can immediately take that the given basis is not orthonormal. In fact, it is not orthogonal at all.

So the question is to get hold of a basis which is orthonormal and with respect to which the matrix of T is upper triangular. So, before we even start solving for this problem, let us revisit the process of Gram-Schmidt Orthnormalization. So, I will first solve the problem in a more general setting and then eventually compute the orthonormal basis that is given here.

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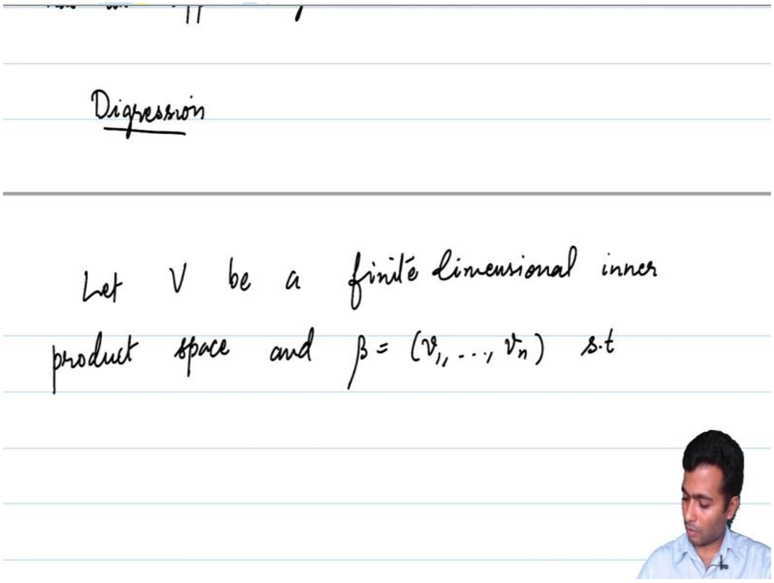

So, more generally so let us come back to the problem a few minutes later, let us look at a digression. So, let v be a finite dimensional inner product space and let us start off with some basis say v 1 to v n such that given E in say L of V the matrix of E with respect to beta is upper triangular.

What does it mean for a matrix of T to be upper triangular? This means that T v k is in the span of v 1 to v k, if you carefully go back and check what the meaning of upper triangular is it means that, if you look at T v k and the column representation of it, it will have entries above the diagonal to be non-zero, all the entries below the diagonal will be 0. That means that the coefficients of the terms v 1, v 2 up to v k contribute in the expansion of T v k in terms of v 1 to v n and the coefficients of v k plus 1 to v n do not contribute. So, therefore T v k belongs to span of v 1, v 2 up to v k.

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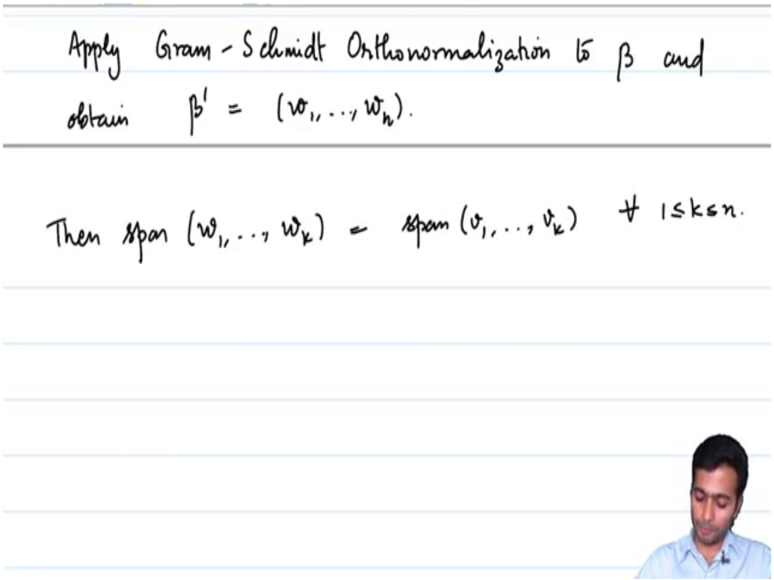

Let us apply Gram-Schmidt Orthonormalization to the basis beta and obtains a beta prime which is equal to say w 1 to w k. So, w k or w n to begin with, it was w n, so let me be careful here yeah. So, beta prime is obtained by Gram-Schmidt Orthonormalization of v 1, v 2 up to v n therefore we get hold of vectors w 1, w 2 up to w n.

But the Gram-Schmidt Orthonormalization was not just giving us orthonormal vectors It was also at every stage ensuring that span of w 1 to w k is also equal to the span of v 1 to v k. This is true for all 1 less than or k less than or equal to n, this is the case that we, the process of Gram-Schmidt Orthonormalization ensure this particular property.

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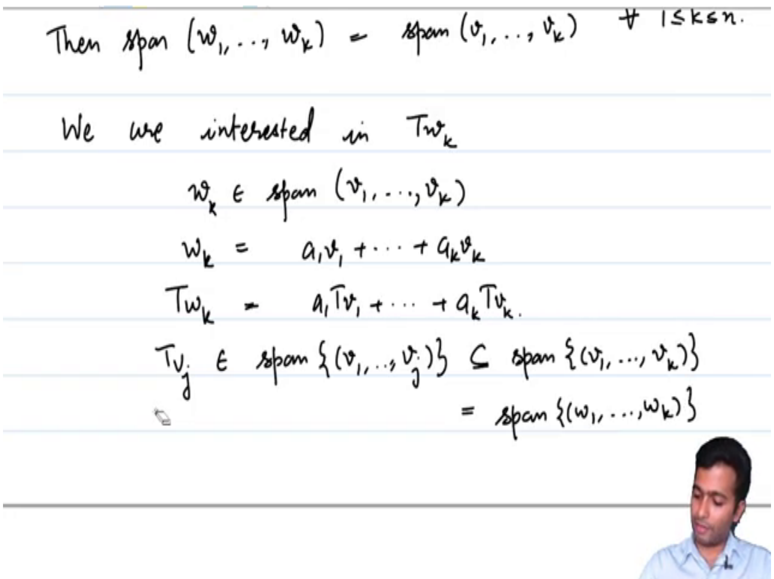

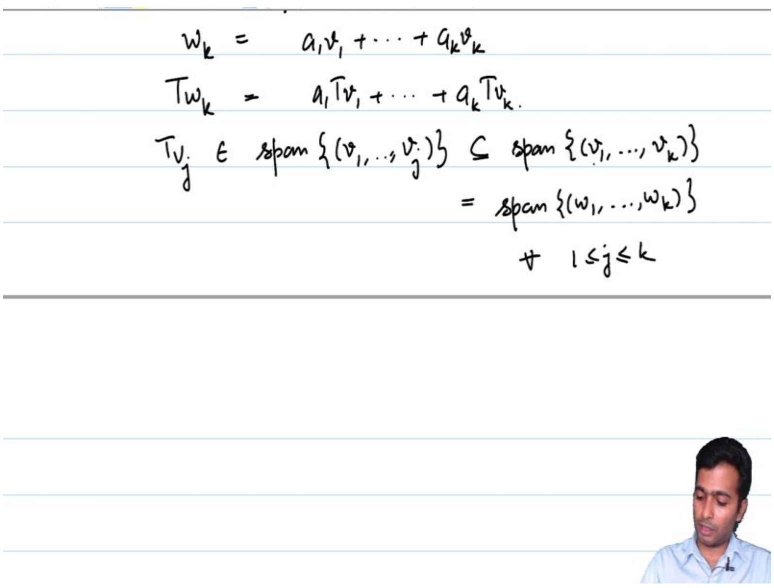

So, if you look at E we, let us look at T w k, now notice that w k in particular we are interested in T w k from 1 to n so that we can compute the matrix of T with respect to beta prime. So, what do we know about T w k? What do we know about w k? We know that wk in particular belongs to the span of v 1 to v k and therefore w k is just a 1 v 1 plus something up to say a k v k. And if you look at T w k, this is just going to be equal to a 1 T v 1 plus up to a k, T v k. But each of these T vi's, T v or rather T v j, this belongs to span of v 1 to v j for each of the j, for all 1 less than or equal to j less than or equal to k.

This would imply that T w k this term belongs to, in particular span of v 1 v 2 up to v j this is contained in the span of w 1. So, let me write it like this, this is contained in the span of v 1 to v k, which is equal to the span of w 1 to w k. So, what do we have? We have each of the T v j belongs to the span of W 1 to W k for all 1 less than or equal to j less than or equal to k. Why is this the case?

This belongs to, this particular relation comes up because of the fact that the matrix of T with respect to beta is an upper triangular matrix, this comes up because the span of v 1 to v k will in particular contain the span of any subset of v 1 to v k and the final one is because of Gram-Schmidt Orthonormalization.

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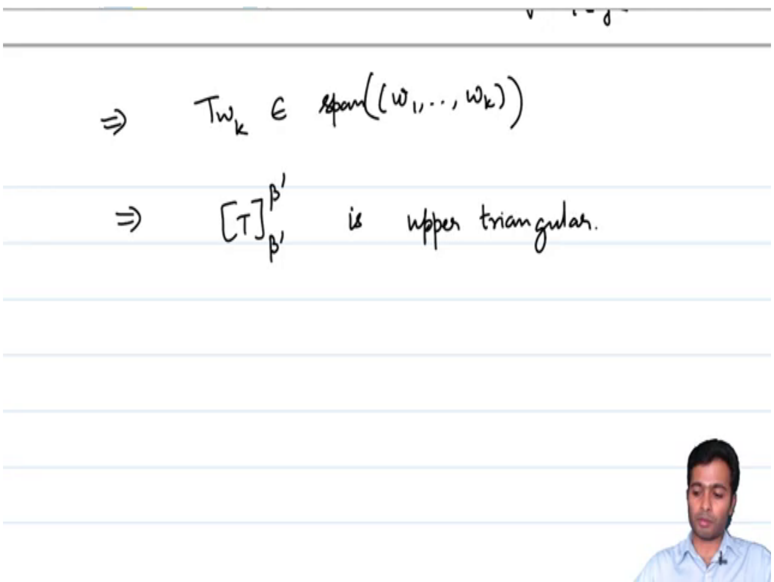

Which implies that each of the T v j belongs to span of v 1, v 2 up to v k and hence T w k which is a 1 T v 1 plus up to a k T v k also belongs to the span of v 1, w 1 to w k. But what does this imply? This implies that the matrix of E with respect to beta prime is upper triangular, this is precisely what it means. So, if we start off with a basis with respect to which are given linear operator is upper triangular, and if we do Gram-Schmidt Orthonormalization of that particular operator, we get a new basis again that will be an orthonormal basis with respect to which our linear operator continues to be an upper triangular matrix. So, now let us get back to our problem.

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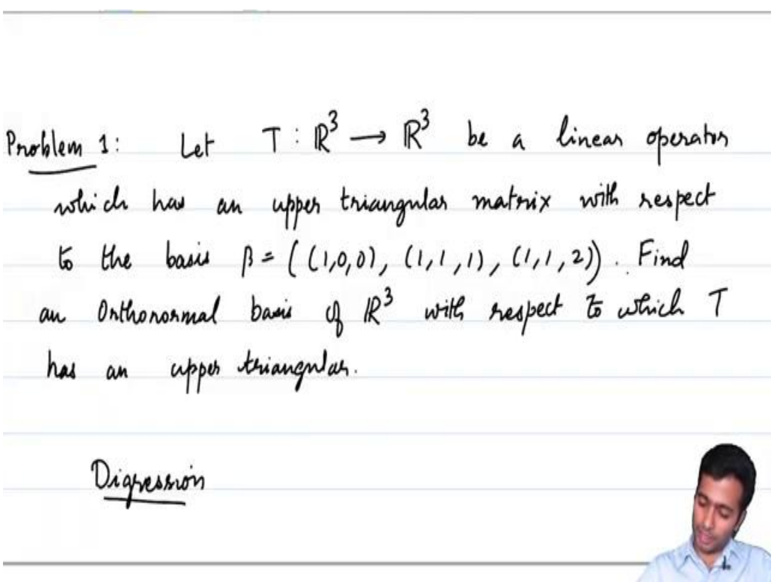

Our problem was to find an orthonormal basis of R 3 with respect to which T has an upper triangular matrix, given the condition that T is already upper triangular with respect to basis 1 0 0, 1 1 1 and 1 1 2.

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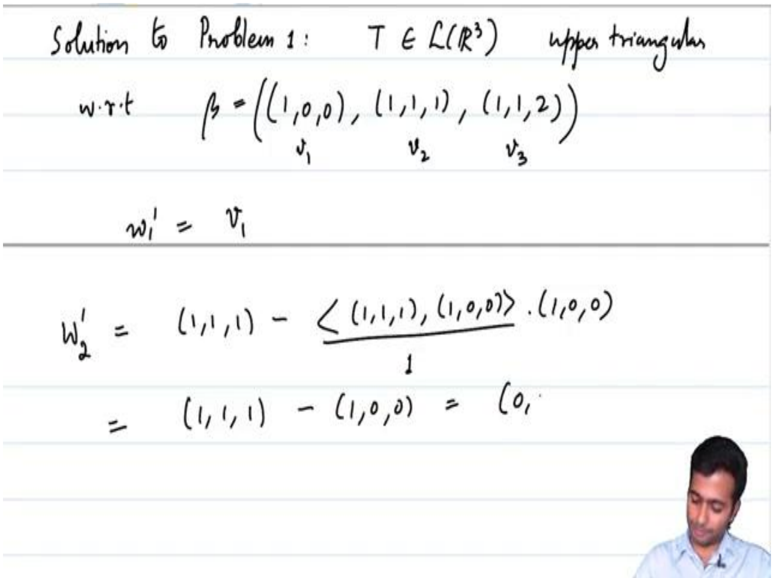

So, let us now solve for our given problem, solution to problem 1, getting back from the digression. So, recall that T belong to L of R 3 upper triangular with respect to beta, which was 1 0 0, 1 1 1 and 1 1 2. So, the only thing that we have to do is to apply Gram-Schmidt Orthonormalization to beta with respect to the standard inner product, of course, the inner product because it was not mentioned, we will assume that the inner product involved is the standard inner product, with respect to the standard inner product let us orthonormalize our given basis here.

So, this is say v 1, v 2 and v 3 so, w 1 prime will be equal to v 1 itself, what will be w 2, w 2 will be w 2 prime, let us calculate w 2 prime that is going to be 1 1 1, which is v 2 minus the inner product of v 2 with w 1 prime by the length of w 1 prime square, which is 1 times w 1 prime and this is just 1 1 1 minus 1 0 0 which is equal to 0 1 1.

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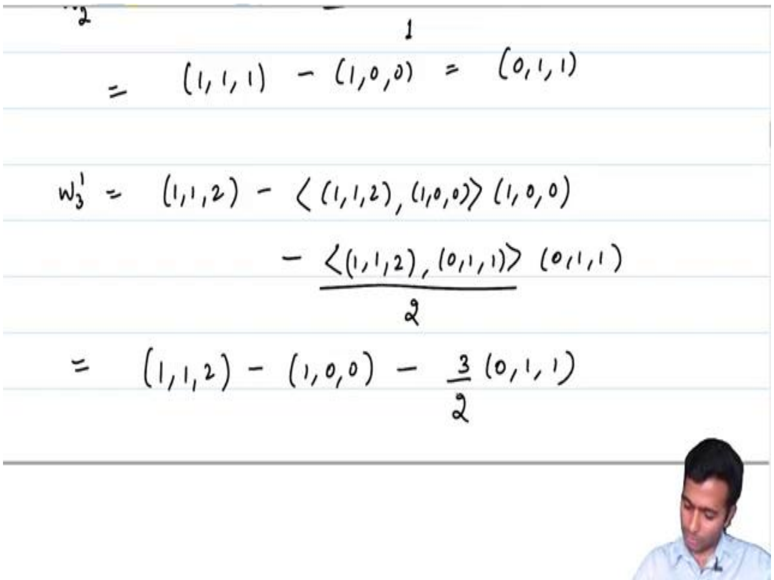

That is w 2 prime and what about w 3 prime? W 3 prime will just be equal to 1 1 2 minus the inner product of 1 1 2 with 1 0 0 times 1 0 0 minus the inner product of 1 1 2 with 0 1 1 by the length of w 2 prime square, which is 2 times 0 1 1. So, this is just 1 1 2 minus, this is just 1 times 1 0 0 again, so this is 1 0 0 minus 1 plus 2 is 3 by 2 0 1 1, let me be careful, 1 plus 2 is 3, yeah 3 by 2.

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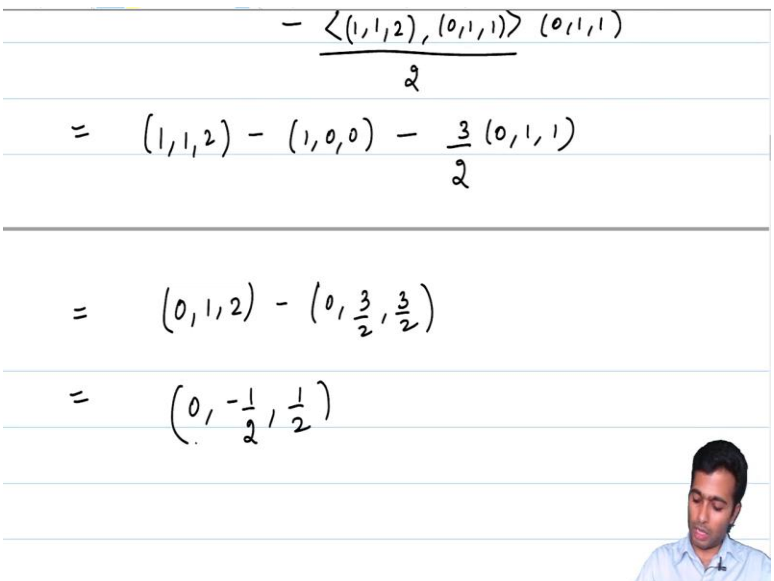

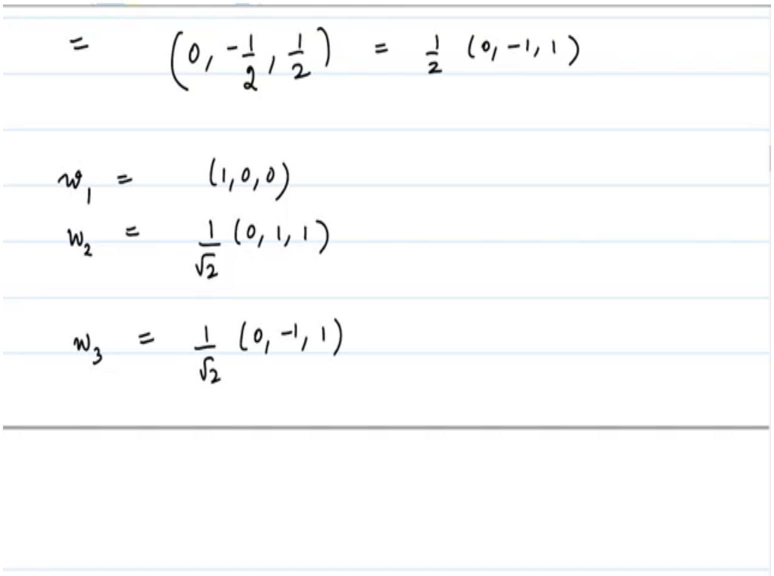

So, this is just going to be equal to 1 minus is 1 0, comma 1, comma 2 minus 0, 3 by 2, 3 by 2, which is equal to 0, 1 minus 3 by 2 is minus of half, 2 minus 3 by 2 is half. And therefore, w 2 prime is obtained in this manner, so what will be our orthonormal basis.

So, w 1 will be after normalizing it is just going to be 1 0 0 w 2 prime will be 0 1 1. And after normalizing, it will be 1 by root 2 times 0 1 1. This is w 2, w 2 is w 2 prime by the length of w 2 prime, and how about w 3? W 3 will be again W 3 prime by so this is half of 0, minus 1, 1, this will be 1 by root 2 times 0, minus 1, 1.

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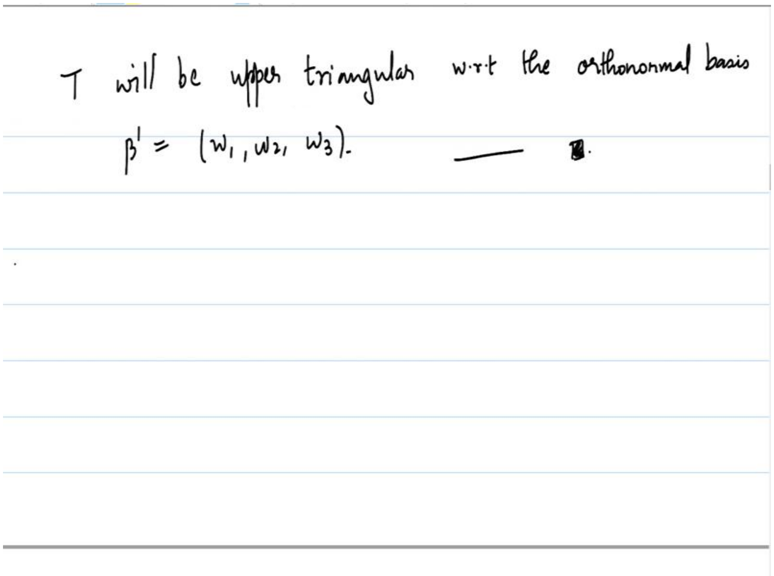

So, this is the orthonormal basis that we obtained by Gram-Schmidt Orthonormalization of a given basis and because T was upper triangular with respect to beta, so T will be upper triangular with respect to beta prime, the orthonormal basis which is given by w 1, w 2, and w 3.

So, I do not even have to sit and compute what the matrix of T will be, because we cannot do that because we do not know what T is, but nevertheless from the theory that we have developed and from the abstract mathematics that we have already seen, we can conclude here that T will be upper triangular with respect to this orthonormal basis.

Okay, so that is more or less the first problem. I would like to give you an exercise here at this point of time, which was already in some sense covered in the digression that we did just a few minutes back check that okay, it will be.

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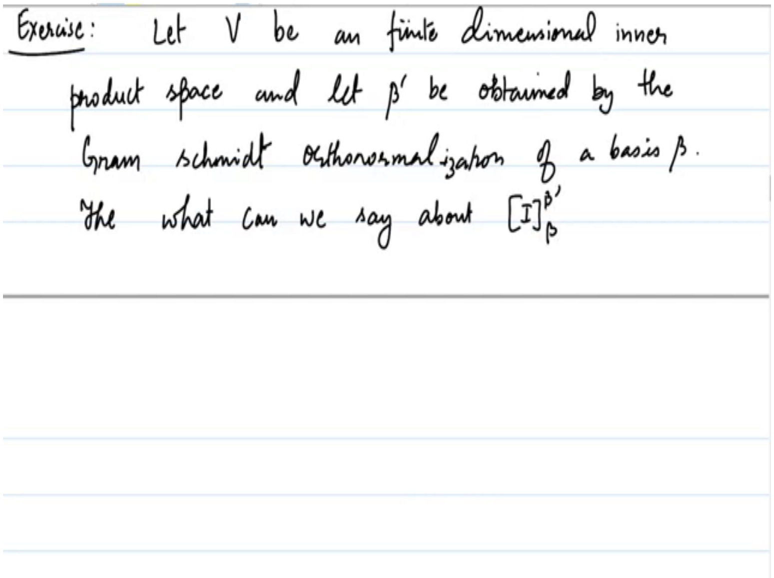

So, let us get back to the setup of let V be an inner product, finite dimensional inner product space. And let beta prime be obtained by the Gram-Schmidt Orthonormalization of a basis beta then what will the matrix change of basis matrix from beta to beta prime look like?

Then what will be, I, what can we say about, what will be will not be? We cannot compute it explicitly because we do not know what beta and beta prime is, it will depend on beta and beta prime but what can we say about the change of basis matrix from beta to beta prime. Well, the digression should tell you that this is going to be an upper triangular matrix.

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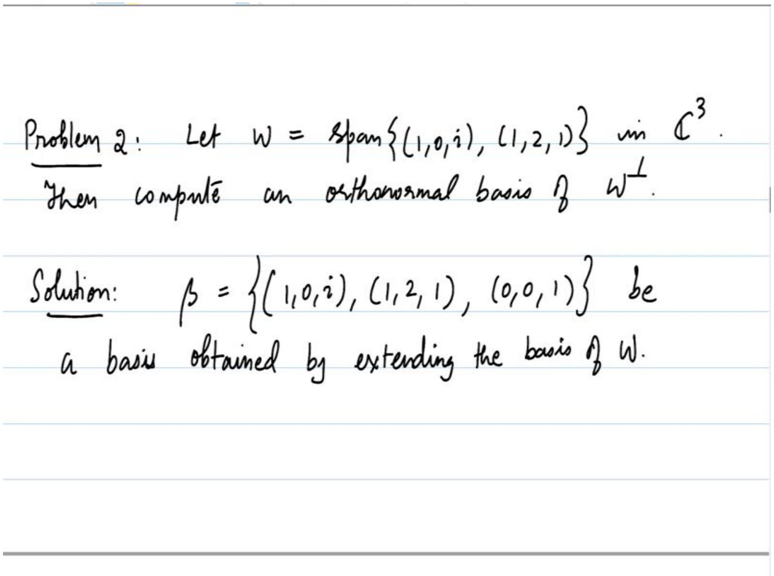

So, the next problem deals with computing an orthonormal basis of the orthogonal complement of a given subspace. So, let w be equal to the span of 1, 0, i and 1, 2, 1 in C 3. Then compute an orthonormal basis of the orthogonal complement of w. So, at this point I would like to remind you that if so, let me call it solution.

So, let us do one thing let us extend beta, let us extend 1, 0, i and 1, 2, 1, notice that that is linearly independent. And if you add 1 more linearly independent vector, we will get hold of a basis of C 3. So, let beta be say 1, 0, i, 1, 2, 1 and let us say 0, 0, 1 be a basis. So, the first thing to check is whether it is a basis obtained by extending the basis of w so, that is what we have done.

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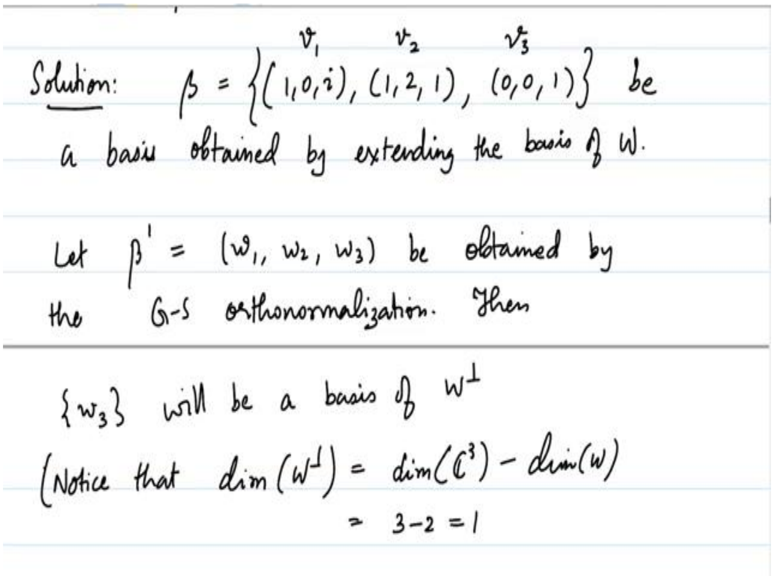

And suppose we have let us call it v 1, v 2 and v 3. And we also normalize this particular basis beta to obtain beta prime. So, let us beta prime equal to w 1, w 2 and w 3 be obtained by the Gram-Schmidt Orthonormalization. Then what do we know? We know that span of v 1, v 2 is the span of w 1, w 2 and therefore, that will be a basis of w, and w 3 will be a basis of the orthogonal complement of w, notice that dimension of orthogonal complement is going to be equal to the dimension of C 3 minus dimension of w.

So, then let me just complete whatever I was writing then w 3 will be a basis of orthogonal complement of w. So, notice that the dimension of the orthogonal complement of w, this is equal to dimension of C 3 minus dimension of w, which is equal to 3 minus 2, which is equal to 1, so it is correct, everything is falling in place, w 3 will be 1 vector which will form a basis of w orthogonal complement of w.

So, let us compute the vector w n here, but before that, I would leave it as an exercise for you to check that so there is 1 thing which you will have to check that vector v 3 is not linearly dependent on v 1 and v 2 that I will leave as an exercise for you. And let us jump to computing the vector w p which is obtained by orthonormalization of beta.

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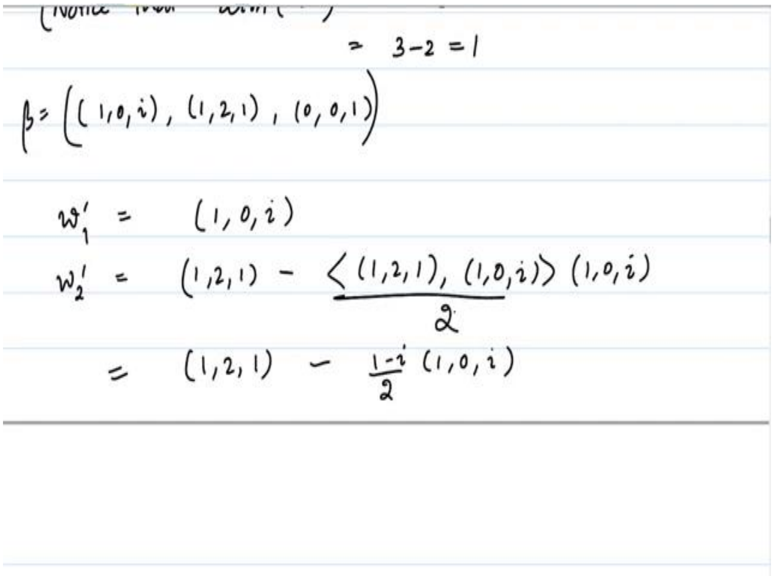

So, recall what beta was, beta was 1,0, i 1, 2, 1 and let us take the next one to be 0, 0, 1 this is our beta ordered set consisting of these 3 vectors. So, what will be w 1 prime? w 1 prime will be just 1, 0, i, w 2 prime will be equal to 1, 2, 1 minus the inner product of 1, 2, 1 and 1, 0, i divided by 2 times 1, 0, i.

The quick computation should tell us that this is minus of 1 minus of i by 2 times 1, 0, i, notice that this is an inner product in C 3 so the complex conjugate will come in from the second vector. And therefore, minus i will come in here, 1 minus i is right so this is this correct.

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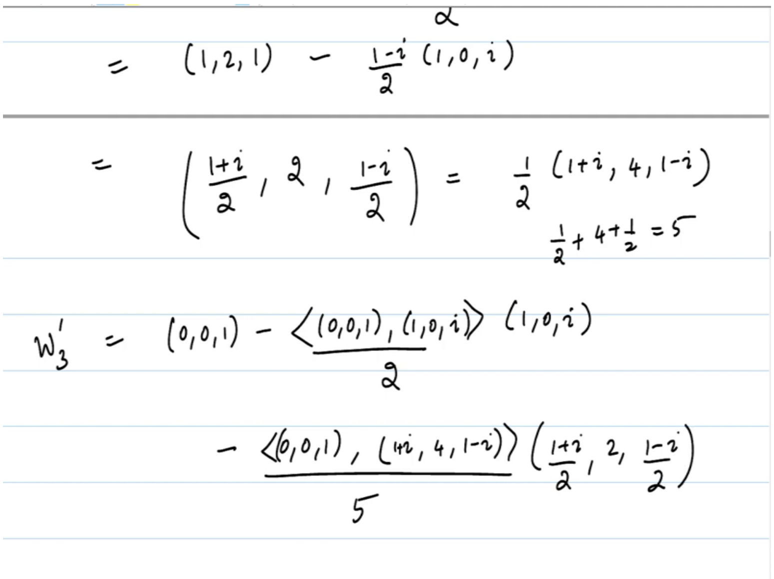

So, this is just going to be equal to 1, minus 1, minus i, which is 1 plus i by 2, and then 2 here. And how about the last one, 1 minus i by 2 times i so this is just going to be equal to 2 minus 1 minus 1, which is going to be 1 minus i by 2, you should that this is indeed correct.

And what about w 3 prime? w 3 prime is what we are interested in, w 3 prime will just be equal to 0, 0, 1 minus the inner product of 0, 0, 1 with 1, 0, i by 2 times 1, 0, i, the 2 below is basically the length of w 1 prime square and there will be 1 more term, this is going to be the inner product of 0, 0, 1 with our w 2 prime, which is, okay, so let us just do one thing.

This is just half 1 plus i, 4, 1 minus i. So, this is going to be 1 plus i, 4, 1 minus i by what is the length of this square rather, this is going to be 1 plus 1 is 2. Maybe I should just compute the length of this here, length of this is just going to be 1 by 4 plus 1 by 4 is 1 by 2 plus 4 plus 1 by 2, which is equal to 5. So, this is just 5 times 1 plus I by 2, 2, 1 minus i by 2.

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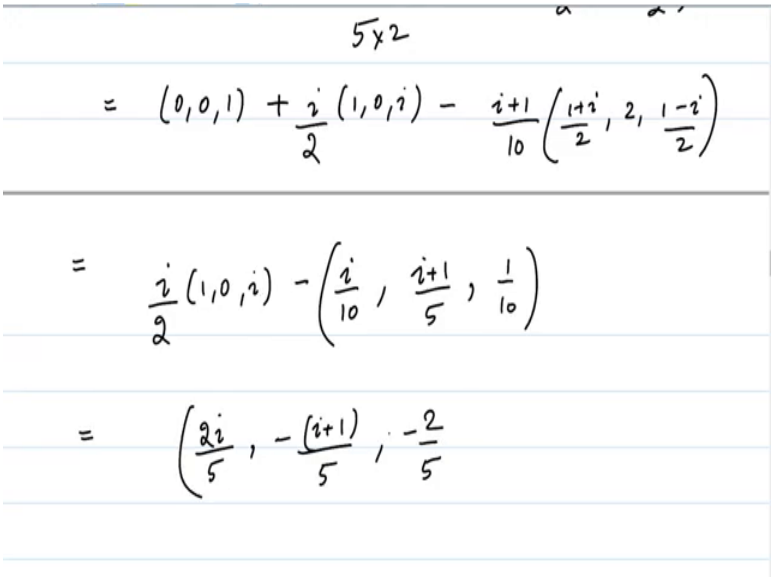

Let us compute what this vector is, it is by computation level as I was indicating a bit earlier, this is going to be 0, 0, 1 minus the term here will be contributing will be minus i so minus minus will be i by 2 times 1, 0, i, minus I hope I am not making a mistake. This is going to be 1 plus i. Oh, there is into 2 that will come in here. There is a half which I missed.

So, this half and yeah, this is right. So, this is going to be 1 plus i by 10 times 1 plus i by 2, 2, and 1 minus i by 2. So, there is some mistake. Yeah, we will see what the mistake is a bit on this one. i by 2 times 1, 0, i minus 1 plus i the whole square is i by 10 and then i plus 1 by 5. And 1 minus i square is basically 2 and this is just going to be 1 by 10.

Let us compute further, this is going to be i by 2 minus i by 10, which is going to be equal to 5 minus 1 4, i so 2 i by 5. And next one will just be equal to minus of i plus 1 by 5 and this is minus of 1 by 2 minus 1 by 10. Did I miss sign here somewhere? i minus 1 minus i square and plus 1 that is 2 by 10 that is right, so this is just going to be minus of 2 by 10, which is going to be 1 by 5. I made a mistake this is not going to be, this is 2 i. This will be minus 1 by 2, this is also going to be 2 by 5.

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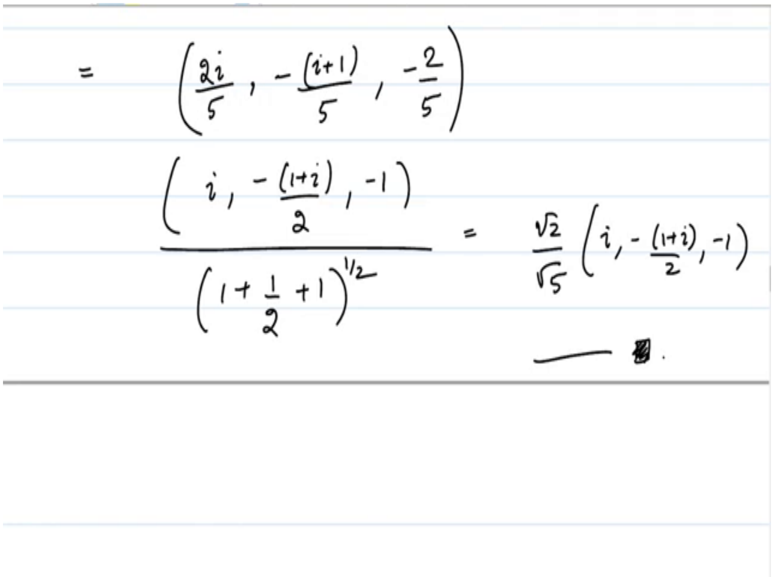

And therefore, so if we have to extract things out, this is just going to be equal to 2 by 5 times i minus of 1 plus i by 2 and minus of 1. And you orthonormalize it by so can forget about this. What is this, this is just going to be equal to square root of 1 plus half plus 1 by 4 plus 1 by 4 is half plus 1, which incidentally turns out to be root 2 by to the power 1 by 2, root 2 by 5 times i minus of 1 plus i by 2, and minus 1, so this is the orthonormal basis of our given vector. So, I should have just left these calculations to you. But once in a while, it is good to do the calculations in front and see how it is coming out. So, that is the basis of the orthogonal complement of R w.

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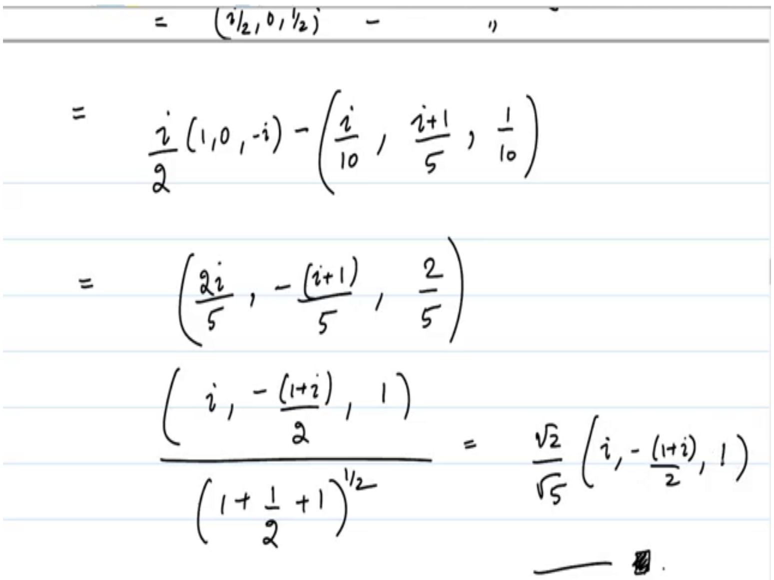

So I made a mistake here. So let me let me just do a minor correction, this is just going to be equal to i by 2, 0 and minus 1 by 2, 1 by 2 so this is going to be 1 by 2 here, minus the same quantity. So, this is just going to be equal to i by 2 and hence here it will just be equal to minus of i, minus of i square will be minus 1 times minus 1 and hence this will be equal to 1 by 2, which makes sense.

So here, this will just be equal to half minus 1 by 10, which is 5 minus 4 so this is going to be plus 2 by 5, it is going to be plus but this does not change and therefore this is going to be plus. I think I was making a minor mistake, I have rectified it.

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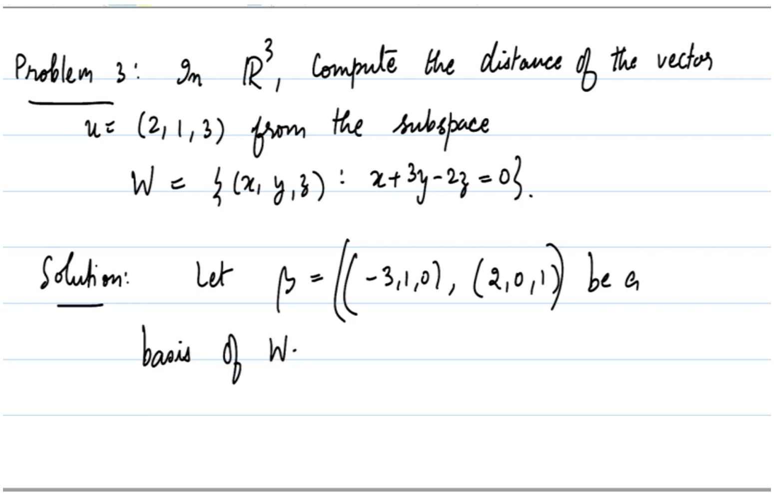

So, the next problem deals with finding the distance of a given vector from a given subspace in R 3. So let me write it down, problem 3. So, in R 3, compute the distance of the vector say u is equal to 2, 1, 3 from the subspace w, which is equal to the set of all x y z such that x plus 3 y minus 2 z is equal to 0. So, yet again, this is going to be a computational problem what the problem demands is to find the distance of the vector 2, 1, 3 from w.

So, how do we go about solving such a problem? So, there are certain theorems which we have already done, which captured the particular vector in our given subspace which is nearest to the given vector, which we had called the projection of 2, 1, 3 on to w.

So, if you recall, what we had done was that we computed the orthonormal basis of w looked at the projection as being the unique vector such that, let us call the unique vector v such that there is unique w such that our given vector can be written as v plus w. And this vector v is the closest vector in w to our given u, we just have to compute the length of u minus v.

So, will write it down explicitly and describe what I just said. So, what we will do is let beta be equal to let us pick 2 vectors from w which are linearly independent and which will turn out to be a basis let beta be equal to say, minus 3, 1, 0 and say 2, 0, 1 be a basis. You should take that this is a basis of w and let us orthonormalize this particular basis to beta prime.

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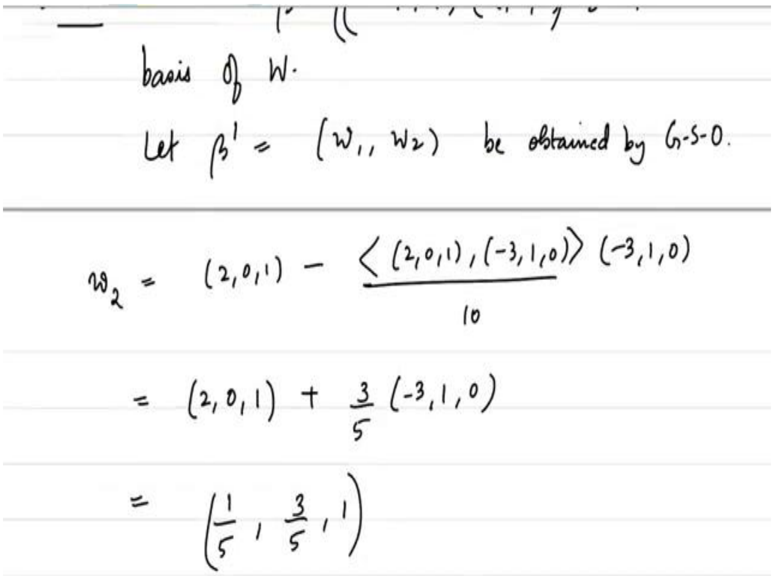

Let beta prime be equal to w 1, w 2, be obtained by auto normalizing it, be obtained by the Gram-Schmidt Orthonormalization let me just write it in short here. So, what will our w 2 be, it is just going to be equal 2, 0, 1 minus the inner product of 2, 0, 1 and minus 3, 1, 0 by the length of w 1 square which is 10 times minus 3, 1, 0.

Okay, so this is equal to 2, 0, 1 minus, let us see, this minus 6 by 10, which is minus 6 will be plus now, 3 by 5 times minus 3, 1, 0, minus 3 into 2 is 6, the other terms do not contribute, so minus 6 by 10 is minus 3 by 5, minus and minus will be plus this this is right. So, this is equal to minus of 9, so this is 1 by 5, 3 by 5 and 1, so this is obtained by orthogonalizing w 2.

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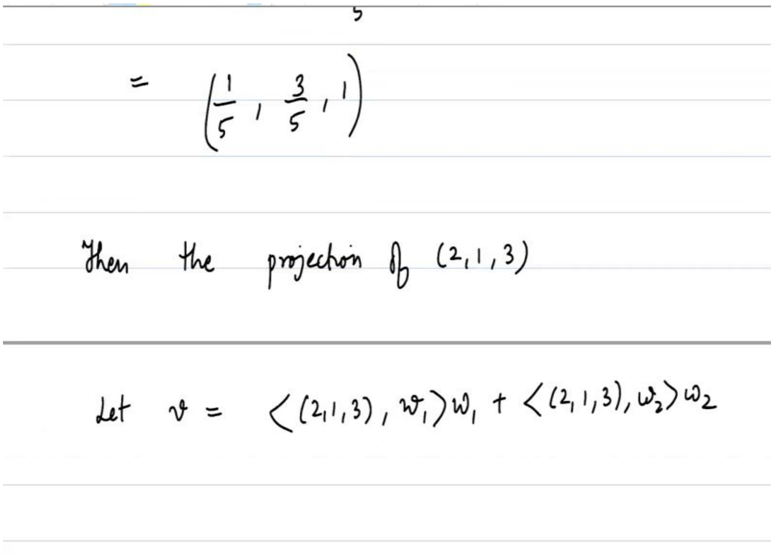

And now let us see what is the projection of 2, 1, 3. Then the projection of 2, 1, 3 which will give you the vector which is closest to 2, 1, 3, this the closest vector in w to 2, 1, 3, the projection will be if you recall that is given by let v be equal to the inner product of 2, 1, 3 with a w 1 times w 1 plus the inner product of 2, 1, 3 with w 2, comma w 2, w 1 and w 2 are obtained by, this is actually prime you have orthonormalie it make it norm 1 and that is how you will be getting hold of the projection, should go back to your lectures and check that is precisely what we had done.

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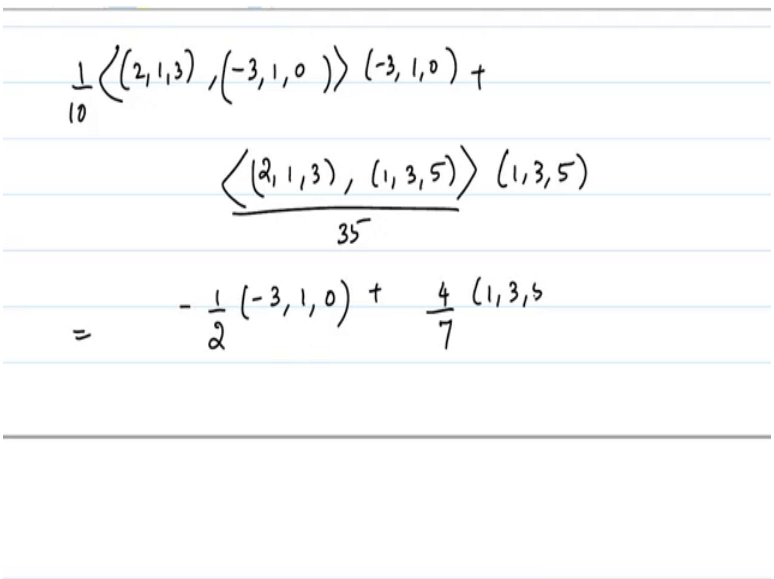

So, here the first vector is minus 3, 1, 0 so, this is going to be minus 3, 1, 0 and a product of 2, 1, 3, with this into minus 3, 1, 0. The normalization has to take place let me write it here, 1 by 10 plus 2, 1, 3. And what was the new vectors that we got 1 by 5 3 by 5 and 1. So, let me just write it as 1, 3, 5 inner product of this with a number by 35, this is precisely the projection onto our subspace w. So, let us just compute this very quickly. Okay, so this is just going to be minus 6 plus 1 minus 5. So, this minus of 1 by 2 times minus 3, 1, 0 plus 2 plus 3, 5 plus 15 is 20. So, 4 by 7 times 1, 3, 5.

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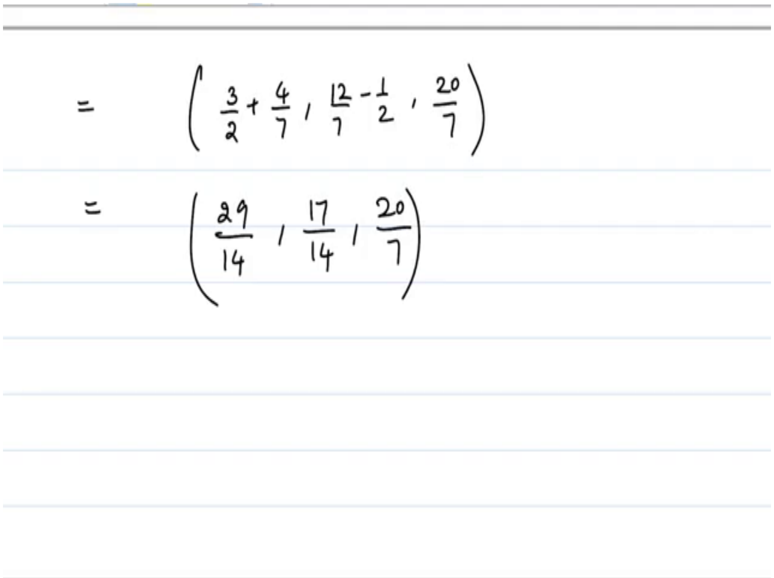

And if we compute this, this is 3 by 2 plus 4 by 7, so 21 plus 829 by 7, 4 by 7 minus 1 by 2, so I will just write it down. Maybe 1 more step will not hurt so this is 3 by 2 plus 4 by 7, 12 by 7 minus 1 by 2 and 20 by 7, this is our projection onto our subspace W. So, let us see what this is, this is 29 by 14, this is 24 minus 7 is 17 by 14 and this is 20 by 7, so this is the vector in W, which is closest to 2, 1, 3.

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$$= \left(\frac{29}{14}, \frac{17}{14}, \frac{20}{7}\right)$$

Hence distance of $(2, 1, 3)$ to W is given by

$$\left\| (2, 1, 3) - \left(\frac{29}{14}, \frac{17}{14}, \frac{20}{7}\right) \right\|$$

$$\left\| (2, 1, 3) - \left(\frac{29}{14}, \frac{17}{14}, \frac{20}{7}\right) \right\|$$

$$= \left\| \left(\frac{-1}{14}, -\frac{3}{14}, \frac{1}{7}\right) \right\|$$

And then what is going to be the distance of our vector to the subspace it is just going to be the distance of our vector to this particular vector, this particular vector. And hence, distance of 2, 1, 3 to W is given by. So this distance is with respect to the length that we are obtaining from our given inner product. This is going to be the length of 2, 1, 3 minus 29 by 14, 17 by 14 and 20 by 7. This is going to be equal to minus of 1 by 14, 14 minus 17 is minus 3 by 14, and 7 into 3 is 21, 1 by 7, the length this.

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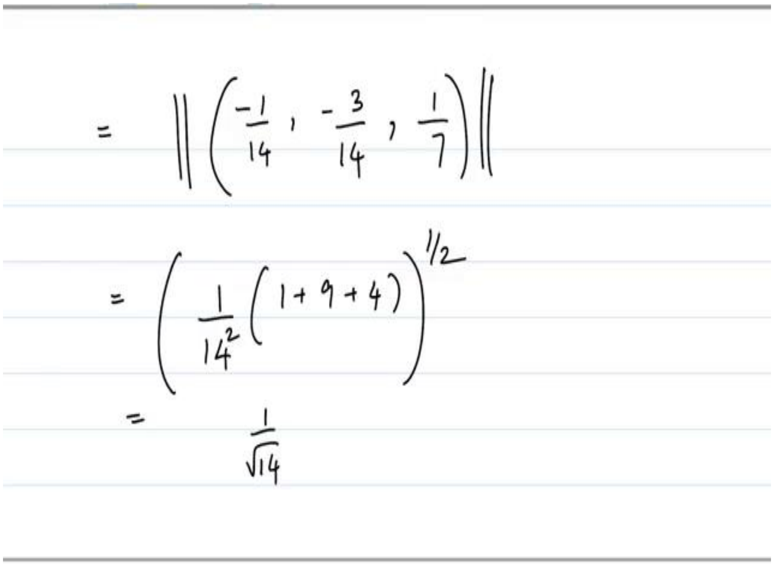

Which is going to be equal to 1 by 14 the square root of 1 by 14 square into 1 plus 9 plus 4, which is equal to 1 by square root of 14. This is precisely the length of a given vector to our given subspace. So, the next problem that we will be solving is what is popularly known as Bessel's inequality.

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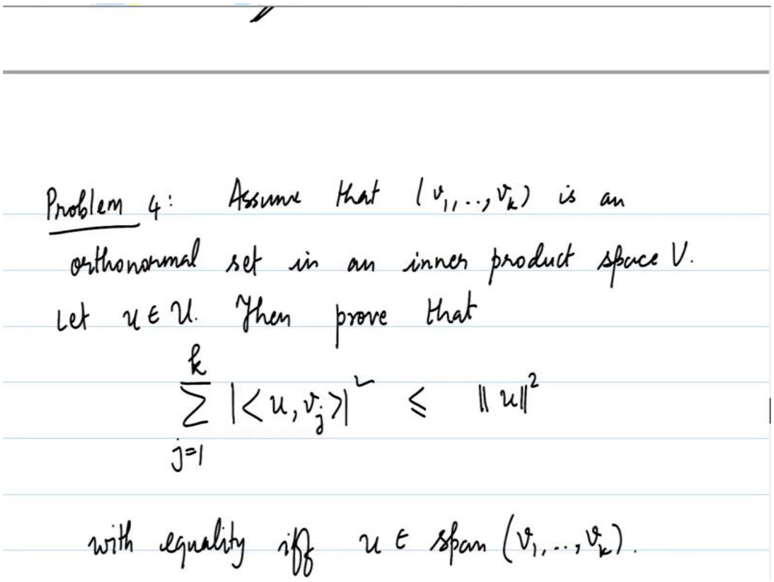

Let us look at the problem. So, assume that v 1 to v k is an orthonormal set in an inner product space V. Let u be some arbitrary vector in capital U then prove that the inner product of u with v j square where the summation is from 1 to k, this is less than or equal to the length of u square, observe that this is a inner product space and this inequality is always satisfied for any collection of orthonormal vectors with equality that is exactly we know exactly when the equality happens. With equality if and only if u belongs to the span of v 1 to v k.

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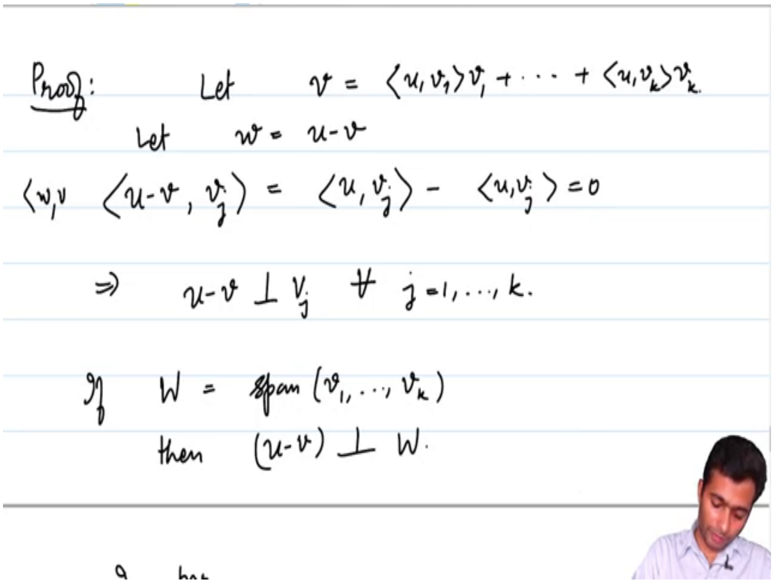

So, let us do one thing, let us give a proof of this. So, let u be equal or rather, v be equal to the vector given by inner product of u with v 1 times v 1 plus up to u with v k times v k. Now, if you observe that u minus v, if you look at the inner product of this with say v j, this is just going to be equal to u with v j minus the inner product of v with v j, which I will just directly write it as u with v j which is equal to 0.

So, this implies that u minus v is orthogonal to v j for all j from 1 to k. That means, if W is equal to the span of v 1 to v k then u minus v is orthogonal to W, in particular. So let me call u minus v so let w be equal to u minus v then inner product of w with v j is going to be this.

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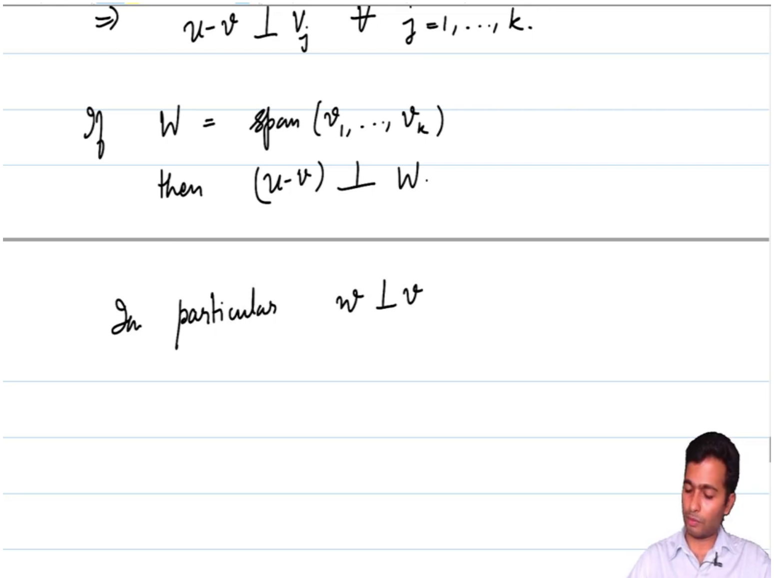

So in particular, w is also orthogonal to our given v.

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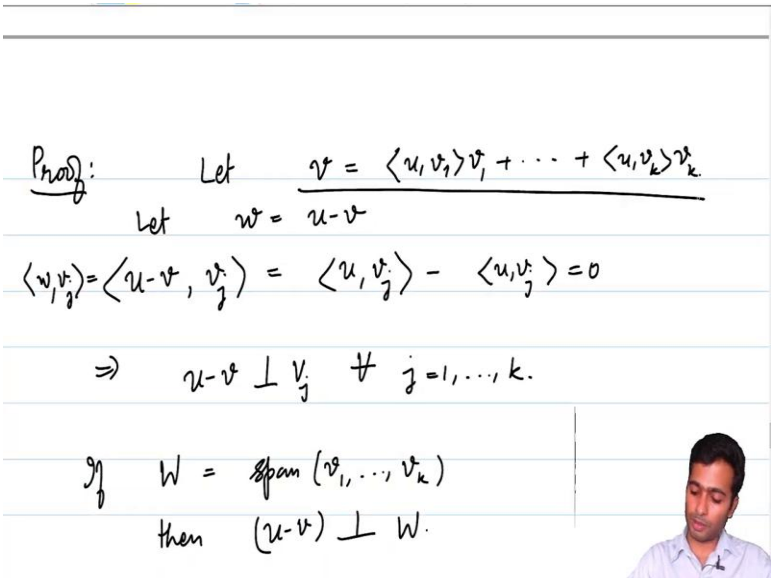

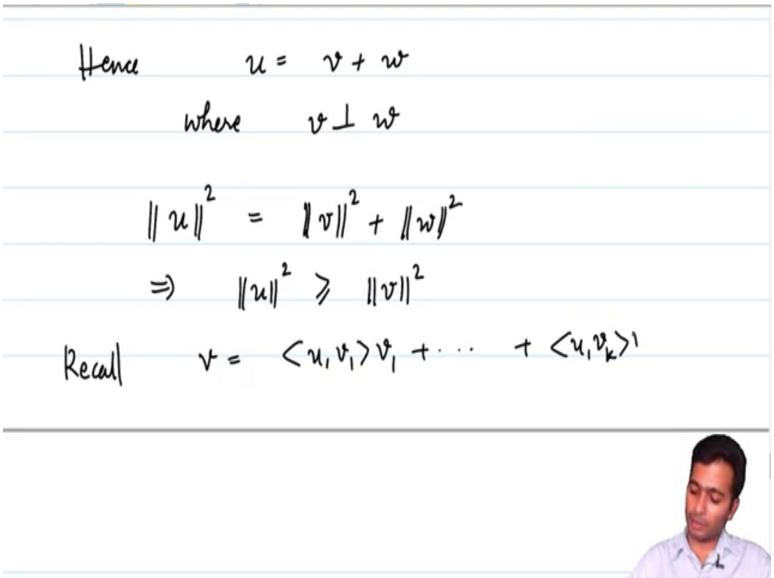

So, what do we have? We have hence u is equal to v plus w where v is also orthogonal to our w, so what can we say about the length of u square? Length of u square, now by the Pythagoras theorem, if you go back and check, there was no assumption of any finite dimensionality to talk about Pythagoras theorem, this is just going to be equal to the length of v square plus the length of w square.

And that would imply that the length of u square is greater than or equal to the length of v square. Or length of u is greater than or equal to the length of v. This is precisely what we had set out to proof, what was length of v square? V was this, so length of v is just going to be the sum of absolute value of the inner product square.

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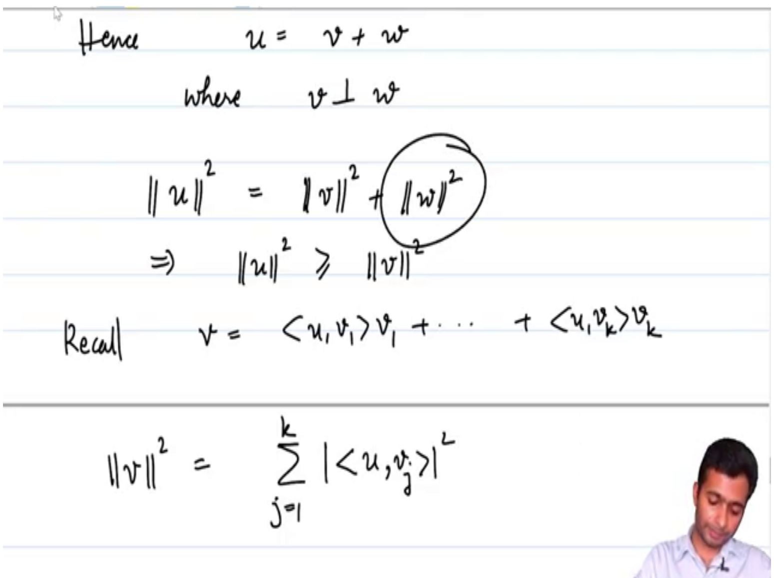

So, recall that v was just inner product of u with v 1 times v 1 plus up to the inner product of u with v k times v k and therefore the length of v square is just equal summation of the absolute value of u with v j square where j is from 1 to k and that establishes Bessels' inequality. When can we say about equality? We can say that the equality happens if this w is 0, right.

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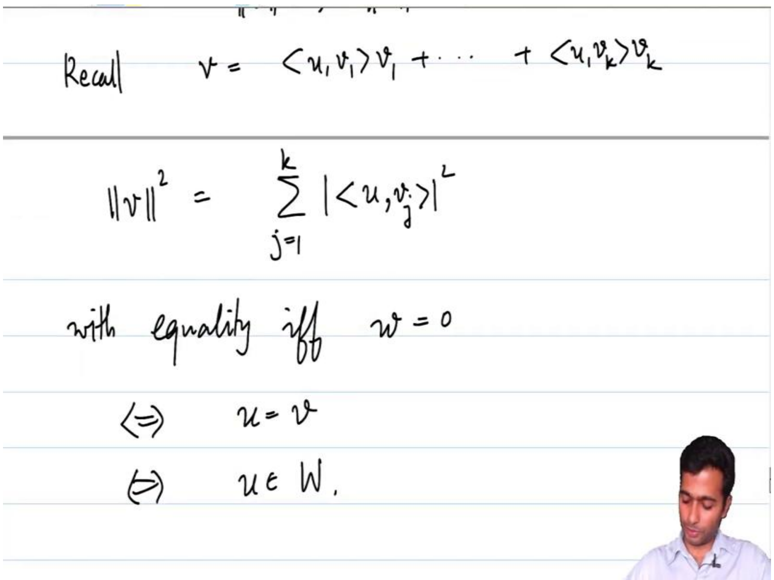

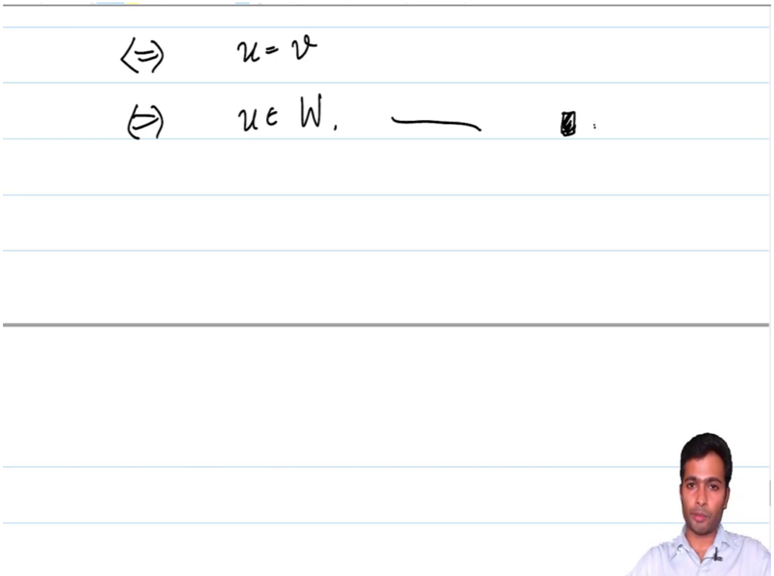

With equality if and only if w is equal to 0, but what does that mean? That means that this is if and only if u is equal to v which is if and only if u belongs to w. So, we had set out to proof this inequality, and the equality is something which we just observed as following when w is equal to 0, and with that we have completed to proof of Bessel’s inequality. So, let me stop here.