Linear Algebra
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Lecture 11.3
Adjoint of a linear transformation

So, we have already seen the Riesz Representations Theorem, which states that if you are given a linear functional T on an inner product space V there exist a unique W in V such that TV which is a scalar is the inner product of V with W for all V in capital V. Our next goal is to study how the inner product interacts with linear transformation between inner product spaces and in order to do that, we develop the notion of what is called as the adjoint of a given linear transformation. So, let us begin by considering a linear transformation between inner product spaces.

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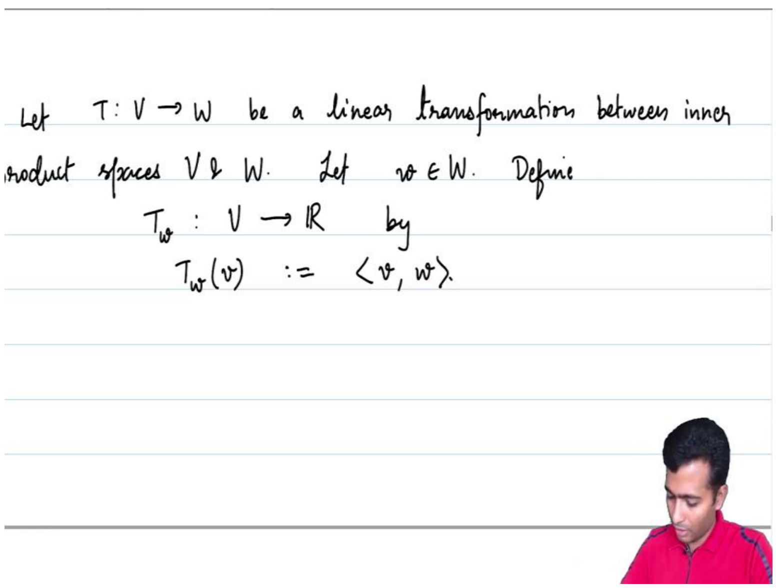

So, let, so, let T from V to W be a linear transformation between inner product spaces V and W. So, let w be some vector in capital W and let us define a function let us define T subscript w, which is from V to R by, what is T w of small v? This is defined to be the inner product of v with w. So, let us look at an example to understand what we are doing.

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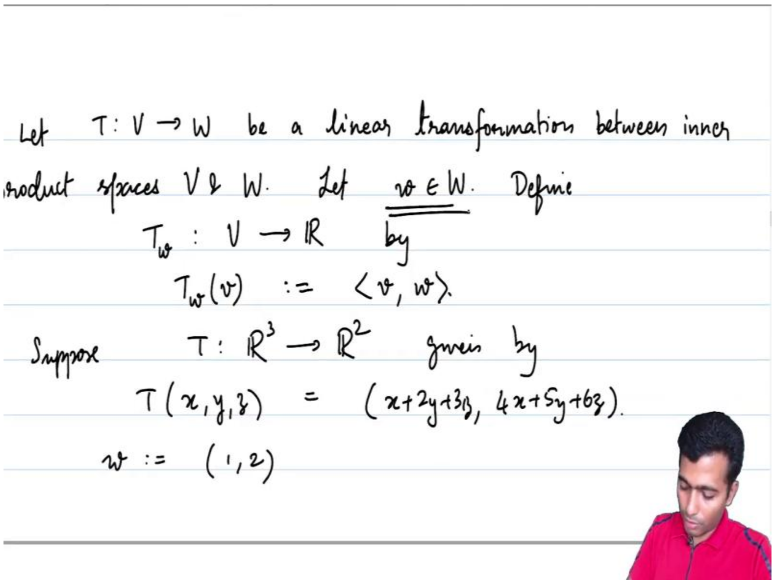

So, suppose, so, suppose T is from say, R, R 3 to R 2, given by T of say x, y, comma z, this will be some vector in say R 2, we would like to see what? Ok, let us put it something x plus y plus z, it will be too simplistic x plus 2 y plus 3 z, 4 x plus 5 y plus 6 z, suppose we have two such coordinates, where x, y, z is being sent to. And we would like to define what is? So, let us fix w, w let us say this is equal to say something like 1, comma 2. Notice that w should be in capital W, right? That is what our choice here was. So, fix one such vector 1, comma 2 in the target, which is R 2.

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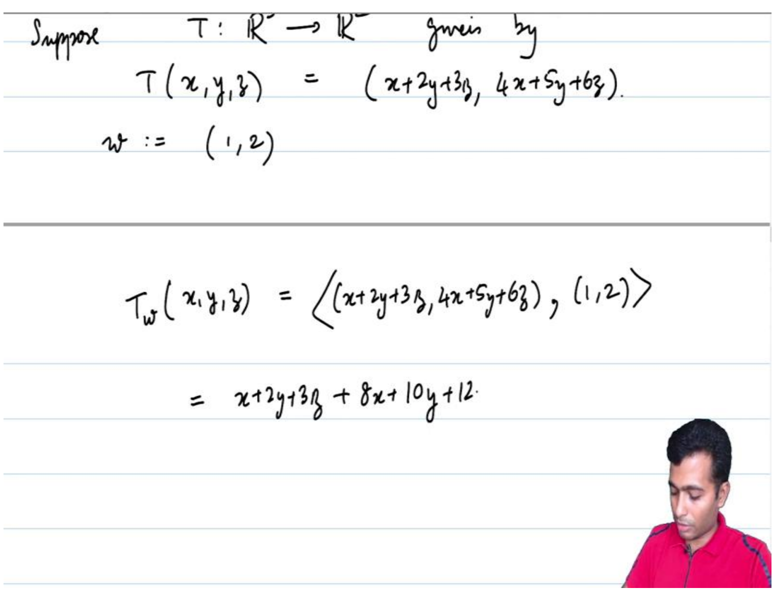

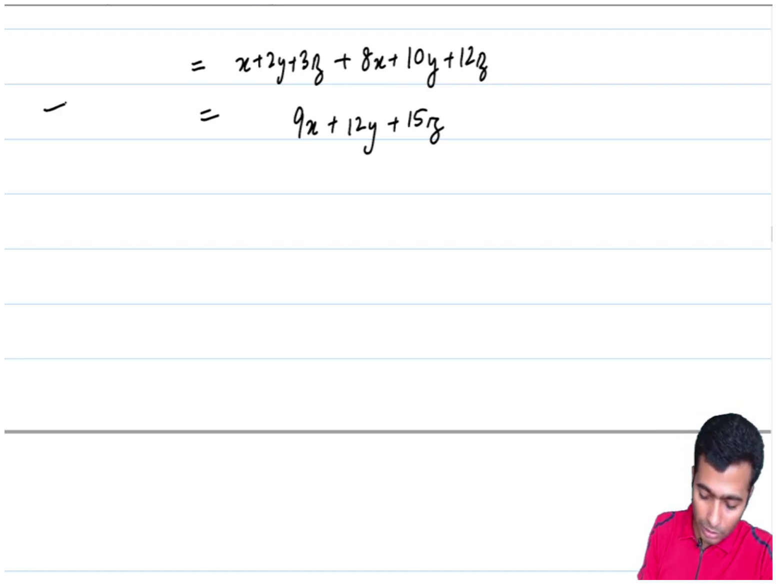

And let us define Tw of x, y, z. What was this? This was basically T of x, y, z. So, basically that is the vector x plus 2 y plus 3 z and 4 x plus 5 y plus 6 z. This vector, inner product of this with W, which is 1, comma 2, which is equal to x plus 2 y plus 3 z times 1 plus 2 times 4 x plus; I will write it down directly, this is 8 x plus 10 y plus 12 z which is equal to 9 x plus 12 y plus 15 z, which happens to be a linear functional.

So, notice that this isTw of x, y, z. So, this happens to be a linear functional on R 3 which is the target, so that is not any mere coincidence.

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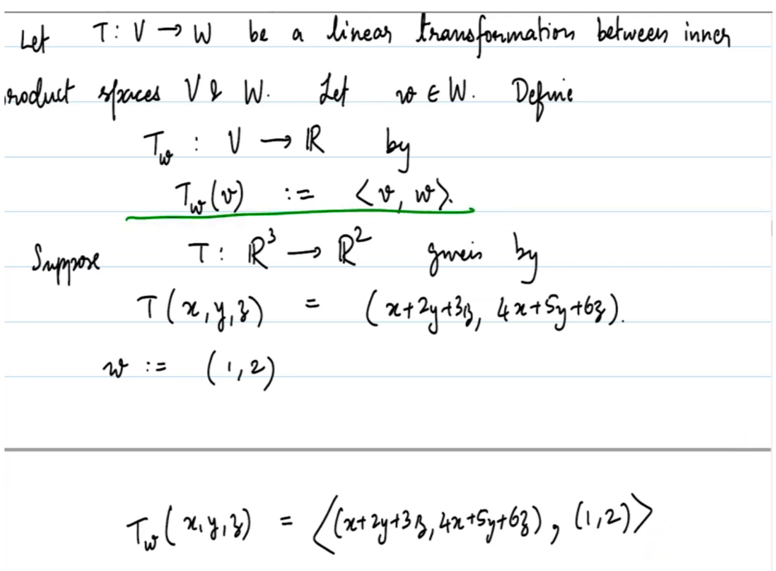

So let us, let us observe that this function which now I am underlining in green, which is from V to R is always going to be linear. So, this is in particular a linear functional, ok. So, I should, let me let me just prove that first.

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$$T_w(x,y,z) = \langle (x+2y+3z, 4x+5y+6z), (1,2) \rangle$$

$$= x+2y+3z + 8x+10y+12z$$

$$T_w(x,y,z) = 9x + 12y + 15z$$

$T: V \to W$ & $w \in W$. $T_w(v) := \langle v, w \rangle$

Claim: $T_w : V \to R$ is a linear functional.

So, claim Tw from V to R. So, we are not in the setup of the example, we are in the set up before that, where T is (())(04:41) So, T from V to W between linear inner product spaces which is a linear transformation and w be in capital W, Tw of v is defined as the inner product of v, comma w, this is the setup. So, the claim is that Tw is a linear functional. So, like in this case the example, it just happened to be linear functional it is going to be linear functional always. So, let us give a quick proof of this.

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Claim: $T_w : V \to \mathbb{R}$ is a linear functional.

$$
\begin{aligned}
T_w(v_1+v_2) &= \langle T(v_1+v_2), w \rangle \\
&= \langle Tv_1 + Tv_2, w \rangle \\
&= \langle Tv_1, w \rangle + \langle Tv_2, w \rangle
\end{aligned}
$$

So, what will be necessary to show that this is, what is that we have to check? The first one is to check that if we have v 1 and v 2 in capital V what is this, by definition this T of v 1 plus v 2 inner product with w. But T is a linear map to begin with. So, this is just the inner product of Tv 1 plus Tv 2 and W which by the properties of the inner product is the inner product of Tv 1 with w plus the inner product of Tv 2 with w but that is by definition.

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$$
\begin{aligned}
T_w(v_1+v_2) &= \langle T(v_1+v_2), w \rangle \\
&= \langle Tv_1 + Tv_2, w \rangle \\
&= \langle Tv_1, w \rangle + \langle Tv_2, w \rangle \\
&= T_w(v_1) + T_w(v_2) \qquad \forall\, v_1, v_2 \in V
\end{aligned}
$$

The first term is Tw of v 1 and the second term is Tw of v 2 that by establishing that Tw s indeed (())(6:02). So, the first property is correct, this is for all v1 and v2 for all v1 and v2 in capital V.

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$$T_w(v_1+v_2) = \langle T(v_1+v_2), w\rangle$$
$$= \langle Tv_1 + Tv_2, w\rangle$$
$$= \langle Tv_1, w\rangle + \langle Tv_2, w\rangle$$
$$= T_w(v_1) + T_w(v_2) \quad \forall\, v_1, v_2 \in V$$

$$T_w(cv) = \langle T(cv), w\rangle$$
$$= c\langle Tv, w\rangle$$
$$= c\, T_w v$$

And for c in the field of scalars and v a vector in capital V, let us see what this is, this is again TCv inner product with w which is equal let me step a few, let me skip a few steps, write it as C times Tv, comma w which is equal to C times Tw of v. Therefore, Tw is always a linear functional.

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$$= c\, T_w v$$

Hence T_w is a linear functional on V.

By Riez Representation theorem, $\exists !$ vector $T^*w \in V$

s.t. $T_w(v) = \langle v, T^*w\rangle$

$T: V \to W$ & $w \in W$: $T_w(v) := \langle Tv, w \rangle$

Claim: $T_w : V \to \mathbb{R}$ is a linear functional.

$$
\begin{aligned}
T_w(v_1 + v_2) &= \langle T(v_1 + v_2), w \rangle \\
&= \langle Tv_1 + Tv_2, w \rangle \\
&= \langle Tv_1, w \rangle + \langle Tv_2, w \rangle \\
&= T_w(v_1) + T_w(v_2) \qquad \forall\, v_1, v_2 \in V
\end{aligned}
$$

$$
\begin{aligned}
T_w(cv) &= \langle T(cv), w \rangle \\
&= c \langle Tv, w \rangle \\
&= c\, T_w v
\end{aligned}
$$

And therefore, hence, Tw is a linear functional on V on an inner product space V. But what do we know about linear functionals on inner product spaces. We know that by Riesz representation theorem there is some vector W, W is already picked. So, T star w with now, which when you take the inner product with V, will give you a linear functional.

So, let me just note that by Riesz Representation Theorem, which we proved in the previous video, Riesz Representation Theorem. There exists a unique vector, let us call it T star w in capital W. So, notice that, ok, not capital, this is in capital V. Notice that this is the inner product, which is being taken in capital V. So, I am not writing down where which inner product has been taken, but the context should make it clear. Such that.

Let us see what so such that Tw of v is equal to inner product have v with T star w. This is precisely the statement of Riesz Representation Theorem. Tw is a linear functional. So, there is a unique vector T star w in capital V such that, if you look at T star v, V inner product with T star w that will give you Tw of V.

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$$\Rightarrow \langle Tv, w \rangle = \langle v, T^* w \rangle$$

But what is our left hand side? Left hand side Tw of v is nothing but the inner product of Tv with w which is equal to v, comma T star of w. So, notice again, now we should be a, bit more careful as usual, the left hand side the inner product is being looked at in the inner product space, w. The right hand side is the inner product in capital V. So, the abuse of notation however, is quite clear from the context and should not create any confusion. All right, so, we have seen an example here.

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$$\Rightarrow \langle Tv, w \rangle = \langle v, T^* w \rangle \quad \text{——} \quad (*)$$

Example (Contd.) $\quad T(x, y, z) = (x+2y+3z,\ 4x+5y+6z)$

$(a, b) \in \mathbb{R}^2$

$$\langle (x+2y+3z,\ 4x+5y+6z), (a, b) \rangle = \langle (x, y, z), T^* w \rangle$$

Let us do one thing, let us compute the explicit formula of T star in our case, let me just write down here the example I will write this as example, about continued, let me write it like this. So, recall that T of x, y, z was equal to x plus 2 y plus 3 z, and 4 x plus 5 y plus 6 z and let us try let us attempt computing. What is a, b? So, T is from R 2, R 3 to R 2. So, T star will be from R 2 to R 3.

So, a, b will let us see what a, b is, a, b in R 2, we would like to compute what is T star of a, b. And let us look at the formula just above T star of a, b will satisfy this formula. So, T star of let me write it like this, Tv is so x plus 2 y plus 3 z, comma 4 x plus 5 y plus 6 z. This inner product with a, b is going to be equal to what was our v, v is in our case x, comma y, comma z, comma T star w. This is precisely the formula we have from star, ok.

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$$a(x+2y+3z)+b(4x+5y+6z)$$
$$=(a+4b)x+(2a+5b)y+(3a+6b)z$$

$$=\langle (x,y,z),(a+4b,2a+5b,3a+6b)\rangle$$

What is the left hand side here? The left hand side is a times x plus 2 y plus 3 z, plus 4 x plus 5 y plus 6 z times b. What is this? Let us just compute what this is, this is equal to a plus 4 b times x plus 2 a plus 5 b times y plus 3 a plus 6 b times z, but because it is in R3 and the inner product, standard inner product of R3 is quite, quite familiar for us, we know that this is the inner product of the vector x, y, z and a plus 4 b, comma 2 a plus 5 b, comma 3 a plus 6 b.

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$$= \langle (x,y,z), (a+4b, 2a+5b, 3a+6b)\rangle$$

$$= \langle (x,y,z), T^* w\rangle$$

i.e. $\langle (x,y,z), T^*(a,b)\rangle = \langle (x,y,z), (a+4b, 2a+5b, 3a+6b)\rangle$

$\forall (x,y,z) \in \mathbb{R}^3$

$\Rightarrow T^*(a,b) = a+4b, 2a+5b, 3a+6b)$

But what is this? This is also equal by star by from here this is also equal to the inner product of x, y, z and T star w. But the uniqueness so, what is this telling us? i.e, let me just write it down clearly so that it is clear, inner product of x, y, z and T star w, this is equal to the inner product of x, y, z. So, w let me just write down what w here is w is a, comma b, is equal to inner product of x, y, z and a plus 4 b, 2 a plus 5 b, 3 a plus 6 b and this is true for all x, comma y, comma z; in R 3, and this implies by the uniqueness the proposition which proved uniqueness in the previous video this proves that T star of a, comma b is equal to a plus 4 b, 2 a plus 5 b, 6, 3 a plus 6 b, ok.

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i.e. $\langle (x,y,z), T^*(a,b)\rangle = \langle (x,y,z), (a+4b, 2a+5b, 3a+6b)\rangle$

$\forall (x,y,z) \in \mathbb{R}^3$

$\Rightarrow T^*(a,b) = (a+4b, 2a+5b, 3a+6b)$

i.e. $T^*: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation.

That is good because if you notice T star i.e, T star from R 2 to R 3 is a linear transformation. Of course, in this case, we have very specifically computed it in this particular example and we obtain that, but this is a good motivation to conjecture that the map T star that we are defining in the general case always happens to be a linear transformation as well. So, that is going to be the content of our first proposition.

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$\Rightarrow \quad T^*(a,b) = (a+4b,\ 2a+5b,\ 3a+6b)$

i.e. $T^*: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation.

<u>Definition of the adjoint</u>

Let $T: V \to W$ be a linear transformation between inner product spaces. Then the adjoint of T, denoted by T^*,

Let $T: V \to W$ be a linear transformation between inner product spaces. Then the adjoint of T, denoted by T^*,
is the map $T^*: W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

So, let us state it and prove it. So, let T from V to W. So, did I define what T star is, ok. So, I have a definition to make the before that, so, before I jump into the preposition, let me give you the definition of the adjoint. All this was the precursor for this definition of adjoint.

So, let T from V to W be a linear transformation between inner product spaces. T star, ok, then the adjoint of T, of T, which is denoted as T star, denoted by T star is the map as defined above from W to V such that inner product of v; Tv, comma w which is in capital W is equal to the inner product of v with T star w. So, this is what we had, we just showed by Riesz Representation Theorem that we can define such a map T star from W to V and that map is called as the adjoint of T.

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$$\langle Tv, w\rangle = \langle v, T^{*}w\rangle$$

Proposition: Let $T: V \to W$ be a linear transformation between inner product spaces. Then $T^{*}: W \to V$ is a linear transformation.

Definition of the adjoint

Let $T: V \to W$ be a linear transformation between inner product spaces. Then the adjoint of T, denoted by T^{*}, is the map $T^{*}: W \to V$ such that

$$\langle Tv, w\rangle = \langle v, T^{*}w\rangle$$

Proposition: Let

The next proposition which I was about to state a few minutes back, let me now state it, it states that the adjoint of given linear transformation between inner product spaces will again be a linear transformation. So, let me write it down.

So, let me just run over the definition for you once more. You start off with a linear transformation from V to W, where V and W are inner product spaces the adjoint T star of T is a map from W to V such that inner product of T v and w is the same as the inner product of v and T star w. This T star w as we had seen is obtained using the Riesz Representation Theorem applied to T subscript w which we had defined a bit earlier, ok.

So, the proposition states that. So, the context of the proposition is that, let T from V to W be a linear transformation between inner product spaces then T star from W to V is a linear transformation, ok. So, let us give a proof of this statement.

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Proof: Let $w_1, w_2 \in W$

$$\begin{aligned} \langle v, T^*(w_1 + w_2) \rangle &= \langle Tv, w_1 + w_2 \rangle \\ &= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle \\ &= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle \\ &= \langle v, T^* w_1 + T^* w_2 \rangle \end{aligned}$$

Let $T: V \to W$ be a linear transformation between inner product spaces. Then the adjoint of T, denoted by T^*, is the map $T^*: W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

Proposition: Let $T: V \to W$ be a linear transformation between inner product spaces. Then $T^*: W \to V$ is a linear transformation.

Proof, (())(17:22) the main tool that we have to go by is this particular equality. The inner product of Tv with w is equal to the inner product of v with T star w. So, let us somehow use this to prove that T star is a linear transformation. So, let w 1, w 2 be in capital W and let us look at what is the property that is satisfied by T star w.

So, T star, T star w 1 plus w 2, w 1 plus w 2. Inner product of this with a vector v in capital V by definition, this is equal to the inner product of Tv with w 1 plus w 2, but inner products are conjugate linear in the second variable as well.

So, in particular, it is additive and hence, this is equal to the inner product of, this is the inner product in w that we are looking into, this is inner product of Tv with w 1 plus the inner product of Tv with w 2. But we know that by the property of the adjoint which we have just defined, this is V inner product with T star w 1. Notice that we have now moved over to an inner product in capital V plus the inner product of v with T star of w 2 and by the properties of the inner product in capital V, this is the inner product of v with T star w 1 plus T star w 2.

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$$\underline{\langle v, T^*(w_1+w_2)\rangle} = \langle Tv, w_1+w_2\rangle$$
$$= \langle Tv, w_1\rangle + \langle Tv, w_2\rangle$$
$$= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle$$
$$= \underline{\langle v, T^*w_1 + T^*w_2\rangle}$$

$$\langle v, T^*(w_1+w_2)\rangle = \langle Tv, w_1+w_2\rangle$$
$$= \langle Tv, w_1\rangle + \langle Tv, w_2\rangle$$
$$= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle$$
$$= \langle v, T^*w_1 + T^*w_2\rangle$$
$$\Rightarrow \quad T^*(w_1+w_2) = T^*w_1 + T^*w_2 \quad \text{(by uniqueness in Riesz-Repn. thm.)}$$

Now, again both the, both these are linear functional, this is also a linear functional, this is also a linear functional when looked at as a function on V and by Riesz Representation Theorem there exist a unique vector in capital V, which when you take the inner product with v will give you the linear functional.

And therefore, by the uniqueness, this implies that by, by the proposition that we proved in the last lecture, this implies that T star of w 1 plus w 2 is equal to T star of w 1 plus T star of w 2, by uniqueness in Riesz Representation Theorem, you should check the uniqueness part, should check very carefully how the uniqueness is being used here to conclude that T star of w 1 plus w

2 is the same as T star of w 1 plus T star w 2, ok. We have seen the case when it is the check for whether T star is additive.

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$$= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle$$

$$= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle$$

$$= \langle v, T^* w_1 + T^* w_2 \rangle$$

$$\Rightarrow \quad T^*(w_1 + w_2) = T^* w_1 + T^* w_2 \quad \text{(by uniqueness in Riesz - Repn. thm.)}$$

Let $c \in \mathbb{F}$ & $w \in W$.

Now, let c be some scalar and w be some vector and capital W. So, notice that we have slowly stopped any reference to our field of scalars being exclusively real numbers. That is not at all the case we are considering there is all these statements are completely true. Even keeping in mind that the possibility of the field of scalars being complex numbers is there.

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Let $c \in \mathbb{F}$ & $w \in W$.

$$\underline{\langle v, T^*(cw) \rangle} = \langle Tv, cw \rangle$$

$$= \bar{c} \langle Tv, w \rangle$$

$$= \bar{c} \langle v, T^* w \rangle$$

$$= \underline{\langle v, c T^* w \rangle} \quad \forall v \in V$$

Let $c \in \mathbb{F}$ & $w \in W$.

$$\begin{aligned}\langle v, T^*(cw)\rangle &= \langle Tv, cw\rangle \\ &= \bar{c}\langle Tv, w\rangle \\ &= \bar{c}\langle v, T^*w\rangle \\ &= \langle v, cT^*w\rangle \quad \forall v \in V\end{aligned}$$

$$\Rightarrow \quad T^*(cw) = cT^*w. \quad (\text{by uniqueness})$$

Therefore, if you look at T star of cw, and if you look at the inner product of this with v, this is equal to the inner product of Tv with cw, by the very definition. And the inner product in W which is being considered to the right, we can take out the scalar out, it is conjugate linear remember that so, this is going to be c bar times Tv, comma w, which now, by definition is the inner product of v with T star w, this is the very definition of the adjoint that we have defined and therefore, this now can be brought back in by using the properties of the inner product in this case in V to conclude that this is c times T star of w, but this is true for all V in capital V. And hence, as linear functionals this and this both are linear functionals which are equal.

And by uniqueness of the Riesz Representation Theorem. This gives that T star cw is equal to c times T star w, by uniqueness. I will just write by uniqueness here you should again think over how uniqueness was used here of Riesz Representations Theorem, ok.

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$$\Rightarrow \quad T^*(cw) = c\,T^* w. \quad (\text{by uniqueness})$$

$$\Rightarrow \quad T^* \text{ is a linear transformation.}$$

$$\langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle = \langle v_1, \bar{c} v_2 \rangle$$

$$= \langle (x,y,z), (a+4b, 2a+5b, 3a+6b) \rangle$$

$$= \langle (x,y,z), T^* w \rangle$$

i.e. $\langle (x,y,z), T^*(a,b) \rangle = \langle (x,y,z), (a+4b, 2a+5b, 3a+6b) \rangle \quad \forall (x,y,z) \in \mathbb{R}^3$

$$\Rightarrow \boxed{T^*(a,b) = (a+4b, 2a+5b, 3a+6b)}$$

i.e. $T^* : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation.

<u>Definition of the adjoint</u>

The both the checks are done this concludes that T star is linear transformation. So, given every, given any linear transformation T from V to W, we are able to get hold of a T star which is again a linear transformation from W to V. So, you might be tempted to think of the adjoint as something like the inverse but let me stop you right there and say that the inverse of a linear transformation can be defined only when the linear transformation is invertible. However, the adjoint can be defined for any linear transformation.

So, the right notion with which you should probably. So, if you look at inner product of say v 1 and cv 2, suppose this is the case this is what is this, this is c bar times v 1, comma or maybe I should start here, let me start with cv 1, comma c 2. This will be c times v 1, v 2 and this is going to be v 1, comma c bar v2. So, c is becoming c bar when it is going to the other coordinating the inner product.

So, T star in some sense captures an analogue of the complex conjugate in the language of operators; linear, linear maps. So, for linear maps the adjoint in some sense is the correct notion of the complex conjugate that we have to consider. We will elaborate on this in a few minutes and to do that let us use the power of V being V and W being inner product spaces. If you notice, if we scroll up a, bit and look at the example that we were evaluating, we did some, some amount of computation to come to the conclusion that our T star here is being defined in this manner.

It might actually be a good point to say that if we are to use the tools of the, say, the orthonormal bases that we have the notion of orthonormal bases that we have in an inner product space, a lot of these things might actually turn out to be far simpler. So, our next goal here is to compute T star the matrix of T star rather in terms of the matrix of T by using the tools of an orthonormal bases, so, let us fix.

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$$\langle cv_1, v_2\rangle = c\langle v_1, v_2\rangle = \langle v_1, \bar{c}v_2\rangle$$

Let $\beta = (v_1, \ldots, v_n)$ and $\gamma = (w_1, \ldots, w_m)$ be orthonormal bases of our inner product spaces V & W.

Recall that $v = \langle v, v_1\rangle v_1 + \cdots + \langle v, v_n\rangle v_n$ $\forall\, v \in V$.

So, let us fix beta, say v 1 to v n and gamma, which is say w 1 to w m. So in this case now, we are in finite dimensional vector spaces till now, to define most of the things we defined we were not. Oh Yes, we did. So, let me be a, bit careful, let us not bother about infinite dimensional inner product spaces right now, we did use Riesz Representation Theorem to get hold of adjoint.

So, in the case of infinite dimensional vector spaces, we do not have a ready analog of Riesz Representation Theorem and we, we will have to do some work if at all it exists to talk about the adjoint of a given linear transformation.

So, in this lecture let us all through assume that our inner product spaces are finite dimensional, we have indeed used it very very strongly to even talk about what T star w is. So, let beta v 1 to v n and gamma w 1 to w m be orthonormal bases of our inner product spaces V and W. What is the good thing about an orthonormal bases? When we have an orthonormal bases v 1 up to v n we know the explicit expression of any vector v in terms of say v 1 to v n.

We know that so let me recall that v is the inner product of v with v 1 times v 1 plus up to v with v n times v n for all v in capital V. We know that any vector we know the explicit linear combination of v 1, v 2, up to v n, which will give us the vector v. So what we will do is let us compute the matrix of T with respect to beta gamma.

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Recall that $v = \langle v, v_1\rangle v_1 + \cdots + \langle v, v_n\rangle v_n$

$\forall\, v \in V.$

Goal: Compute $[T]_\beta^\gamma$

$$\left([Tv_1]^\gamma,\ [Tv_2]^\gamma,\ \ldots,\ [Tv_n]^\gamma \right)$$

So, goal. To compute T beta gamma, and for that we will be considering what is Tv 1 with respect to gamma, Tv 2 with respect to gamma and so on, Tv n with respect to gamma and this will be our columns of the matrix T beta gamma, ok. But we know explicitly what this is.

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Hence

$$[T]_\beta^\gamma = \begin{pmatrix} \langle Tv_1, w_1\rangle & \langle Tv_2, w_1\rangle & \cdots & \langle Tv_n, w_1\rangle \\ \langle Tv_1, w_2\rangle & \langle Tv_2, w_2\rangle & & \langle Tv_n, w_2\rangle \\ \vdots & \vdots & & \vdots \\ \langle Tv_1, w_m\rangle & \langle Tv_2, w_m\rangle & \cdots & \langle Tv_n, w_m\rangle \end{pmatrix}$$

So, this is going to be so, hence T beta gamma will just be equal to the inner product of Tv 1 with w 1, Tv 1 with w 2, Tv 1 with w n. This will be the column representation of Tv 1 with respect to gamma. How about the second column? Tv 2 with w 1, Tv 2 with w 2, Tv 2 with w, oh this is not w n, this is w m. After all it is an m cross m matrix, right. Because W is of dimension m.

And let us go to the final column final column will be Tv n w 1, Tv n w 2, this will be Tv n w m. So, this is our m cross n matrix of T with respect to beta and gamma. So, what is the i, j th entry, let us explicitly write down what the i, j th entry is.

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$$\left([Tv_1] , [Tv_2] , \ldots , [Tv_n] \right)$$

Hence

$$[T]_\beta^\gamma = \begin{pmatrix} \langle Tv_1, w_1 \rangle & \langle Tv_2, w_1 \rangle & \cdots & \langle Tv_n, w_1 \rangle \\ \langle Tv_1, w_2 \rangle & \langle Tv_2, w_2 \rangle & & \langle Tv_n, w_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle Tv_1, w_m \rangle & \langle Tv_2, w_m \rangle & \cdots & \langle Tv_n, w_m \rangle \end{pmatrix}$$

Hence the (i, j) entry of $[T]_\beta^\gamma = \langle Tv_j, w_i \rangle$

$$\left([Tv_1] , [Tv_2] , \ldots , [Tv_n] \right)$$

Hence

$$[T]_\beta^\gamma = \begin{pmatrix} \langle Tv_1, w_1 \rangle & \langle Tv_2, w_1 \rangle & \cdots & \langle Tv_n, w_1 \rangle \\ \langle Tv_1, w_2 \rangle & \langle Tv_2, w_2 \rangle & & \langle Tv_n, w_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle Tv_1, w_m \rangle & \langle Tv_2, w_m \rangle & \cdots & \langle Tv_n, w_m \rangle \end{pmatrix}$$

Hence the (i, j) entry of $[T]_\beta^\gamma = \langle$

Hence the i, j entry of T beta gamma, this is nothing but let us go back to this matrix, check it out, the i is here j is here. So, the ith one is with respect to w i and the jth one is with respect to v j. So, this will be Tv j wi. This is our matrix i, jth entry of the matrix of T with respect to beta gamma.

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$T^*: W \to V$

What is $[T^*]_\gamma^\beta$

$$(i, j) \text{ entry} = \langle T^* w_j, v_i \rangle$$
$$= \langle w_j, T v_i \rangle$$
$$= \overline{\langle T v_i, w_j \rangle}$$

$$= \langle w_j, T v_i \rangle$$
$$(i, j) \text{ entry of } [T^*]_\gamma^\beta = \overline{\langle T v_i, w_j \rangle}$$

Now, we know where T star is from T star is a map from W to V and we have the same orthonormal bases; gamma and beta of W and V respectively. Let us see what will be the expression of T star with respect to beta gamma. So, what is T star, not beta gamma, gamma beta, gamma beta.

But we already did the hard work to show that right the i, jth entry is equal to T star of in this case the role of v and beta and gamma is reverse. So, this is going to be T star of w j and v i, this is precisely the i, j entry of. If you do the same process that we did above, this is what the i, j entry of T star gamma beta will be. But what is T star w j v i, that is nothing but by the definition of the adjoint this is w j and Tv i and by the properties of the inner product, this is going to be Tv i w j bar.

So, let us just write down what T star. So, this is the i, jth entry of T star gamma beta. So, let us see what will be the 1 jth entry? What will be the first column? 1 j entry will be just Tv 1 w j bar, Tv 1 w 1 bar, Tv 1 w.

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$$[T^*]_\gamma^\beta = \begin{pmatrix} \overline{\langle Tv_1, w_1\rangle} & \overline{\langle Tv_1, w_2\rangle} & \cdots & \overline{\langle Tv_1, w_m\rangle} \\ \overline{\langle Tv_2, w_1\rangle} & \overline{\langle Tv_2, w_2\rangle} & \cdots & \overline{\langle Tv_2, w_m\rangle} \\ \vdots & & & \\ \overline{\langle Tv_n, w_1\rangle} & \overline{\langle Tv_n, w_1\rangle} & \cdots & \overline{\langle Tv_n, w_m\rangle} \end{pmatrix}$$

Let me write it down, the first row for example, when i is equal to 1, this is just going to be Tv 1 w 1 inner product of this bar, Tv 1 w 2 bar and the nth this is going to be Tv 1 w m. This is the first row. Second row similarly will be Tv 2 w 1 bar, Tv 2 w 2 bar, Tv 2 w m bar. And finally, the last row will be Tv n w 1 bar, Tv n w 1 bar. So, this is our matrix. So, I have just written down all the entries, let me just put the brackets. This is the matrix of T star gamma beta and if you notice this is just the adjoint of the conjugate transpose of our matrix T beta.

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$$[T^*]_{\gamma}^{\beta} = \begin{pmatrix} \overline{\langle Tv_1, w_1\rangle} & \overline{\langle Tv_1, w_2\rangle} & \cdots & \overline{\langle Tv_1, w_m\rangle} \\ \overline{\langle Tv_2, w_1\rangle} & \overline{\langle Tv_2, w_2\rangle} & \cdots & \overline{\langle Tv_2, w_m\rangle} \\ \vdots & & & \\ \overline{\langle Tv_n, w_1\rangle} & \overline{\langle Tv_n, w_1\rangle} & \cdots & \overline{\langle Tv_n, w_m\rangle} \end{pmatrix}$$

$$= \left([T]_{\beta}^{\gamma}\right)^{\dagger} \quad \left(\text{matrix adjoint of } [T]_{\beta}^{\gamma}\right)$$

Let me just note down that this is just T beta gamma adjoint. If you recall this was the symbol we used for adjoint the let me just write it down in brackets adjoint of the matrix adjoint, matrix adjoint of T beta gamma. Remember that the, recall that the matrix adjoint of a matrix A will be the conjugate transpose of the conjugate of the matrix. So, you first take the conjugate of the entries and you look at the transpose of that, that is what is being defined as the adjoint of our given matrix.

So, if you notice carefully, the choice of the word adjoint for the matrix adjoint was not arbitrary. It just turns out that when we are using, when we are defining adjoint of a linear transformation in the way we have just defined and if we compute the matrix of T and if you look at the matrix of T star, it turns out to be the adjoint of the matrix of T. So, these notions are clearly well motivated and, and rightly defined.

So, a note again back on the similarity of our notion of T star to that of taking conjugates of a complex number, if you notice that if T is a map from c to c, the conjugate transpose will just turn out to be the conjugate. So, here when the case of 1 dimensional complex vector space to 1 dimensional complex vector space is considered and if you look at linear transformation, the adjoint is just going to be the conjugate. So, this is the right notion which generalizes the idea of the complex conjugate.

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Example (contd.) $T(x, y, z) = (x+2y+3y, 4x+5y+6z)$

Let β & γ be the std. basis of $\mathbb{R}^3$ and $\mathbb{R}^2$

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Maybe we should just go back to the example that we were considering. If you recall this is again, I will let put a continued to remind you that we were already looking at this, this is what was this, this was x plus 2 x plus 3 y, 4 x plus 5 y plus 6 z. And with respect to the standard basis, so let beta and gamma be the standard basis of, let us just pick R 2 and R 3 and R 2.

Notice that such a linear map could also have been defined between c 3 and c 2 as complex vector spaces. But let us focus on R 3 to R 2 and then what will be the matrix of T beta gamma from R 3 to R 2 with respect to beta and gamma, this will just turn out to be equal to let us see what is 1 0 0 that is just going to be equal to 1, comma 4, 2, comma, this is x plus 2 y, 2, comma 5, and 3, comma 6 and what will be the conjugate transpose?

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$$[T^*]_\gamma^\beta = \left([T]_\beta^\gamma\right)^\dagger = \begin{pmatrix}\bar{1} & \bar{4}\\ \bar{2} & \bar{5}\\ \bar{3} & \bar{6}\end{pmatrix} = \begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}$$

The adjoint of beta gamma will be the first the transpose, so 1 4, 2 5, 3 6 and the conjugate of all this, but these are all real numbers and hence this is just going to be equal to 1 4, 2 5, 3 6. So, this is going to be the matrix of T star. So, this is our T star with respect to gamma beta.

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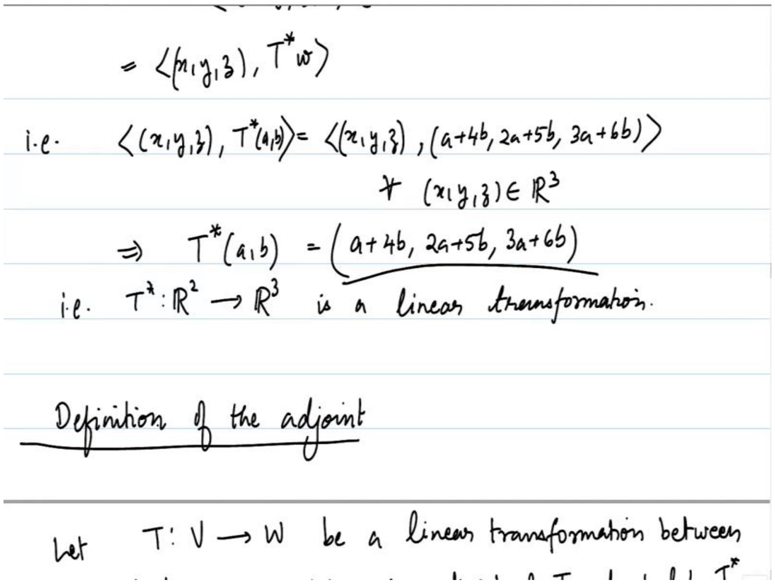
$$[T^*]_\gamma^\beta = \left([T]_\beta^\gamma\right)^\dagger = \begin{pmatrix}\bar{1} & \bar{4}\\ \bar{2} & \bar{5}\\ \bar{3} & \bar{6}\end{pmatrix} = \begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}$$

$$T^*(a,b) = \begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a+4b\\ 2a+5b\\ 3a+6b\end{pmatrix}$$



So, let us see what is T star of say a, comma b. This is just going to be multiplication by this matrix with respect to the standard coordinates. This is going to be a plus 4 b, 2 a plus 5 b, 3 a plus 6 b, which is the column representation of T star a b, which we have already computed, let us go back and say a plus 4 b, 2 a plus 5 b and 3 a plus 6 b. So, yeah. So, it is a cross check of what we have done and it is indeed the case.

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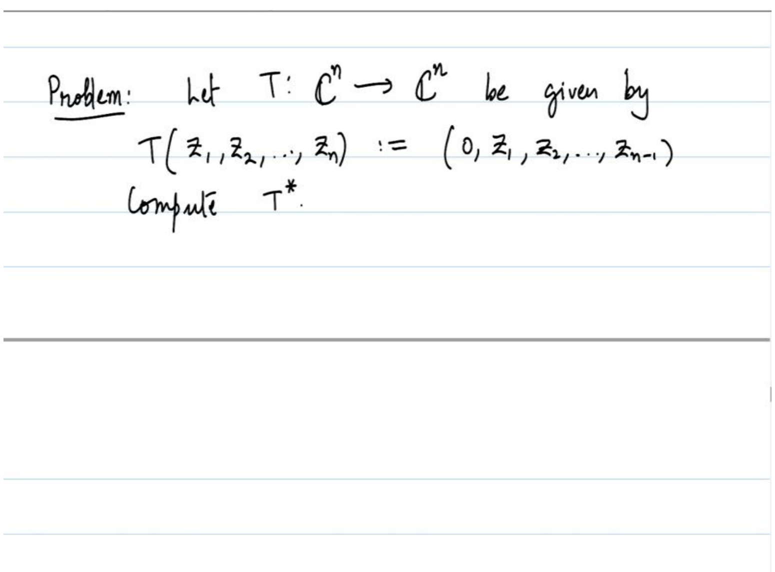

Let us look at one more example just to be in it, just to take that this is let me just call it a problem now. So that so in this week and the next the problem session is kind of integrated into the lecture. So let me just call it a problem. So, the problem is that let T from C n to itself be a linear operator be given by T of z 1, z 2 up to z n, this is defined to be 0, z 1, z 2 up to z n minus 1. So the problem is to compute T star. So, there are two ways to go about doing it. One is to check directly what it will, what the answer would be, and the one is to the next. The second option would be to compute to the matrix of T and look at the conjugate transpose.

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Problem: Let $T: \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$T(z_1, z_2, \ldots, z_n) := (0, z_1, z_2, \ldots, z_{n-1})$$

Compute T^*.

Proof: $\langle (z_1, \ldots, z_n), T^*(w_1, \ldots, w_n) \rangle =$

$$\langle T(z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle$$

Let us do both. It is not a complicated problem. So, let us see what T star of say w 1 to W n, this is what we are interested in. Oh, but yeah, so basically this will be a vector in T star will be from Cn to Cn and so let us see what will be the inner product of z 1, z 2 up to z n, and T star of w 1, w 2 up to w n. Let us see what this is, by definition, this is just going to be equal to T of z 1 to z n, comma w 1 to w n.

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Problem: Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$T(z_1, z_2, \ldots, z_n) := (0, z_1, z_2, \ldots, z_{n-1})$$

Compute T^*.

Proof: $\underline{\langle (z_1, \ldots, z_n), T^*(w_1, \ldots, w_n) \rangle} =$

$$\langle T(z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle$$

$$= \langle (0, z_1, \ldots, z_{n-1}), (w_1, \ldots, w_n) \rangle$$

$$= z_1 \overline{w_2} + \cdots + z_{n-1} \overline{w_n}$$

$$= \underline{\langle (z_1, \ldots, z_n), (w_2, \ldots, w_n, 0) \rangle}$$

$$= z_1 \overline{w_2} + \cdots + z_{n-1} \overline{w_n}$$

$$= \langle (z_1, \ldots, z_n), (w_2, \ldots, w_n, 0) \rangle \quad \forall (z_1, \ldots, z_n) \in \mathbb{C}^n$$

But we know explicitly what our T is, which is equal to 0 z 1 up to z n minus 1, comma w 1 to w n. And we know the standard inner product in Cn, this is just z 1 times w 2 bar plus up to z n minus 1, 0 times w 1 bar is 0 so I did not write it plus w n bar, but what is this, this is just the inner product of the z 1 to z n with 0, comma w 1 to w n. The inner product in Cn is quite straightforward.

So, maybe I am making a mistake, let me be a bit careful. Yes, it is a mistake. So, this is the z1 to this thing and w 2 to w n, comma 0. This is precisely what we are looking at if you carefully observe. But, this is equal to this. Now, again by the uniqueness in Riesz Representation Theorem, this is true for all z 1 to z n, so, let me just note it for all z 1 to z n, in Cn.

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$\in \mathbb{C}^n$

$$\Rightarrow \quad T^*(w_1, \dots, w_n) = \underline{(w_2, \dots, w_n, 0)}$$

$$T^*(a,b) = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+4b \\ 2a+5b \\ 3a+6b \end{pmatrix}$$

<u>Problem</u>: Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$T(z_1, z_2, \dots, z_n) := \underline{(0, z_1, z_2, \dots, z_{n-1})}$$

Compute T^*.

<u>Proof</u>: $\langle (z_1, \dots, z_n), T^*(w_1, \dots, w_n) \rangle =$

$$\langle T(z_1, \dots, z_n), (w_1, \dots, w_n) \rangle$$

And this implies that T star of w 1 to w n is equal to w 2 up to w n, comma 0. So, in some sense what is happening if you consider the right shift, this is in some sense the right shift operator of course, we are not doing it strictly the z n is being thrown out. So, I am not calling it the right shift, it is like the right shift operator and the adjoint is giving us something like the left shift operator here, ok. So, that is one way to compute T star. The other way is to compute the matrix of T.

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Let β be the std basis of $\mathbb{C}^n$

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & & \vdots & \\ \vdots & 0 & & & \vdots \\ & \vdots & & 0 & \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

So, let beta, beta be the standard basis of Cn. Standard basis if you recall, e 1, e 2 up to e n where e i is having 1 in the ith position and 0 elsewhere. So, let beta be one such standard basis, so let us try to see what is the matrix of T with respect to beta. Well, it will be T of 1 0 0 with respect to 1 0 0, T of 1 0 0 is 0 1 0 0, which is just 0 1 0 0 0, T of 0 2, 0 1 0 0 will be 0 0 1 0 0. After all this is with respect to the standard basis, the n minus 1th column would be 0 0 0 0 and 1 and one in the n th one and the final.

If you notice, T of e n is going to be 0 because it has only 1 in the n th, coordinate and 0 elsewhere by shifting it to the right, it is been thrown out. So, this is going to be the matrix of T with respect to beta.

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$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & & \vdots & \vdots \\ \vdots & 0 & & 0 & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

$$\left([T]_\beta^\beta\right)^T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & 1 \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

$$= \langle (0, z_1, \ldots, z_{n-1}), (w_1, \ldots, w_n) \rangle$$

$$= z_1 \overline{w_2} + \cdots + z_{n-1} \overline{w_n}$$

$$= \langle (z_1, \ldots, z_n), (w_2, \ldots, w_n, 0) \rangle \quad \forall (z_1, \ldots, z_n) \in \mathbb{C}^n$$

$$\Rightarrow T^*(w_1, \ldots, w_n) = \underline{(w_2, \ldots, w_n, 0)}$$

Let β be the std basis of $\mathbb{C}^n$

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & & \vdots & \vdots \end{pmatrix}$$

The conjugate transpose the adjoint of T beta beta. In this case, the entries are real so the conjugate will be the same it is just going to be 0 1 0 0 0, 0 0 1 0 0 0. The third, the second last one will be 0 0 0 0 1 and the final column would be 0 0 0 and if you multiply w 1, w 2 up to w n with this, you are precisely going to end up with this particular vector. So yeah, both these obviously it should be giving the same vector and yeah, we have just cross checked that in does. Let us do a couple more, a couple of problems more, ok.

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Problem: Let $v_0 \in V$ and $w_0 \in W$ be fixed vectors in an inner product space. Define $T: V \to W$ by

$$Tv := \langle v, v_0 \rangle w_0$$

Compute T^*.

Let us get more familiarized with the notion of adjoint by computing the adjoint explicitly. So, let T from V to W be a linear transformation, be the linear transformation, ok. So, to talk about T let us fix so let v 0 be in capital V, and w 0 be in capital W, be fixed vectors.

Where are these fixed vectors? These fixed vectors are in an inner product space. And we can define a linear transformation T from V to W by Tv is defined as the inner product of v with v 0 times w 0, we should check that this is indeed a linear transformation from V to W. Compute T star.

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$$Tv := \langle v, v_0 \rangle w_0$$

Compute T^*.

Solution: $\langle v, T^* w \rangle = \langle Tv, w \rangle$

$$= \langle \langle v, v_0 \rangle w_0, w \rangle$$

$$= \langle v, v_0 \rangle \langle w_0, w \rangle$$

Of course, multiple approaches here, I think the first approach might be better let us check. We would like to compute what is T star of W. Let us do the standard trick. Let us look at the inner product of a vector v with T star of w. This is just going to be inner product of Tv with w and we know what Tv is explicitly this is just the inner product of v with v naught times w naught into w. Which is in particular inner product of v with v naught times w naught with w. That is interesting, because both are scalars, the first one is an inner product which happened in V, the second one is an inner product which happened in W. Nevertheless, both are scalars.

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$Tv := \langle v, v_0 \rangle w_0$

Compute T^*.

Solution: $\langle v, T^* w \rangle = \langle Tv, w \rangle$

$$= \langle \langle v, v_0 \rangle w_0, w \rangle$$

$$= \langle v, v_0 \rangle \langle w_0, w \rangle$$

$$= \langle v, \overline{\langle w_0, w \rangle} v_0 \rangle$$

We can pull this scalar in by using the linearity in the first variable, this is going to be w naught w or let me be more careful, this is going to be what do we want? We want T star of w. So, this is going to be v and w naught, comma w bar times v naught. So, let me just go back and see if there is any mistake there is a good possibility that there is a mistake. So, this is Tv, comma w, Tv is v, comma v naught times w and this comes out as a normal one and this goes in. Yeah, all this is right.

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$$= \langle v, \overline{\langle w_0, w\rangle} v_0 \rangle$$

Hence $T^*w = \langle w, w_0 \rangle v_0$.

So, hence T star w is equal to the inner product of w 0, w bar which is the same as w, w 0 times v naught. So, this is the explicit expression for T star.

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Hence $T^*w = \langle w, w_0 \rangle v_0$.

Problem: Suppose $T: V \to V$ be a linear operator on an inner product space V & $\mathcal{U}$ is a subspace of V. Then prove that $\mathcal{U}$ is T-invariant iff $\mathcal{U}^{\perp}$ is T^*-invariant.

So, let me do one more problem before I conclude this video. So, the problem, problems is the following. Suppose, V is an inner product space T from V to V be a linear transformation, linear operator on an inner product space V and suppose U is a subspace of V. Then prove that U is T-invariant if and only if the orthogonal complement of U is T star invariant. So, if you recall that a subspace U is T invariant if for any vector u in capital U, T, U is also in capital U, ok. So, there are two directions to prove.

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an inner product space V & $\mathcal{U}$ is a subspace of V. Then prove that $\mathcal{U}$ is T-invariant iff $\mathcal{U}^{\perp}$ is T^*-invariant.

Proof: $\Rightarrow$ suppose $\mathcal{U}$ is T-invariant

let $u' \in \mathcal{U}^{\perp}$

For $u \in \mathcal{U}$, consider $\langle u, T^* u' \rangle$.

So, let me just prove the forward direction. So, suppose U is T-invariant, let us prove that T star u orthogonal complement of U is T star invariant. So, let us start with some vector. So, let u prime be in the orthogonal complement of U and let us see what is the situation of T star u, we would like to see that it is in the orthogonal complement of U. So, for u in capital U consider inner product of u with T star u prime.

If we show that this is equal to 0 for all, all such u in capital U then we will, we would have proved that T star u prime is in the orthogonal complement of u as well. But what does this mean this what is u?

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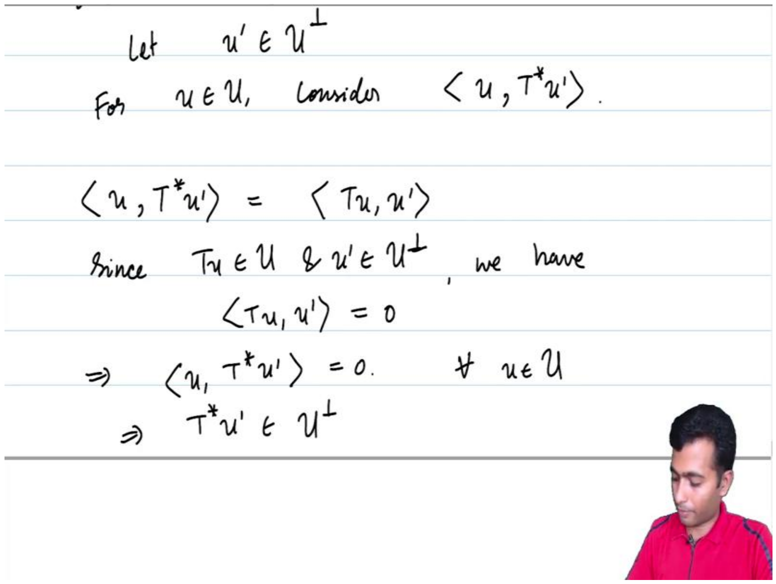

This what is u, this is a comma it is not u 1, u and T star of u prime, this is just equal to by the property of the adjoint T u and u prime. So, recall that u prime belongs to the orthogonal complement u is in capital U and capital U is T-invariant.

Since Tu belongs to capital U and u prime is in the orthogonal complement of U, we have Tu and u prime when you look at the orthogonal inner product they are orthogonal it will give us 0. This implies that u T star u prime is equal to 0, but that is true for all u in capital U. For all u in capital U, this implies that T star u prime belongs to the orthogonal complement of U.

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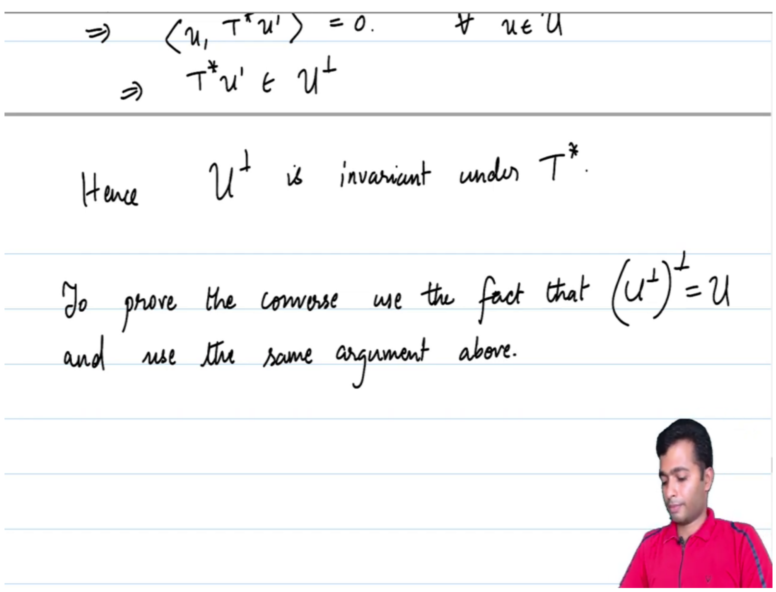

And hence U, orthogonal complement of U is invariant under T star. To prove the converse, let me just say that the orthogonal complement of the orthogonal complement of u is equal to u and the same proof works there as well. So, to prove the converse, this was an assignment problem, I hope you have done it to prove the converse. Use the fact that U orthogonal complement of the orthogonal complement of U is equal to U since U is a subspace and hence and apply the, and use the same argument above. Alright, so let me stop here.