Linear Algebra
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Lecture 11.2
Riesz Representation Theorem

So, in the last couple of weeks we defined what an inner product is what an inner product space is and we studied many properties of the inner product. So, in this week we will be studying the interaction of this inner product with linear transformations which are defined on the inner product space. We begin by studying the impact of these inner products on linear functionals. So, let us start by, let us begin by recalling what a linear functional was.

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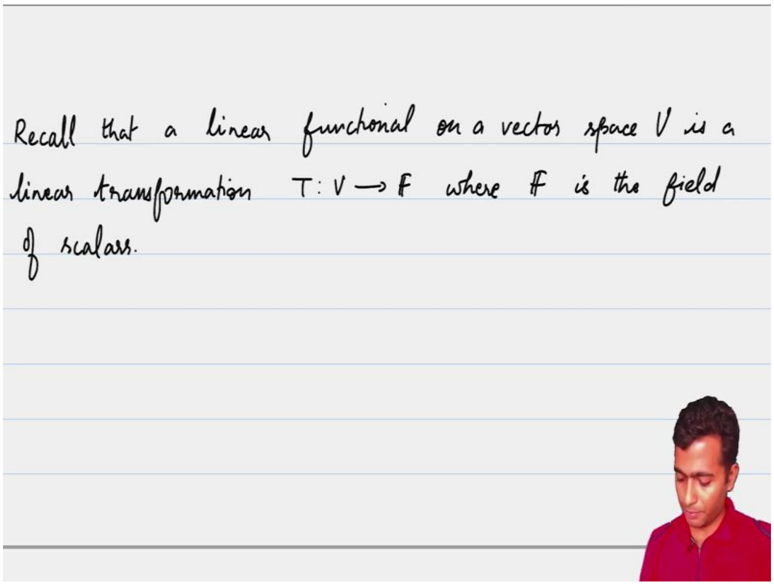

So, recall that a linear functional on a vector space V is a linear map linear transformation T from V to F, where F is the field of scalars. So, we have studied these objects in detail, we studied the dual of given vector space. Nevertheless, it is good to recall, in this context what a linear transformation is because it is useful.

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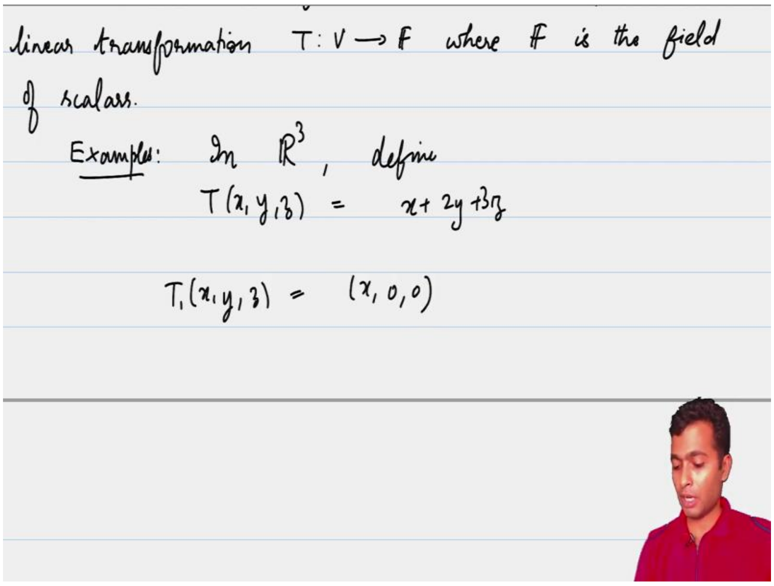

So, recall that there were quite a few examples which we have already seen recall that (())(1:57), so let us see few examples, let us look at. Let us look at a few examples of linear functionals as well. So, examples. So, let us first consider R3. So, in R3 let us define T of say x, y, z to be something like say, x plus 2y plus 3z, this is one linear functional, this is an example of a linear functional, another linear functional would be T of maybe T1 of x, y, z is equal to say x, comma 0, comma 0, this is another linear functional. Okay, how about some other examples.

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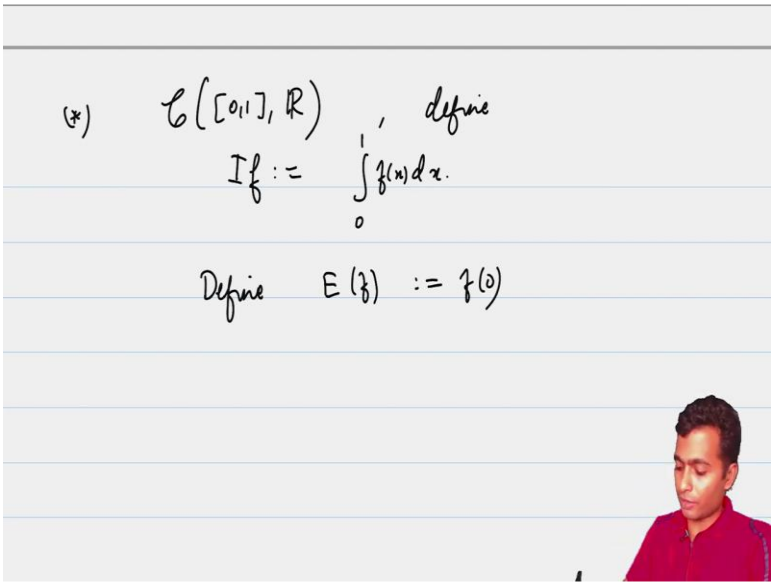

So, example 2, maybe C 0, 1, R, be space of all continuous functions on the interval 0, 1 and define, we define I of f, the integral of f to be the integral from 0 to 1, fx dx. And get checked or we did not take rather we assumed it from the knowledge of real analysis course and this is going to be a linear transformation. And yeah, so this is another example of a linear functional.

Another examples, maybe on the same space, consider define E of f to be equal to the evaluation functional at 0. So, every function f is sent to f of 0. That is another example of a linear functional. In inner product spaces, we have something special, we have some special linear functionals that we can talk about or rather we can use the inner product to define linear functions. Remember that the inner product is linear in the first variable and conjugate linear in the second variable.

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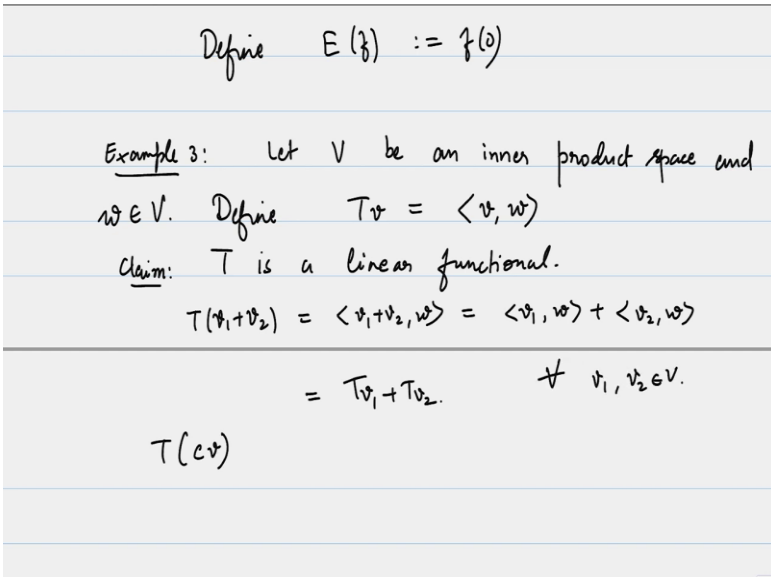

So, example. So, let me not, this example 3. So, let V be an inner product space. That means we have an inner product at our disposal. And let us fix some vector w and w be a vector in capital V. Define T v to be equal to the inner product of v with w. My first claim is that claim. T is a linear functional.

Let us see T of v1 plus v2, this is just the inner product of v1 plus v2, comma w. But what is this, this is by linearity of inner product, inner product of v1 with w plus the inner product of v2 with w. But that is precisely T v1 plus T v2. So, the additive property easily gets satisfied. How about T of say c times v this is for all v1, comma v2 in capital V this is what the linearity is about, the additive properties about.

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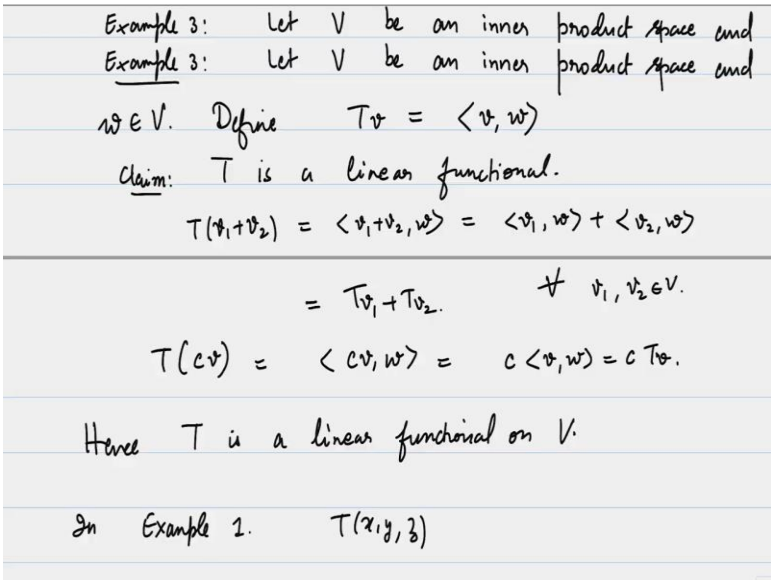

T of c v in this case be the inner product of cv with w. And the linearity property of our inner product tells us that this is c times the inner product of v with w which is c times T v. So, yes. So, hence, T is indeed a linear transformation and therefore a functional. So, let me just directly write it now as a linear functional on V. If you slowly go back to the examples we were looking at for example, say example 1.

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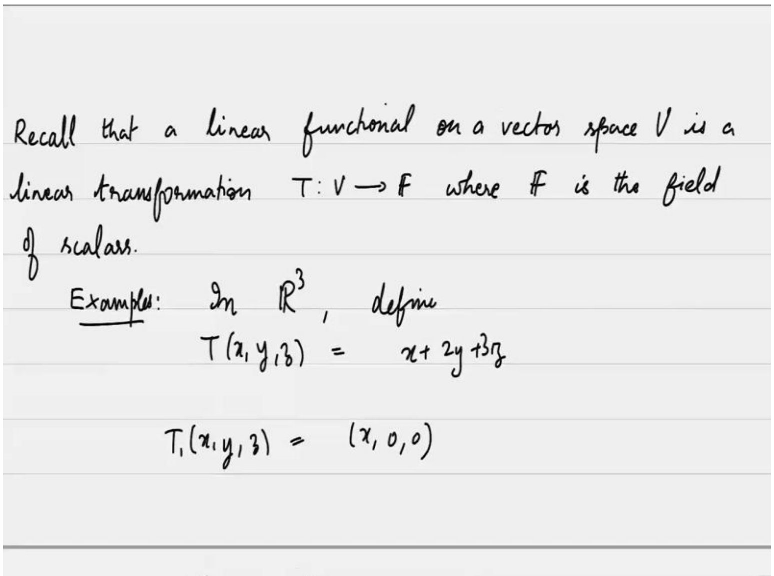

In Example 1. $T(x,y,z) = x+2y+3z$

$= \langle (x,y,z), (1,2,3)\rangle$.

$T_1(x,y,z) = (x,0,0) = \langle (x,y,z), (1,0,0)\rangle$.

In example 1, what was our T, our T was T of x, y, z if you go back and check it was x plus 2y plus 3z is the first example that came to my head yes. So, T of x, y, z was x plus 2y plus 3z and if you are to treat R3 as an inner product space, this is nothing but the inner product of x, y, z with 1, 2, 3.

So, ironically, the linear functional which we started off with in the first example turns out to be inner product with the vector 1, 2, 3. In fact, the next example that we consider T x, y, z being centred to say x, comma, 0 comma 0. This is also the inner product of x, y, z with the vector 1, 0, 0.

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In example 2: $If = \langle f,1\rangle$ where

$\langle f,g\rangle = \int_0^1 fg$.

Let us look at the next example that we considered in example 2, I of f is just the inner product of f, comma 1, where the inner product of f, comma g is defined to be integral 0 to 1 fg. This is also actually the inner product of the integration operation is inner product of f with the constant function 1.

So, this (())(7:55) that in an inner products space maybe every linear functional can be realized in this manner. So, the conjecture if we have to conjecture it like this, it is partially true in the sense that on finite dimensional vector spaces we can certainly say that this is the case. So, let me state and proof what is known as the Riesz Representation Theorem.

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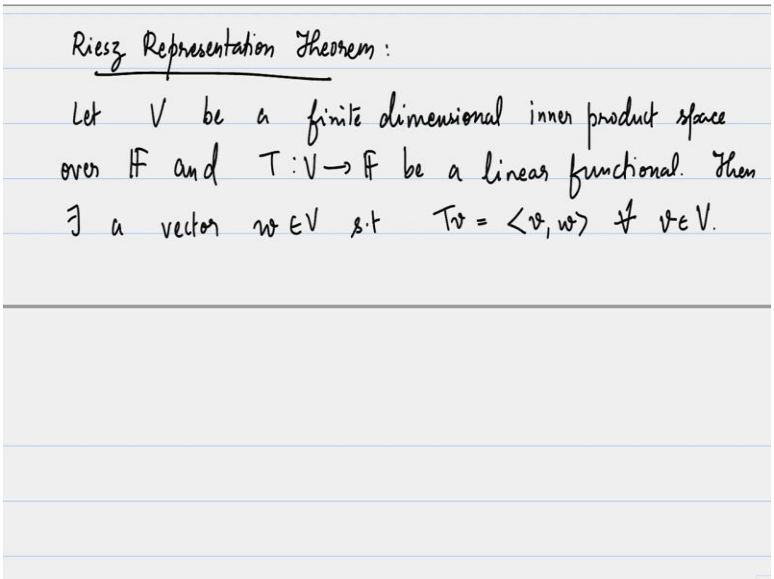

So, let me write it down. Riesz Representation Theorem. So, Riesz Representation theorem says that in a finite dimensional vector space, inner product space V, every linear functional can be realized as the inner product with a fixed vector. So, let me write it down. Let V be a finite dimensional inner product space over the field of scalars F.

So, what is that putting F by default now, F means either the field of real numbers or the field of complex numbers. So, their inner product spaces either complex inner product spaces or the linear product spaces. This result is true in both the scenarios. So, let V be a finite dimensional inner product space and T be in V star, T from V to F be a linear transformation, be a linear functional.

Then there exists a vector w in capital V such that T v is equal to the inner product of v with w for all v in capital V. So, whatever linear functional you come up with, there is some w in the finite dimensional inner product space such that T v is the inner product of v with w. So, let us look at proof of this particular theorem. Proof is revealing in the sense that it will also give us an algorithm to get hold of one such w.

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Proof: Let $(v_1', \dots, v_n')$ be an ordered basis of V.

Apply Gram-Schmidt orthonormalization to obtain an orthonormal basis $(v_1, \dots, v_n)$ of V.

So, let us look at the proof. So, V is an inner products, so since v is an inner product space. Let v1 to, let me call it v1 prime to v n prime be an ordered basis of v so it is finite dimensional so n be the dimension of capital V, so v 1 prime to be n prime be an ordered basis of Capital V.

We apply Gram-Schmidt Orthonormalization to this Gram-Schmidt and then we obtain Orthonormalization and then we obtain orthogonal normalized unit vectors as basis to obtain apply the Gram-Schmidt Orthonormalization to obtain an orthonormal basis v1 to v n. So, I put the prime in the previous case just to ensure that finally I have v1 v2 up to v n as a basis. So, there is an orthonormal basis v1 to vn of capital V.

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an orthonormal basis $(v_1, \dots, v_n)$ of V.

For $v \in V$, we have

$$v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \cdots + \langle v, v_n \rangle v_n.$$

Then $Tv = T(\quad " \quad)$

$$= \langle v, v_1 \rangle Tv_1 + \langle v, v_2 \rangle Tv_2 + \cdots + \langle v, v_n \rangle Tv_n.$$

Okay, so, let us pick some vector v in capital V for a vector v in capital V we have by one of the results we have proved earlier V is the inner product of v with v 1 times v 1 plus v, comma v 2 times v 2 plus the inner product of v with v n times v n. So, this is something which we have from the last week's lectures.

Now, let us look at what is T v then T v is equal to T of whatever is written about, but then T is a linear transformation, so in particular, this is just going to be equal to inner product of I will write down the final stage directly considering the fact that we are coming towards the end of this course. The properties of a linear transformation is going to give us that this is not v 1 this is v, comma v n. So, a bit careful. Even this is not 1 (())(13:07) confusing times T v n, this is precisely what our linear transformation, the properties of linear transformation will give us.

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then $Tv = T(\qquad)$

$$= \langle v, v_1 \rangle Tv_1 + \langle v, v_2 \rangle Tv_2 + \cdots + \langle v, v_n \rangle Tv_n.$$

$$= \langle v, \overline{Tv_1}\, v_1 \rangle + \langle v, \overline{Tv_2}\, v_2 \rangle + \cdots + \langle v, \overline{Tv_n}\, v_n \rangle.$$

$$= \langle v, \overline{Tv_1}\, v_1 + \overline{Tv_2}\, v_2 + \cdots + \overline{Tv_n}\, v_n \rangle.$$

But this is inner product and if you look at inner product of v comma v 1 that is a complex number or that is a real number it is a scalar rather, and T v 1 because T is a linear functional T v 1 is also a scalar. So, this is a product of scalars. So, I can just pull this into the inner product into the second coordinate and write this as V, comma T v 1 bar times v 1 plus v, comma T v 2 bar times v 2.

So, notice that whatever is being written inside they are just star multiplication of the scalar T v 1 bar with v 1 T v 2 bar with v 2. And finally, in the last term, it will be v, comma T v n bar times v n. Now, by the properties of the inner product, this is just the inner product of v with T v 1 bar v 1 plus T v 2 bar v 2 plus up to T v n bar v n.

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$$= \langle v, \overline{Tv_1}\, v_1 + \overline{Tv_2}\, v_2 + \cdots + \overline{Tv_n}\, v_n \rangle.$$

Define $w = \overline{Tv_1}\, v_1 + \overline{Tv_2}\, v_2 + \cdots + \overline{Tv_n}\, v_n.$

Then $Tv = \langle v, w \rangle.$ ——— ■

$$v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \cdots + \langle v, v_n \rangle v_n.$$

Then $Tv = T(\quad " \quad)$

$$= \langle v, v_1 \rangle Tv_1 + \langle v, v_2 \rangle Tv_2 + \cdots + \langle v, v_n \rangle Tv_n.$$

$$= \langle v, \overline{Tv_1}\, v_1 \rangle + \langle v, \overline{Tv_2}\, v_2 \rangle + \cdots + \langle v, \overline{Tv_n}\, v_n \rangle.$$

$$= \langle v, \overline{Tv_1}\, v_1 + \overline{Tv_2}\, v_2 + \cdots + \overline{Tv_n}\, v_n \rangle.$$

Define $w = \overline{Tv_1}\, v_1 + \overline{Tv_2}\, v_2 + \cdots + \overline{Tv_n}\, v_n.$

Then $Tv = \langle v, w \rangle.$ ——— ■

So, let us define w to be this vector. What is w is, T v 1 bar times v 1 plus T v 2 bar times v 2 plus up to T v n bar times v n. And we can notice that then T v is the inner product of v, and w. And that is precisely what we had set out to proof. As you can see, as I was observing earlier, this is a revealing proof in the sense that it is also telling us the procedure to get hold of a w, which satisfies the condition that T v is the inner product of v and w.

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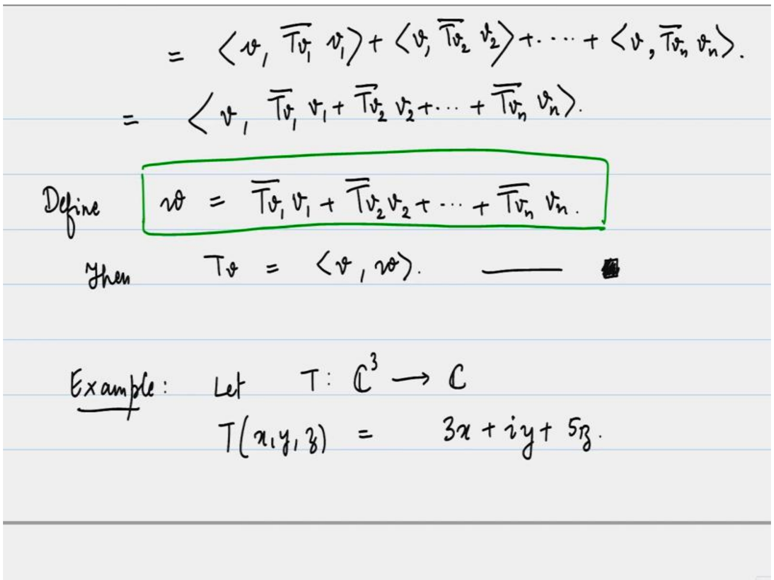

So, let us look at maybe one more example. And maybe use this technique to maybe a couple of examples. We will use this technique to get hold off the vector with which you have to look at the inner product to get hold of the linear functional. So, let T be from, let us look at example in a Complex Vector space, so C3 to C, let us define a linear functional to be say x, y z is equal to say, 3x plus i y plus 5z. Let us look at this example.

So, we can actually guess directly from here, what the w would be in the case of this linear functional T. But nevertheless, let us do the calculation and see exactly whether we get what we are guessing. So, what was the procedure to do that, let us go back let us go up and check that w which I am going to now put in green the box. This is exactly the element that we are going to define. In order to do that, we first need to have an orthonormal basis of our given vector space.

So, we will just notice so this is the C3 has the standard inner product which was given by inner product of z1, z2 and z3 and w1 w2 w3 is z1, w1 bar plus z1, z2 w2 bar plus z3 w3 re bar.

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Example: Let $T : \mathbb{C}^3 \longrightarrow \mathbb{C}$

$$T(x, y, z) = 3x + iy + 5z.$$

Note that $(v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1))$ is an orthonormal basis w.r.t the standard inner product

So, I just note that beta so v1 is equal to 1, 0, 0 v2 is equal to 0, 1, 0 and v3 is equal to 0, 0, 1. This is an orthonormal basis with respect to the standard inner product is an orthonormal basis with respect to the standard inner product.

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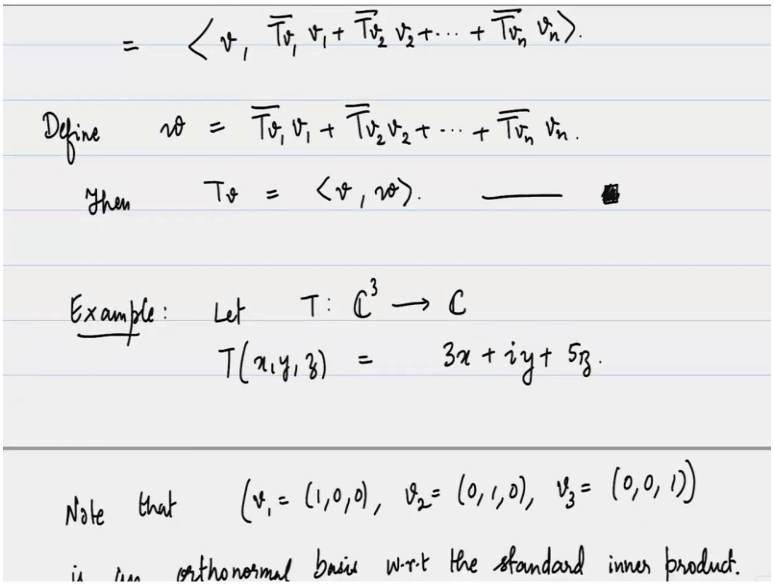
$$\begin{aligned}
\text{Then } \quad w &= \overline{Tv_1}\, v_1 + \overline{Tv_2}\, v_2 + \overline{Tv_3}\, v_3. \\
&= \bar{3}\,(1,0,0) + \bar{i}\,(0,1,0) + \bar{5}\,(0,0,1) \\
&= (3, -i, 5)
\end{aligned}$$

$$\langle (x, y, z), (3, -i, 5) \rangle = 3x + iy + 5z. = T(x, y, z).$$



But then if you look at the candidate for w then what was our w, w is just equal to T v1 bar v1 plus T v2 bar v2 plus T v3 bar v3 and what is T v1, T of 1, 0, 0. If you recall was 3x plus i y plus 5z, so, this is just going to be equal to inner product T of v1 is T of 1, 0, 0 which is equal to 3 bar times v1 which is 1, 0, 0. And what is T of v2, T of v2 is in the same vein, this is i bar the conjugate of i times 0, 1, 0. And how about the third one that is going to be t of v3 bar is 5 bar times 0, 0, 1.

But notice that 3 and 5 are real numbers, its conjugate is the same and the conjugate of i bar is minus i. So, this is just going to be equal to 3, comma minus i, comma 5. And that is precisely what we were expecting, why? Because if you look at inner product of x, y, z with 3 minus i, comma 5, remember that the second vector, the conjugate is what will be contributed.

So, this is going to be x times 3 bar, which is 3 plus y times minus i bar, i bar is minus i. So, minus of i bar is going to be i. So, this is just going to be i times y plus 5 bar is just 5z and this is precisely equal to our T of x, y, z, so yes, we have managed to get hold of the exact vector from the Riesz representation theorem. Let us look at one more example. So, this is the example in P2 of R. So, we have already seen the case in P2 of R okay, let me just write it down.

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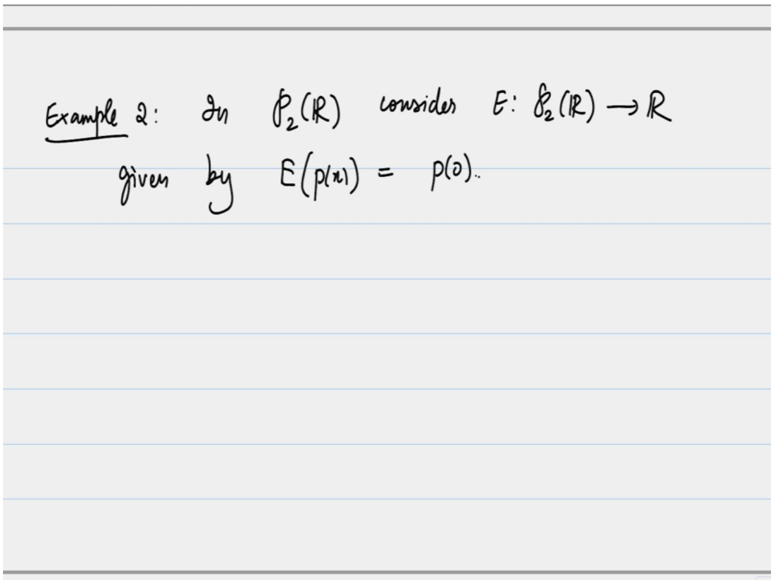

So, Example 2. So, in P2 of R, consider the evaluation map consider E, from P2 of R to R given by E of P of x is equal to the evaluation at 0. So, recall that this is indeed a linear transformation. In fact, it is a linear functional because it is onto R. Now, let us see P2 of R has dimension space a finite dimensional inner vector space. Let us look at in the inner product with which we would like to consider P2 of R.

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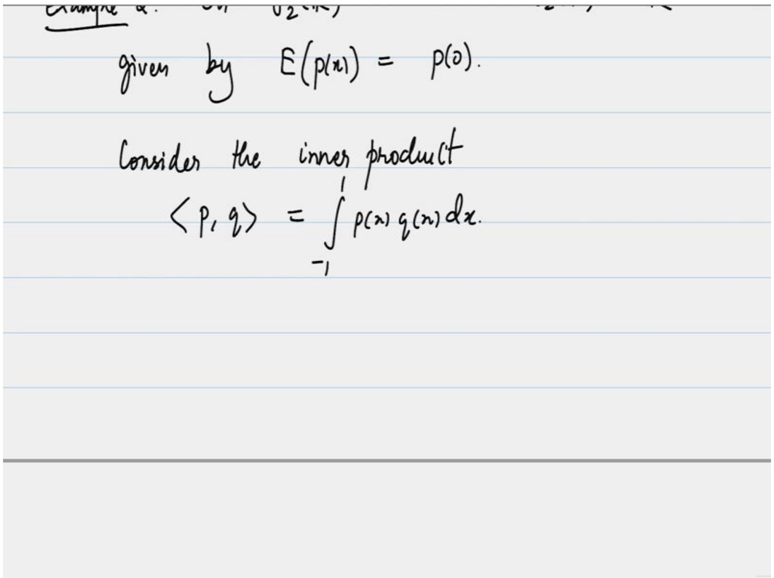

So, consider the inner product integral of p, comma q as being equal to integral from sorry inner product of p, comma q is equal to integral minus 1 to 1, p of x, q of x dx. And let us try to get hold of that particular vector, that particular polynomial, which will give us the evaluation at 0. So, that is our goal.

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$$\langle p, q\rangle = \int_{-1} p(x)\, q(x)\, dx.$$

From the previous week, we can apply Gram-Schmidt orthonormalization to $(1, x, x^2)$ to obtain

$$\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{45}}{\sqrt{3}}\left(x^2-\frac{1}{3}\right)\right)$$

Okay, so to do that, let us get hold of an orthonormal basis. So, we have already done that from the previous week. We have we can apply the Gram-Schmidt Orthonormalization so to what will we be applying this, we will be applying the Gram-Schmidt Orthonormalization to the ordered basis 1, x, x square to obtain 1 by root 2 the first one, root 3 by root 2 times x and the final one which was not fully calculated but nevertheless given as a, left as an exercise. This is root 45 by root 3 the term here yeah times x square minus 1 by 3.

So, this is the orthonormal basis that we obtain by applying Gram-Schmidt Orthonormalization. So, now what is the vector w with respect to which, if you look at the inner product, it will give us the evaluation map.

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$$\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{45}}{\sqrt{8}}\left(x^2-\frac{1}{3}\right)\right)$$

$$w = \overline{E\left(\frac{1}{\sqrt{2}}\right)}\frac{1}{\sqrt{2}} + \cancel{\overline{E\left(\frac{\sqrt{3}}{\sqrt{2}}x\right)}\frac{\sqrt{3}}{\sqrt{2}}x} + E\left(\frac{\sqrt{45}}{\sqrt{8}}\left(x^2-\frac{1}{3}\right)\right)\frac{\sqrt{45}}{\sqrt{8}}\left(x^2-\frac{1}{3}\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{\sqrt{45}}{\sqrt{8}\times 3}\right)\frac{\sqrt{45}}{\sqrt{8}}\left(x^2-\frac{1}{3}\right)$$

$$= \frac{1}{2} - \frac{15}{8}\left(x^2-\frac{1}{3}\right)$$

So, we know what the procedure is w is basically E of 1 by root 2 times bar times 1 by root 2 plus E of root 3 by root 2 into x bar times root 3 by root 2 x plus E of root 45 by it is not root 3 I think it is root 5 by, oh sorry this is root 8 I guess times x square minus 1 into root 45 by root 8 times x square minus 1.

Okay, so what is E? E is just the evaluation functional so 1 by root 2 evaluated at 0 will give you back 1 by root 2 so this is going to be and the conjugate of a real number is the same so this is going to be 1 by root 2 the whole square, root 3 by root 2 times x when evaluated at 0 is going to give a 0 so this term will just vanish, this term will vanish.

And finally, E of root 45 by root 8 times x square minus 1 when evaluated at x is equal to 0 will give us minus of root 45 by root 8 times root 45 by, this is not 1 this is 1 by 3 minus 1 by 3 by root 8 times x square minus 1 by 3. Oh yes, that is a times 3. Some mistake, but yeah, taken care of rectifies. So, this will be equal to 1 by 2 times 45 by 3 is 15, which is minus 15 by 8 into x square minus 1 by 3.

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$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{\sqrt{45}}{\sqrt{8}\times 3}\right)\frac{\sqrt{45}}{\sqrt{8}}\left(x^2 - \frac{1}{3}\right)$$

$$= \frac{1}{2} - \frac{15}{8}\left(x^2 - \frac{1}{3}\right)$$

$$= -\frac{15}{8}x^2 + \frac{9}{8}$$

So, in particular, this is just going to be equal to minus of 15 by 8 times x square plus 5 by 8 plus 1 by 2 or 4 by 8 which is 9 by 8 let me just put it like that. This is the vector. This is polynomial, which will give us the Riesz Representation Theorem, the vector with respect to which when you take the inner product, you will get the linear functional.

In all these examples, so what we were doing was that we were getting hold of the w by getting hold of an orthonormal basis first and then using that orthonormal basis, we calculated what the w is. So, there is a strong indicator that this w is very heavily dependent on what the orthonormal basis is.

But we will now prove a result which says that the dependence is just superficial. It is actually unique, there is no dependence on the orthonormal basis that we have made.

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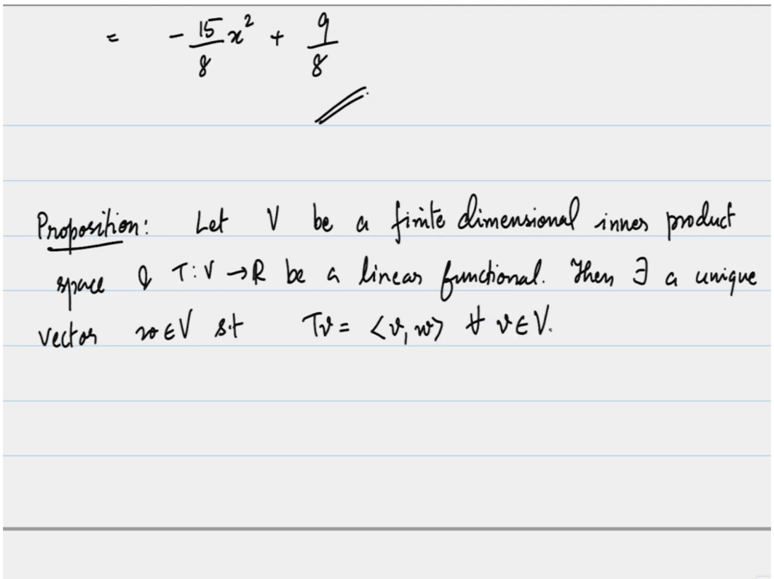

So, let me just write down a proposition for you. So, let V be a finite dimensional inner product space and let T be a linear functional, let w, okay let me rephrase it, this is a linear functional, then there exists unique vector w in capital V such that Tv is equal to inner product of v, comma w for all v in capital V, we know that there exists at least one such vector by the Riesz Representation Theorem, or this proposition is telling us is that vector is necessarily unique.

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Proposition: Let V be a finite dimensional inner product space & $T: V \to \mathbb{R}$ be a linear functional. Then $\exists$ a unique vector $w \in V$ s.t $Tv = \langle v, w \rangle \ \forall \ v \in V$.

Proof: Let w and w' be two vectors in V s.t

$$Tv = \langle v, w_1 \rangle = \langle v, w_2 \rangle$$

Let us give a proof of this statement. The proof of this statement is by considering the possibility of two such vectors w. So, let w and w prime be two vectors in capital V such that the inner product T v is equal to the inner product of v with w1 which is the same as the inner product of v with w2. Suppose, we have two such vectors w1 and w2, this is for all V, right.

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$$\Rightarrow \langle v, w_1 - w_2 \rangle = 0 \quad \forall \ v \in V$$

For $v = w_1 - w_2$, we have

$$\langle w_1 - w_2, w_1 - w_2 \rangle = 0$$

$$\Rightarrow w_1 - w_2 = 0$$

$$\Rightarrow \quad w_1 - w_2 = 0$$

$$\Rightarrow \quad w_1 = w_2 \qquad \text{———} \quad \blacksquare.$$

But then the properties of inner product implies that v w1 minus w2 is equal to 0. This is for again for all v in capital V, because v w 1 minus v w 2 is v w 1 minus w 2. That is what I have written down inside. But in particular, this is also true for w1 minus w2 for v is equal to w1 minus w2 we have the inner product of w1 minus w2, comma w1 minus w2 being equal to 0.

But this is equal to 0 this is 0 only when the vector itself is 0, but this implies that w1 minus w2 is equal to the 0 vector. Which implies w1 is equal to w2. That is what we have set out to proof, so that is just one vector the vector w does not depend upon the orthonormal basis at all. Okay, that is quite nice. So, the Riesz Representation Theorem is quite powerful in some sense. We have proved it for finite dimensional vector inner product spaces, there is a version of it for infinite dimensional inner product spaces or a Hilbert spaces as well. However, it is not in general true.

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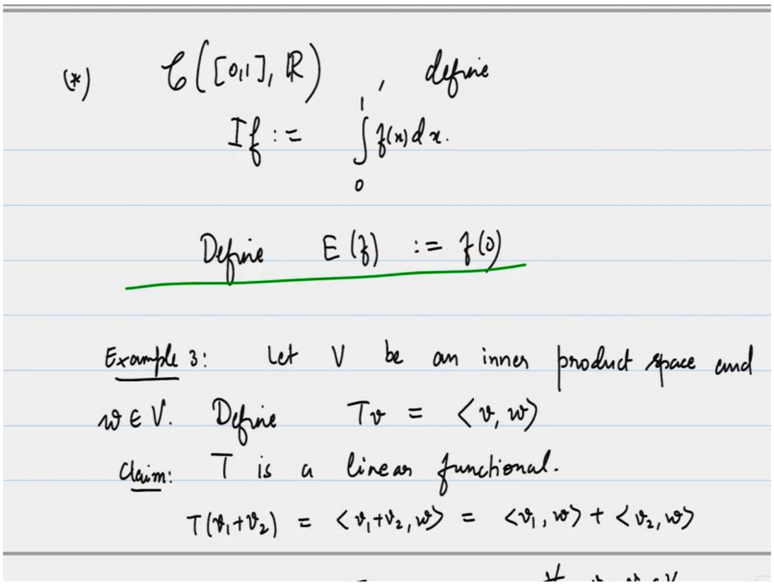

So, for example, one of the examples which was considered about so let me show it to you as a in a problem here that the example here which I am now underlining in green, the evaluation functional on C 0, 1, R that cannot be obtained as an inner product with a vector g say in C 0, 1, R. Okay. So, this is a problem which I would like to work out. So, in the material of this week the problems will be taken care of in this manner in the middle of the lectures, we will try to address some of them.

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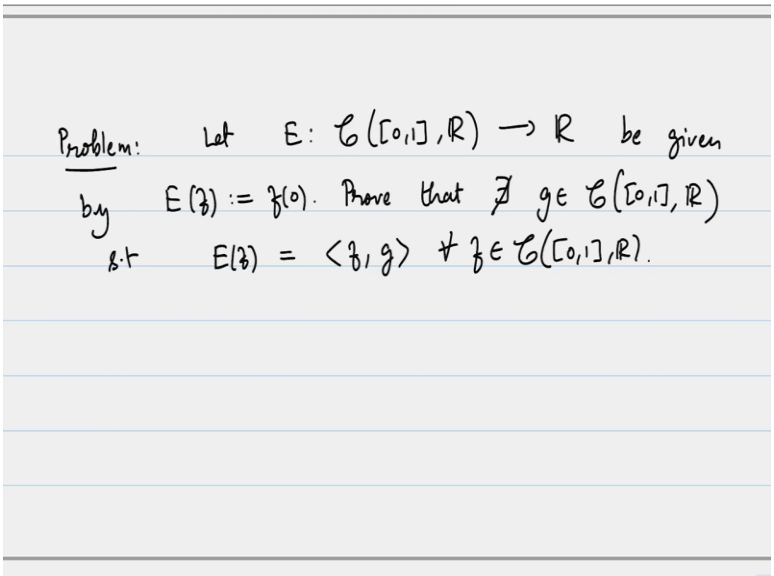

Let E, some, C 0, 1 R to R be given by E of f is defined as being equal to f of 0. Prove that there does not exist g in C 0, 1 R such that E of f is equal to the inner product of f with g for all f in C 0, 1, R. Let us look at a proof of this. Suppose there is one such g.

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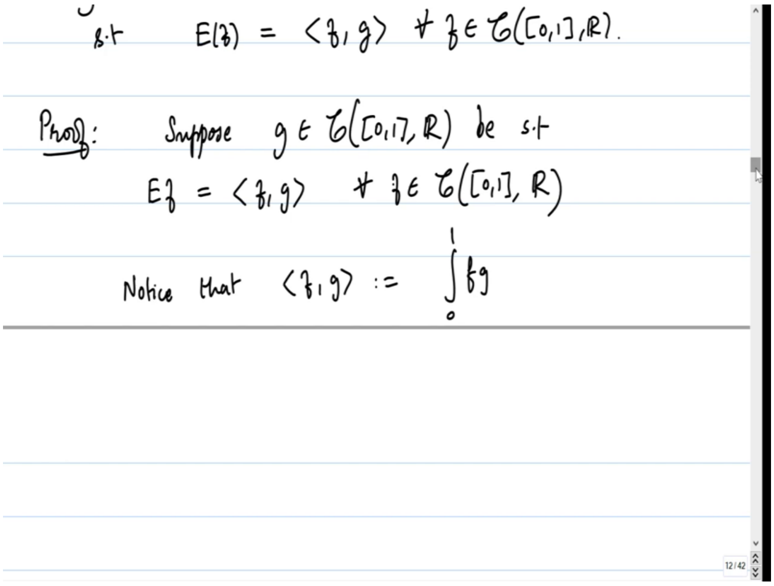

So, proof. Suppose g in C 0, 1, R be such that E f is equal to inner product of f, comma g for all f in C 0, 1, R. If that is the case then what we will do is we will construct certain functions fn such that we will end up with some kind of prediction contradiction. So let us see.

So, notice that the inner product is given by the integral from 0 to, notice that inner product of f g is defined to be the inner product of f g from 0 to 1, this is the inner product with which we are, with which we are working on C 0, 1, R. So, let us define certain functions fn in the following manner.

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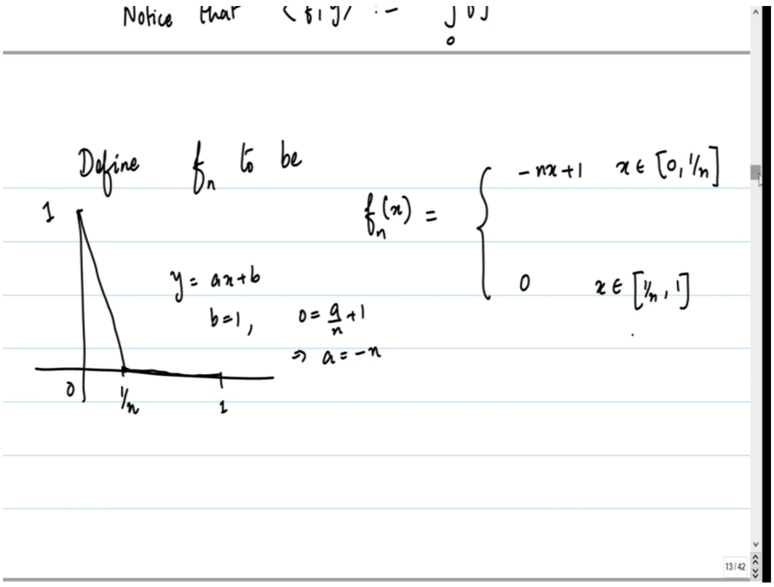

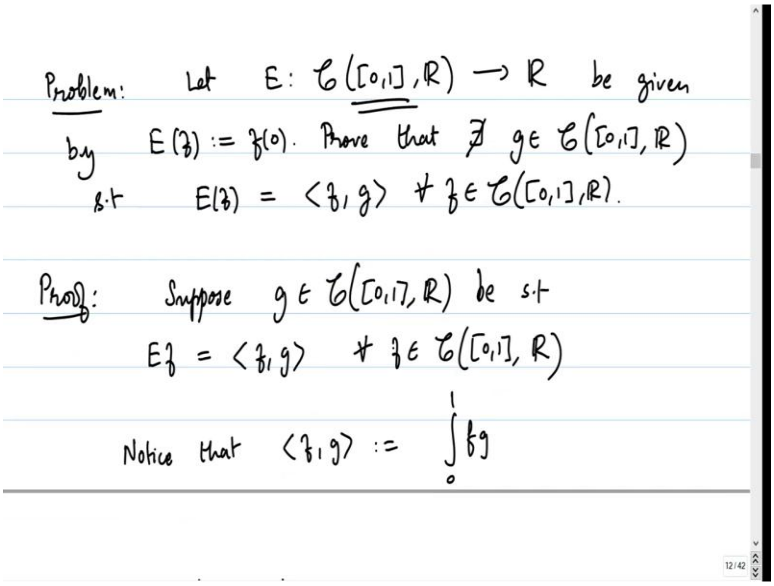

So, let define fn to be let us first draw the graph of f n and then we will define what fn is. So, this is the Cartesian coordinates here, so this be, 1, 0 and this be 1 and let this be 1 by n. So, the function f n is going to be the straight line joining 0, comma 1, 2, 1 by n, comma 0, that will be the graph of the function till here, and from here it will be the 0 function.

So let us define it, specifically f n of x is equal to notice that this is a function defined on 0, 1 that will be the domain here. So, f n of x will, what will be f n of x. Let us see that is going to be equal to well, it will be 0 for x in 1 by n, comma 1, that is for sure. Let us see what will be the equation of this line y be equal to ax plus b be the equation of this line at x is equal to 0 this is 1 that would imply that b is equal to 1 and that x is equal to 1 by n by 0. So, 0 is equal to a by n plus 1 which implies that a is equal to minus of n.

So, f n of x is just minus of n x plus 1, when x is in 0 to 1 by n, yes you can check that this is a continuous function. And let us see what is it that we are looking at we are looking at, we are trying to come up with the contradiction if there exists such a g, right.

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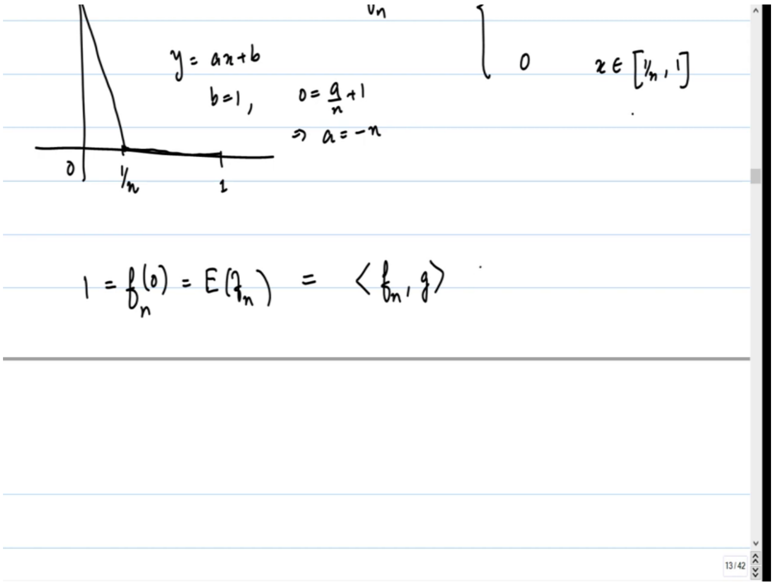

So, let us look at E of f n, notice that this is just f n of 0, write here, E of f n, in our cases, it is just f n of 0, which is equal to 1, but suppose this is also the inner product of f n with g. So, let us see this is going to be, let us now study what the inner product of f n with g is going to be.

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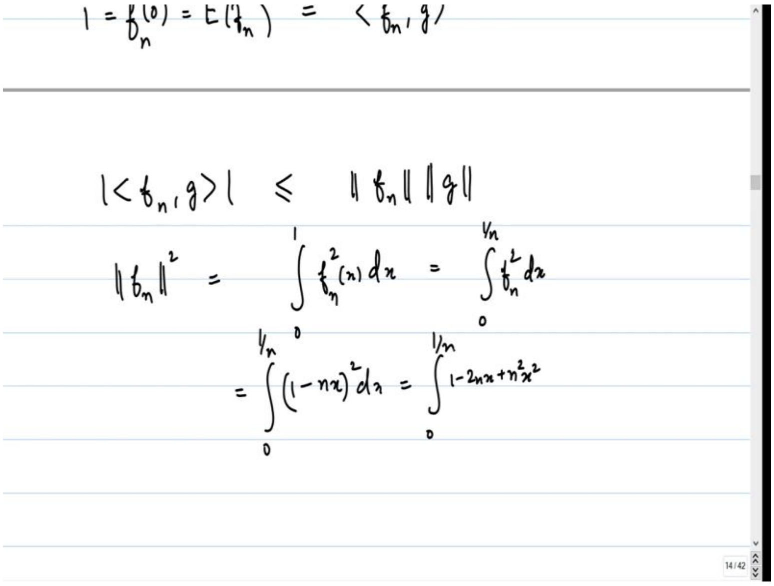

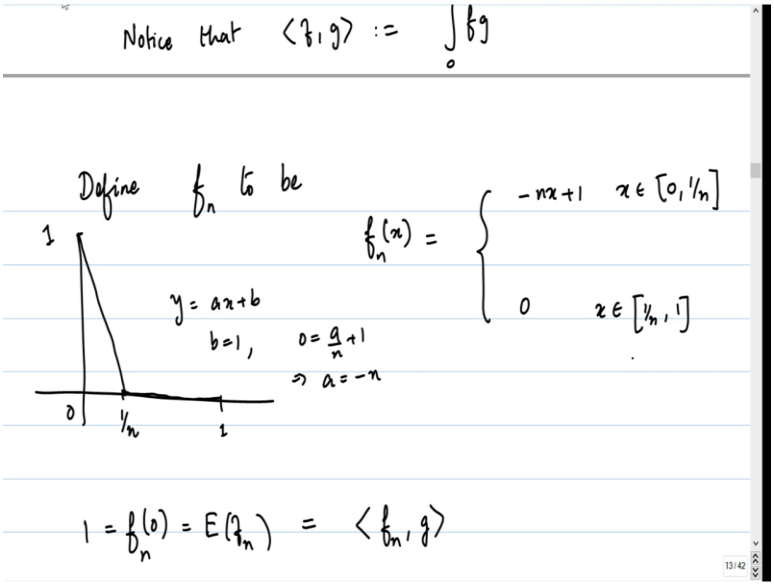

So, we will now use Cauchy–Schwarz inequality here to conclude that the inner product of f n with g. If you look at the model as of this, this is less than or equal to the length of f n times the length of g. Now, length of g is fixed, let us call it some number, C. And let us see what is length of f n. Let us look at what is length of f n square, this is just going to be equal to integral from 0 to 1 f n square of x dx.

So, I will not in fact write down what this is explicitly. I will just notice that, observe that f n is 0. In this interval from 1 by n to 1, that is no contribution. So, this is just going to be integral from 0 to 1 by n. And there is an explicit expression of f n, which does not matter. Let me just write it like that. Just one thing to note is that it is less than 1 so f n square will also be less than 1, so this is going to be less than integral 0 to 1 by n dx.

So, this or maybe I can just compute it, anyway does not matter. But let us just for the sake of completion let us compute it, what is this going to be? This is going to be equal to minus of n x plus 1, which is 1 minus n x the whole square, 0 to 1 by n dx, which is equal to integral 1 minus 2 n x plus n square x square 0 to 1 by n.

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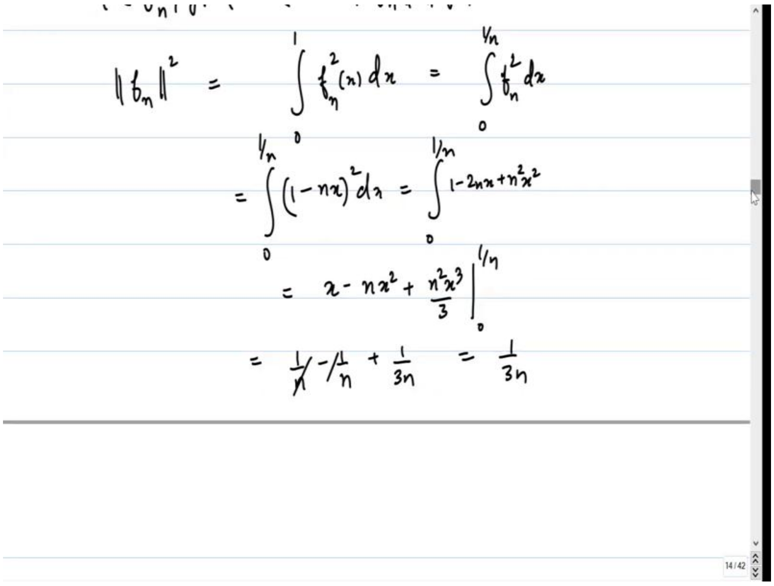

Notice that this is just x minus n x square plus n square x cubed by 3 from 0 to 1 by n. This is 1 by n minus 1 by n plus n square by n cube 3 so 3 times n. This cancels off, this is just 1 by 3n, this is the exact value which is.

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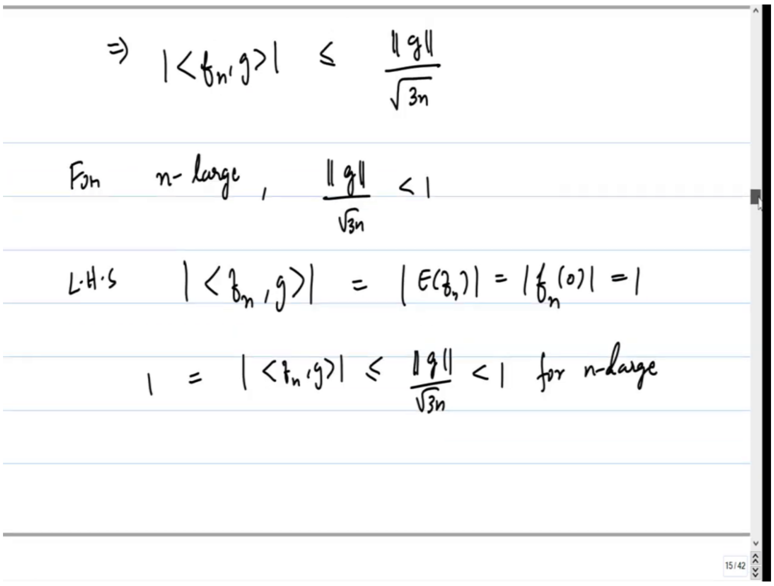

So, what does this mean? This implies that mod of the inner product of f n times g, this is less than or equal to length of g times length of f, which is explicitly equal to the square root of 3n. But notice that for n large for n very large, what do we got, what can be conclude? Norm of oh sorry length of g by square root of 3n is less than 1, but what is LHS.

The absolute value of fn, comma g, what should it be this is exactly equal to E of f n, which is equal to the absolute value of fn of 0 which is 1. So, we get 1 is equal to absolute value of f n, comma g which is less than or equal to length of g by square root of 3n, which is strictly less than 1 for n-large which is a contradiction.

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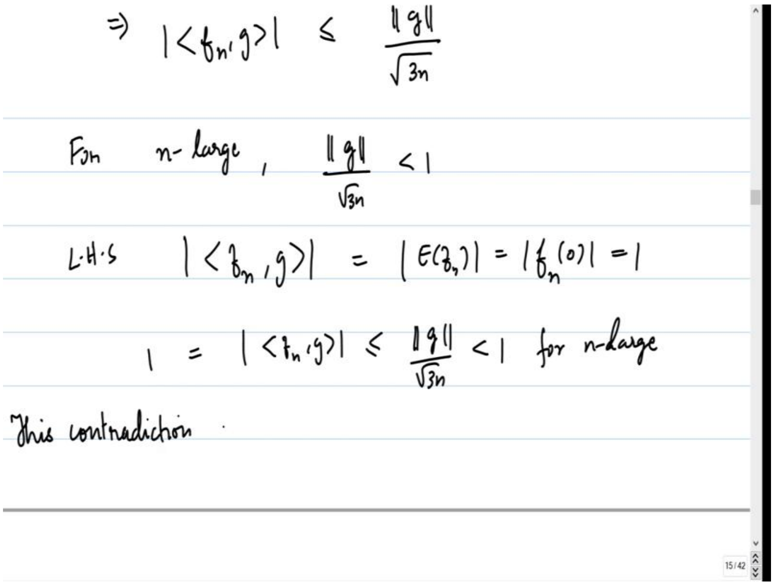

This contradicts, this contradiction. So, this is a contradiction.

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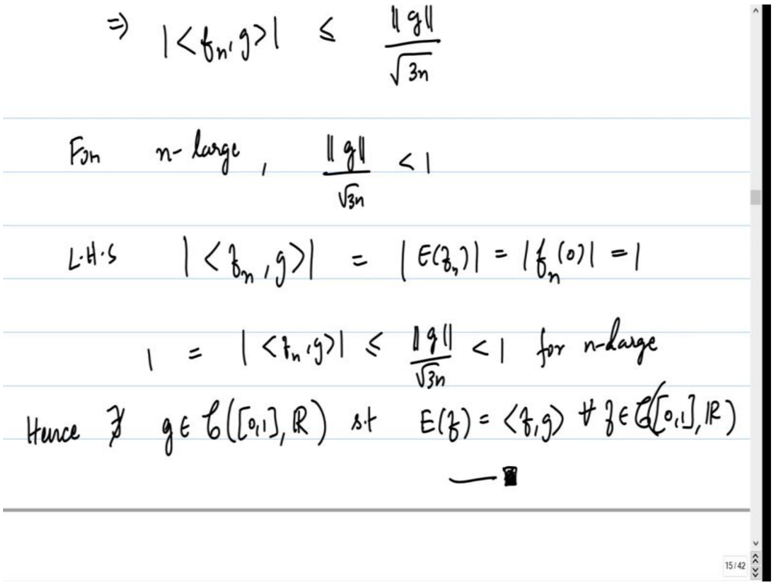

Hence, our assumption has to be false hence there does not exist g in C 0, 1, R such that E f is equal to inner product of f g for all f in C 0, 1, R. So, let me conclude this video by observing that Riesz Representation Theorem is off course proved for finite dimensional inner product spaces in this lecture. However, in the infinite dimensional case it is a bit more subtle, there is a version which is proved, which can be proved, well there are many versions in fact, which can be proved but we will not go into enough this. Let me stop here.