Linear Algebra
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Lecture 11.1
Problem Session

So, this is a problem session which is based on the eighth and the ninth week of this course, the assignment problems have already been given to you and these problems are meant to supplement the problems which were given to you in the assignment. Hope you have done the problems in your assignment. So, let us begin today by solving a problem on the diagonalizability of few given operators.

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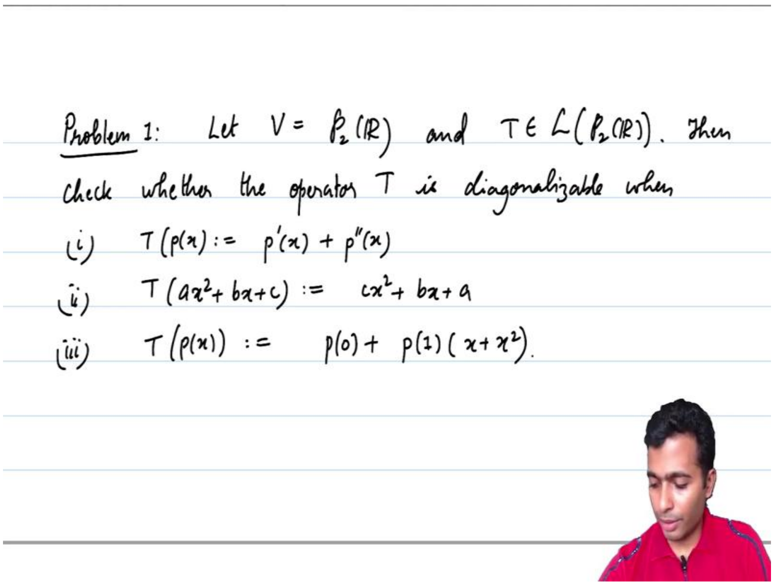

So problem 1, let V be the vector space p2 of R, which is the polynomial vector space of polynomials over R of degree less than or equal to **two** and let T be a linear operator on p2 of R, it is a linear transformation from p2 of R to itself, then check whether the operator T is diagonalizable diagonalizable when. So, let us see, one is when T of say p of x is equal to p prime of x plus p double prime of x, p is sent to prime plus p double prime.

The second is let us take some generic polynomial it will look like ax square plus bx plus c, this polynomial is sent to so this is by definition, cx square plus bx plus a. And how about the third one? Third one is just you have p of x is the evaluation of p at 0 plus evaluation of p at 1 times x plus x square.

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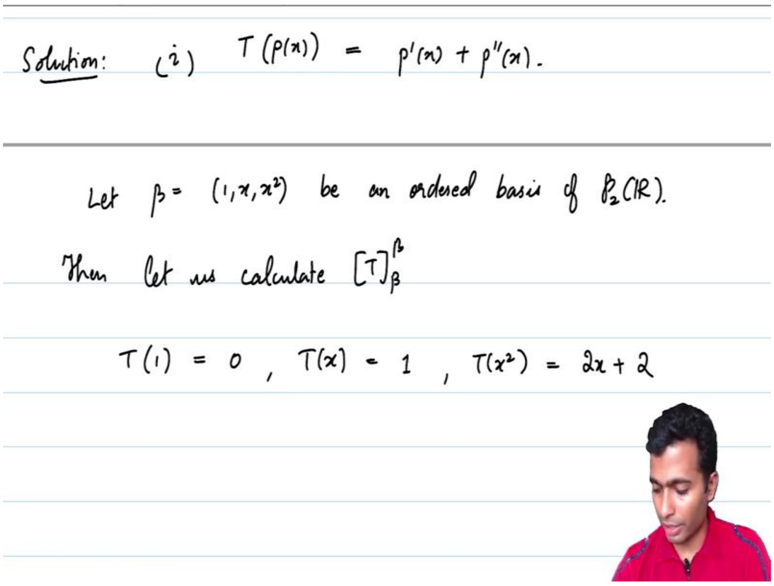

Okay, so we will see whether these 3 linear operators that is just given, whether it is diagonalizable or not. So, let us start with a with the first one, solution. Namely, p of x is sent by this operator to p prime plus p double prime of x. So, I will not bother checking whether T is linear transformation. That will be an easy check, that will be an easy check.

Rather, let us discuss the strategy which we will be using to decide whether, to check whether T is diagonalizable or not. So, when do we say that linear operators is diagonalizable, we say that a linear, we say that a linear operator is diagonalizable, if we can get hold of a basis of V consisting of eigenvectors of T. But to talk about eigenvectors, we will have to talk about the Eigen values and Eigen values can be realized as the roots of the characteristic polynomial of T.

Now, the roots of the characteristic polynomial of T is the same as the roots of the characteristic polynomial of the matrix of T with respect to a basis and we have seen that similar matrices will have the same characteristic polynomial and hence there will be no dependency on the basis. So, we will use all this knowledge to get hold of a matrix of T with respect to given basis we will get hold of the characteristic polynomial and then we will see whether it is diagonalizable or not.

Okay, so that is the strategy, so let us fix a basis, so let beta equal to 1 x x square be a basis, an ordered basis, that way, an ordered basis of p2 of R. Now, let us calculate what is the T beta beta, the matrix of T with respect to beta, then T beta beta. Let us find, then let us calculate T beta beta, so to do that, let us see what is the column representation of each of the vectors in beta. So, notice that T of 1 by definition is the 0 polynomial because the derivative and the second derivative of any constant for that matter is 0. However T of x, T of x will be x prime is 1 plus x double prime is 0, so this is going to be 1, and T of x square will be 2x, which is p prime plus p double prime is 2. This is exactly what our polynomials look like.

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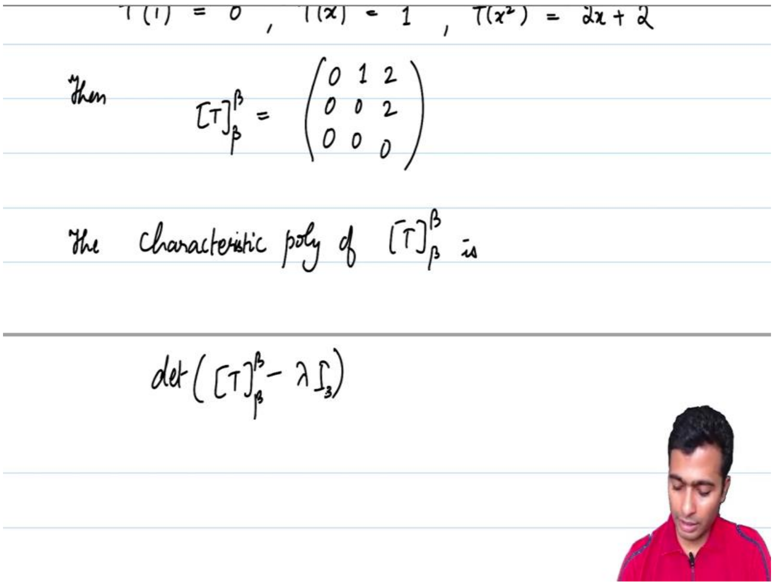

And from this, you can directly write the polynomial of T with respect to beta then, is 0 0 0 in the first column because 0 is represented in this manner. How about 1, 1 is represented as 1 times 1 plus 0 times x plus 0 times x square, and 2 x plus 2 is represented by 2 times 1 plus 2 times x plus 0 times x square, good.

So, what is the characteristic polynomial? The characteristic polynomial of T beta beta is the determinant this is equal to the det of T beta beta minus lambda I. So, notice that p2 of R is a dimension E vector space, so this is I3. So, this matrix is a 3 cross 3 identity matrix.

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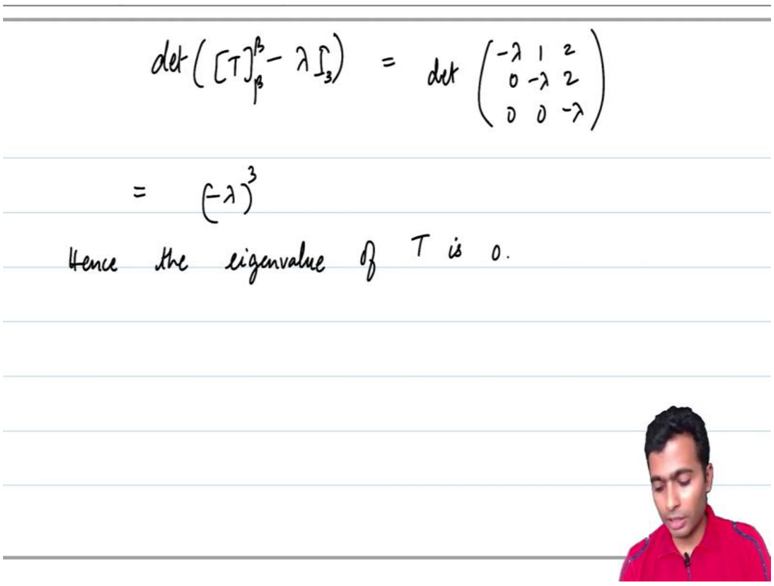

And we will be able to write this easily as the determinant of minus of lambda 0 0 1 minus of lambda 0 2 2 minus of lambda, which is nothing but minus of lambda to the power 3. And therefore, the roots of the characteristic polynomial is just 0. The Eigen values which are the roots of the characteristic polynomial is just 0. Eigen value of T is this scalar 0, 0 is the only Eigen value of T.

But then if we want a basis consisting of Eigen vectors, the Eigen vectors should have some Eigen value and we now know that 0 is the only Eigen value. Therefore, if the Eigen space of 0 is not the entire vector space then T is not diagonalizable.

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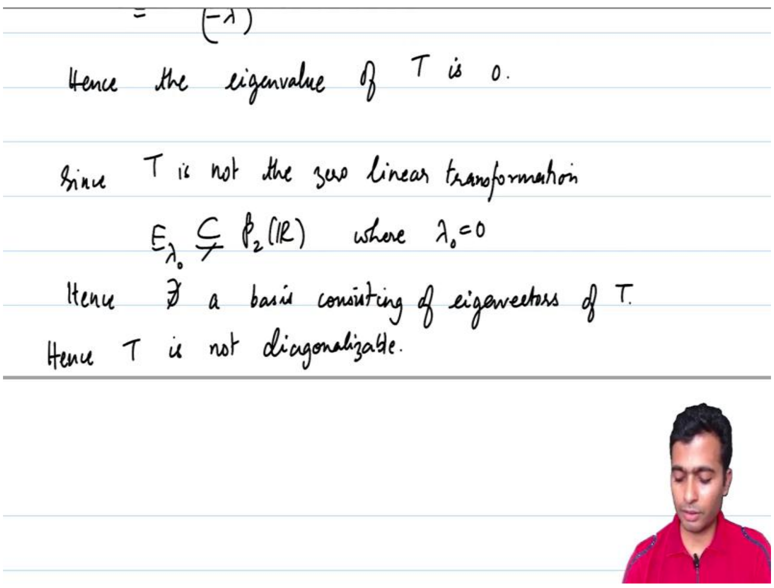

So, notice that since, T is the, T is not the zero linear transformation the Eigen space corresponding to lambda naught is strictly contained in p2 of R, where lambda naught is equal to 0 because if E lambda naught is equal to p2 of R for lambda naught equal to 0 that would mean that T kills every vector in p2 of R, but that is certainly not the case because for example, T of x square was 2 x plus 2 which is not 0. So, hence, we will not be able to get hold of a basis consisting of just eigenvectors.

Hence, there does not exist a basis consisting of eigenvectors of T. Hence, T is not diagonalizable. Okay, so that solves the first part. How about the second one? The second one, dealt with the following linear transformation.

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(ii) $T(ax^2+bx+c) = cx^2+bx+a$

$T(1) = x^2, \quad T(x) = x, \quad T(x^2) = 1.$

Hence $[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

char. poly $\det([T]_\beta^\beta - \lambda I_3)$

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char. poly $\det([T]_\beta^\beta - \lambda I_3)$

$= \det\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = \lambda^2(1-\lambda) - (1-\lambda)$

$= (\lambda^2-1)(1-\lambda) = -(\lambda+1)(\lambda-1)^2$

Hence the eigenvalues of T are $\lambda_1 = 1 \quad \lambda_2 = -1$

If $T(ax^2+bx+c) = ax^2+bx+c$

Two, T of ax square plus bx plus c, this was defined as cx square plus bx plus a. So, in particular, let us again beta be the same standard basis that we had considered earlier, T of 1 is just equal to 0 x square plus 0 x plus 1, which turns out to be x square here. Similarly, T of x is equal to x and T of x square ax square plus bx plus c becomes cx square plus bx plus a, so this is just equal to **one**.

So hence, what would be the matrix of T with respect to beta, that will just turn out to be equal to 0 0 1, T of 1 is 0 times 1 plus 0 times x plus 1 times x square 0 1 0 in the case of T of x, and T of x square is 1 0 0, well that is good, because now we will be able to look at the characteristic polynomial.

Characteristic polynomial is the determinant of the matrix T beta beta minus lambda I again 3 here, which is just going to be equal to minus lambda 0 1 0 1 minus lambda 0 1 0 minus lambda, the determinant of this matrix, the determinant of this matrix is quite straightforward this is just, so this is equal to minus of lambda into minus of lambda, which is lambda square times 1 minus lambda plus 1 times minus of 1 minus lambda. So, minus of 1 minus lambda is basically lambda square minus 1 into 1 minus lambda. So, if we look at the roots of the characteristic polynomial we get the eigenvalues.

Hence the eigenvalues of T are lambda 1 is equal to 1 and lambda 2 equal to minus 1, because this is lambda, this is just minus of lambda plus 1 into lambda minus 1 by whole square. So, there are two roots and they are going to be the eigenvalues. So, to check whether T is diagonalizable or not, we have to get hold of, if at all T is diagonalizable, then there exists an basis, there exists a basis of R3 consisting, in this case p2 of R consisting of Eigen vectors.

So, let us see if whether we will be able to do that, so, let us consider let us focus on lambda 1 equal to 1 and let us get hold of the eigenvectors corresponding to lambda 1 equal to 1, what will be the eigen vectors corresponding to lambda 1 equal to T the set of all ax square plus bx plus c such that T of ax square plus bx plus c is equal to ax square plus bx plus c.

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Hence the eigenvalues of T are $\lambda_1 = 1$ $\lambda_2 = -1$

If $ax^2+bx+c = T(ax^2+bx+c) = cx^2+bx+a$

$\Rightarrow$ $a = c$ Hence if ax^2+bx+c is an eigenvector then $a=c$. If $a=c$, $T(ax^2+bx+a) = ax^2+bx+a$

$\Rightarrow E_{\lambda_1} = \{ax^2+bx+c : a=c\} = \{a(x^2+1)+bx : a,b\in\mathbb{R}\} = \text{span}\{(1+x^2), x\}$.

If $-(ax^2+bx+c) = T(ax^2+bx+c) = cx^2+bx+a$

$\Rightarrow$ $a=-c$, $2b=0$ $E_{\lambda_2} = \{a(x^2-1) : a\in\mathbb{R}\}$.

Hence $\beta = \{(1+x^2), x, x^2-1\}$ are eigenvectors of T.

$\Rightarrow$ β is a basis of $P_2(\mathbb{R})$ $\Rightarrow$ T is digonalizable.

And, if this is the case, then what do we have? Then we have this is equal to by, okay by definition, this is equal to so let me write it like this, ax square plus bx plus c will be equal to T of ax square plus bx plus c. If ax square plus bx plus c is an Eigen vector corresponding to the eigenvalue 1, but we know that T of ax square plus bx plus c is cx square plus bx plus a and by equating coefficients, this gives that Eigen vectors, so this gives that a is equal to c.

So hence, if ax square plus bx plus c is an eigenvector, then a is equal to c, if a is equal to c also if a is equal to c, then what will happen T of ax square plus bx plus a will just be equal to ax square plus bx plus c, which gives that E lambda 1 is just the set of the eigen space corresponding to lambda 1 is the set of all ax square plus bx plus c such that a is equal to c. Let me just leave it for you to check that this is just let me write it like this a times x square plus 1 plus bx where a, comma b belongs to R, so we know exactly what the spanning set, what is the basis of this, which, which is equal to the span of 1 plus x square, comma x.

So, this is going to be the eigen space responding to lambda 1, how about lambda 2? So, let me just see if minus of ax square plus bx plus c that is what is the eigens oh I wrote it, it does not matter let me just be careful this is ax square plus bx plus c, but this is equal to cx square plus bx plus a.

So, if the polynomial ax square plus bx plus c belongs to the eigen space corresponding to minus 1, this would imply that a is equal to minus c and b is equal to minus b, 2b equal to 0 which implies that b is equal to 0, so yeah, 2b equal to 0 that is what we will be able to. So, in in along, in the same line of arguments, E lambda 2 will be just equal to the span of this is just going to be ax square minus 1, so a times x square minus 1, where a belongs to R.

So hence, we have 3 vectors, 1 plus x square, x, and x square minus 1 are eigenvectors of T. And if you notice, if you go back and check this is going to be a linearly independent set. What can we say about a linearly independent set of size 3 in a vector space of dimension **three**, that has to be a spanning set as well. So, this implies that beta, beta is a basis p2 of R, so I leave the check for the fact that beta is linearly independent as an exercise for you, and that will help us conclude that beta is the basis, which implies that T is diagonalizable.

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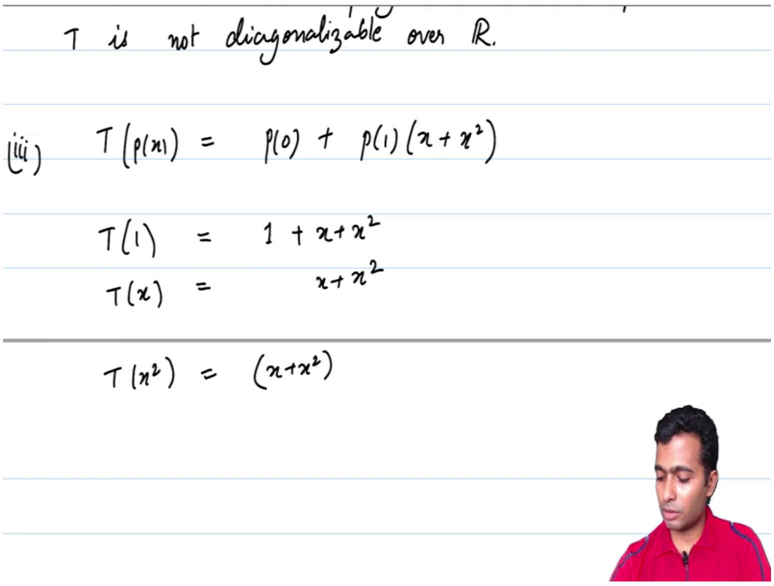

So, T of p of x is equal to p prime, or maybe not p of 0 plus p of 1 times x plus x square. Okay, let us see what is T of 1, T of 1 is just 1 of 0 is 1, p of 1 again, it does not matter, it is 1 times x plus x square. So, this is going to be 1 plus x plus x square. How about T of x, x of 0 is just 0. And p of 1 in this case will be again, x is 1 times x plus x square. So, just write it as x plus x square, how about T of x square. That will also be x plus x square as you can check.

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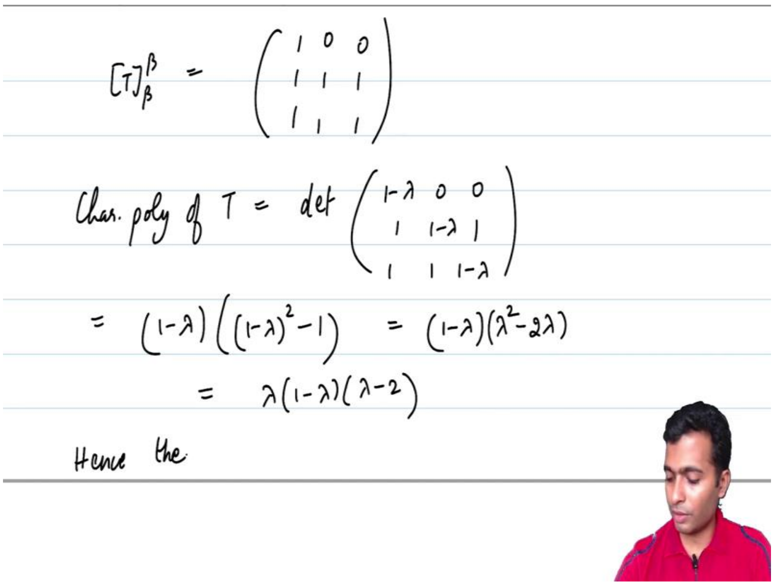

So this is now going to be a an interesting matrix, T beta beta is just 1 1 1 0 1 1 and 0 1 1. Okay, so what will be the characteristic polynomial? This will be the determinant of, I will just skip steps slowly. This is just going to be 1 1 minus lambda 0 not 0 it is 1, 1 1 1 minus lambda, which is equal to 1 minus lambda into 1 minus lambda the whole square minus 1, which is equal to 1 minus lambda times lambda square plus 2 lambda plus 1 minus 1, which is lambda square minus 2 lambda, which is equal to lambda times 1 minus lambda into lambda minus 2. What are the roots of the characteristic polynomial?

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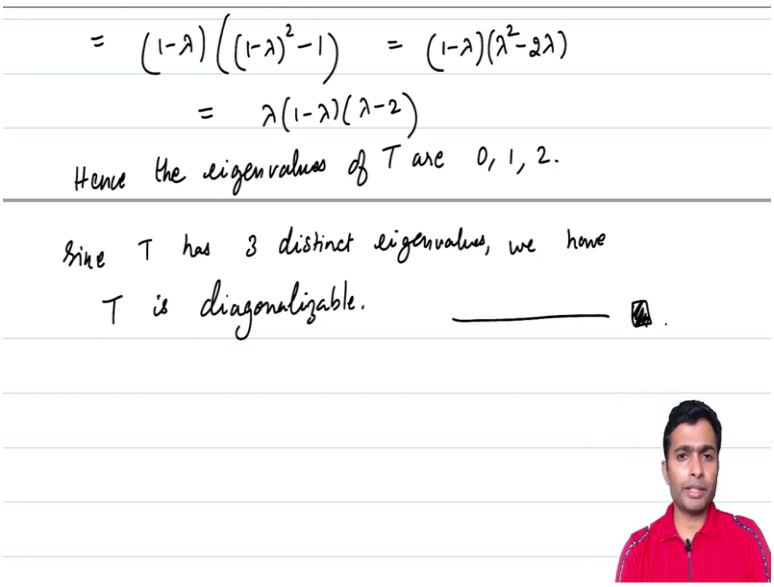

Hence, the eigen values, which are the roots of the characteristic polynomial of T are 0 1 and 2. So, notice that there are 3 distinct eigen values of T and the dimension of p2 of R is also 3.

So, I will not even bother calculating or getting hold of the eigen vectors by invoking 1 of the theorems we have proved in the lectures since, T has 3 distinct eigenvalues, we have that T is diagonalizable.

So, the first example was not diagonalizable in the problem that was given however if we tweak the problem, change the vector space to p2 of c, which is a vector space over c and define the linear operator T similarly, then it turns out to be diagonalizable. And the third problem is a problem which indeed is diagonalizable, so it is a linear operator, which indeed is diagonalizable. So, with that we complete the first problem.

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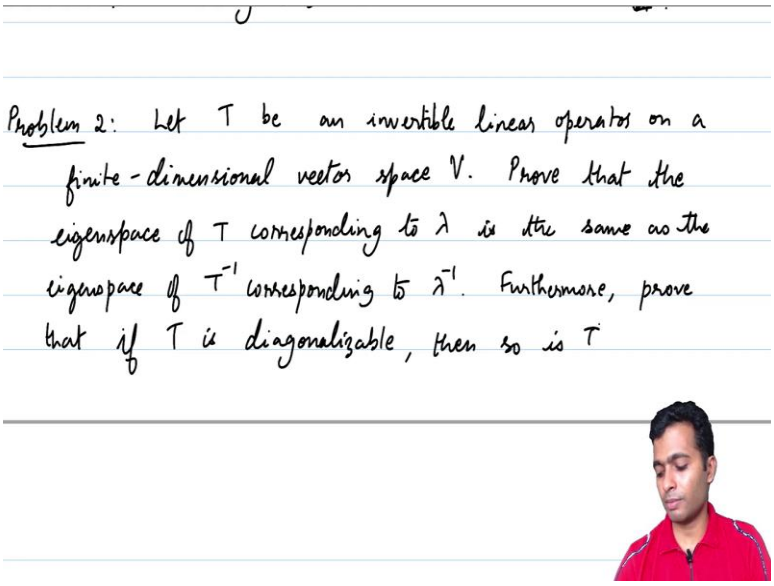

Okay, the next problem, the next problem discusses the relationship between T and T inverse about whether what what can we say about the eigenspaces and eigenvalues of T inverse, when we know about the eigenvalues and eigenspaces of T, further whether we can conclude anything about diagonalizablility of T inverse, when the diagonalizablility of T is not.

So, let me write down the statement of the problem. So, let T be an invertible linear operator on a vector space V, let me impose a finite-dimensional T here, finite-dimensional. Suppose, that lambda is an eigenvalue of T and prove that lambda inverse is also an eigenvalue of T.

So, let me write it like this, prove that the eigen space of T corresponding to lambda, suppose lambda is an eigenvalue of E and suppose the eigenspace of T corresponding to lambda is known, then the problem is to prove that we the same subspace will be an eigenspace of T inverse corresponding to lambda inverse is the same as the eigenspace of T inverse corresponding to lambda inverse. Furthermore, prove that if T is diagonalizable and T inverse is, well if we solve the first part of this problem, then the second part easily follows, so is T inverse.

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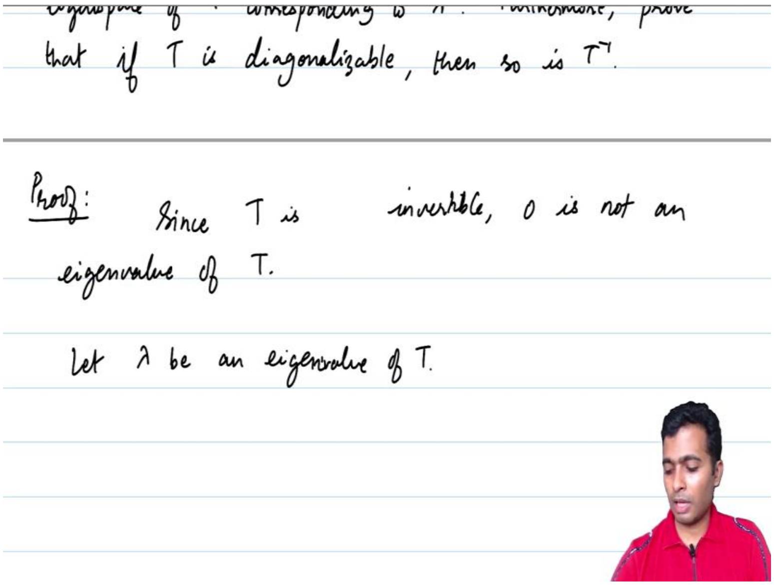

So it is a proof problem, let me write it down as proof. So, first observation is to see that lambda inverse is indeed an eigenvalue of T. So, let lambda be an Eigen value, okay. So, very careful, we should be very careful about it, before we even begin notice that T is an invertible linear transformation.

What can we say about invertible linear transformations? We can say that the invertible linear transformations will not have any element in the null space and therefore, 0 cannot be its eigenvalue since T is not invertible, 0 is not or is invertible rather is invertible, 0 is not an eigenvalue of T. And that tells us that if lambda is an eigenvalue, we can talk about 1 by lambda. So, let lambda be an eigenvalue of T.

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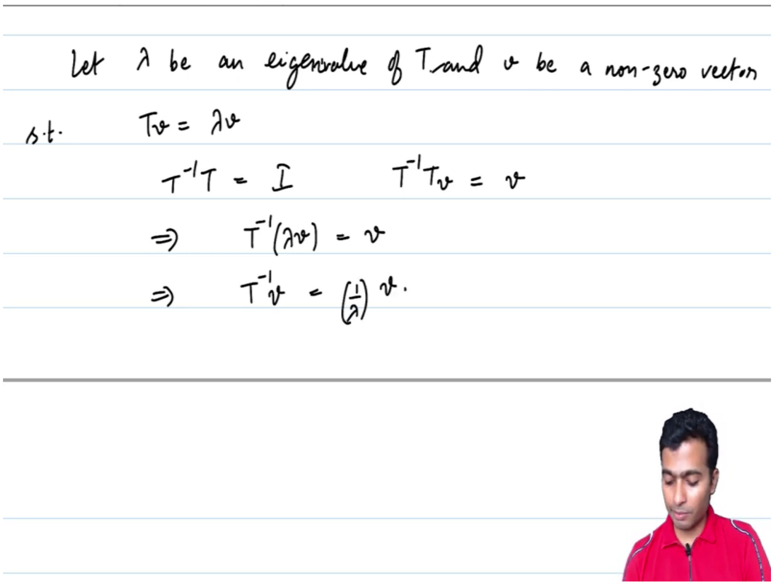

And the moment there is a lambda which is an eigenvalue that v be some non-zero vector such that T lambda, Tv is equal to lambda v and v be non-zero vector such that Tv is equal to lambda v.

What do we know about T inverse? T inverse is the inverse of T. So, T inverse T is the identity map and T inverse Tv the same eigenvector that we just took that is Iv, which is equal to v. But we know what Tv is Tv is lambda v that means T inverse of lambda v is equal to v.

That implies inverse as a linear map, lambda can be taken out, and it can also be inverted, I will just put everything together and say that this is 1 by lambda times v, that means that 1 by lambda is an eigenvalue of T and v is an eigenvector of T inverse, 1 by lambda is an eigenvalue of T inverse and v is also an eigenvector of T inverse. So, what we have proved here just now is that, if v is an eigenvector of T corresponding to lambda and v is also an eigenvector of v inverse of T inverse corresponding to 1 by lambda.

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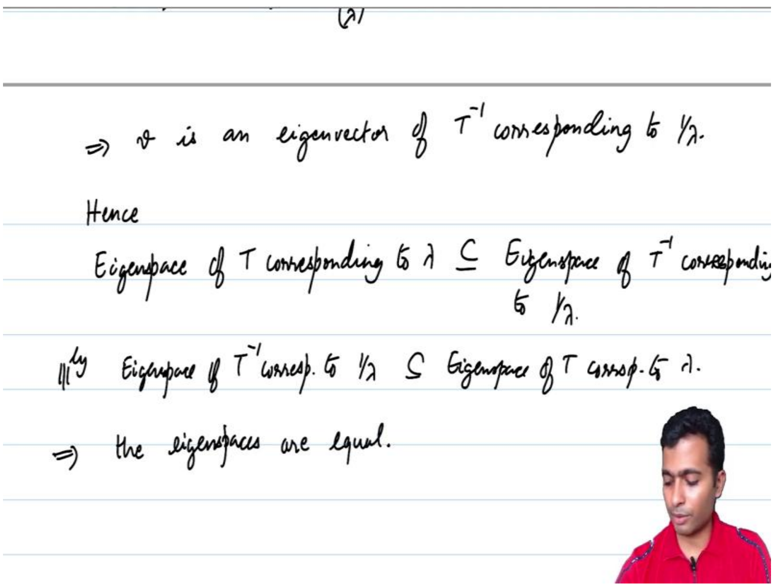

So let me just note that down. This implies v is an eigenvector of the T inverse corresponding to lambda inverse, so that it is not that we are just shown that if lambda is an eigenvalue of T 1 by lambda is an eigenvalue of T inverse, we have shown more, we have shown that v is also the eigenvector of T corresponding to lambda will also be the eigenvector of T inverse corresponding to 1 by lambda. So, now I should be able to say this, the eigenspace which, hence eigenspace of T responding to lambda is contained in the Eigenspace of T inverse corresponding to 1 by lambda.

But then, this is a symmetric argument if we had started off with E inverse in place of T and E in place of T inverse, we would have got the other way, other side containment. Similarly, Eigenspace of T inverse corresponding to 1 by lambda is contained in Eigenspace of T corresponding to lambda, which implies that they are equal, the Eigenspaces are equal, that is what we had set out to prove.

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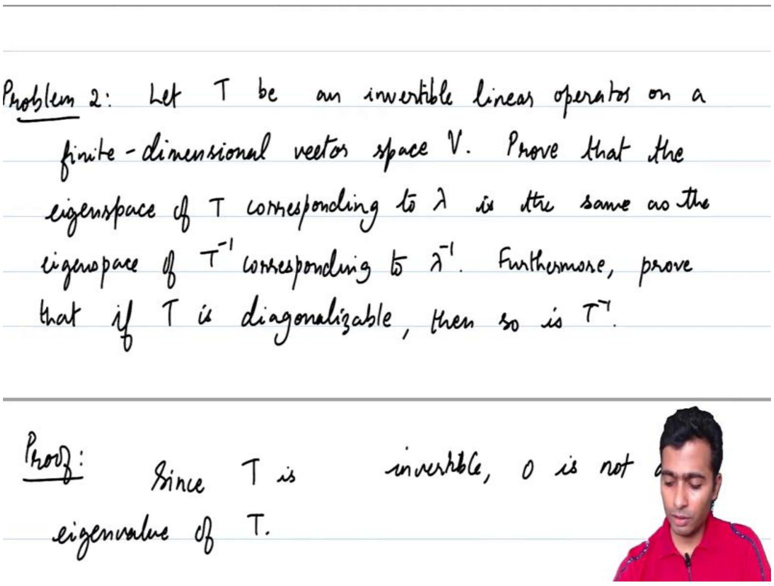

But then there is a second part to the problem, if you recall, it not that we were attempting to only prove that eigenspaces are the same. We also wanted to show that if T is diagonalizable and so is T inverse, but that is quite straightforward because if T is diagonalizable, then we have a basis consisting of eigenvectors of T. We just showed that every eigenvector of T is also an eigenvector of T. inverse. Therefore, the same basis will give you what we want.

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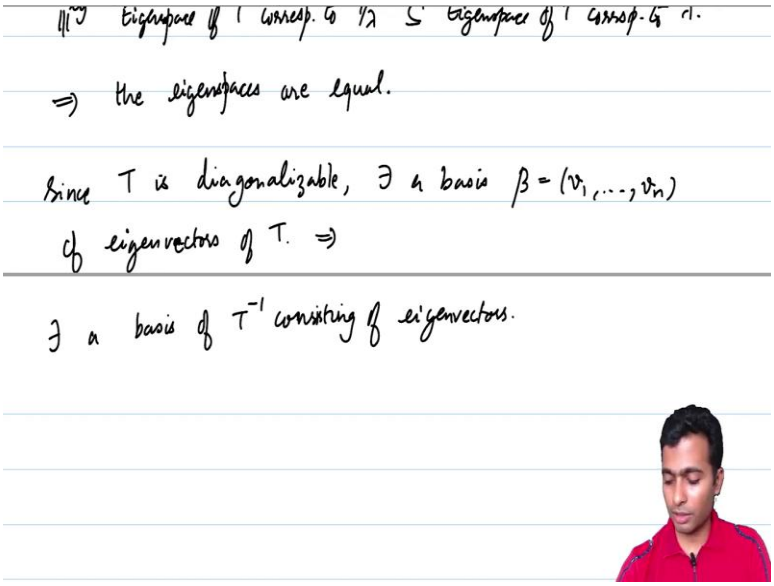

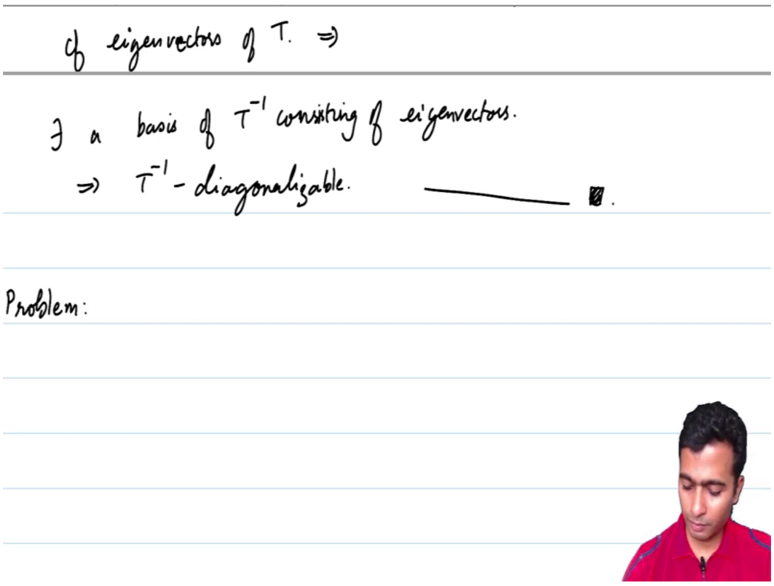

Since, let me just note that T is diagonalizable, there exists eigenvectors there exist a basis beta equal to v1 to vn of eigenvectors of T, but these are also eigenvectors of T inverse, this implies that there exists a basis of T inverse consisting of eigenvectors. That is precisely, what it means to say that a given the linear operators diagonalizable, which implies that T inverse is diagonalizable. So, we have shown all the parts you are in the second part.

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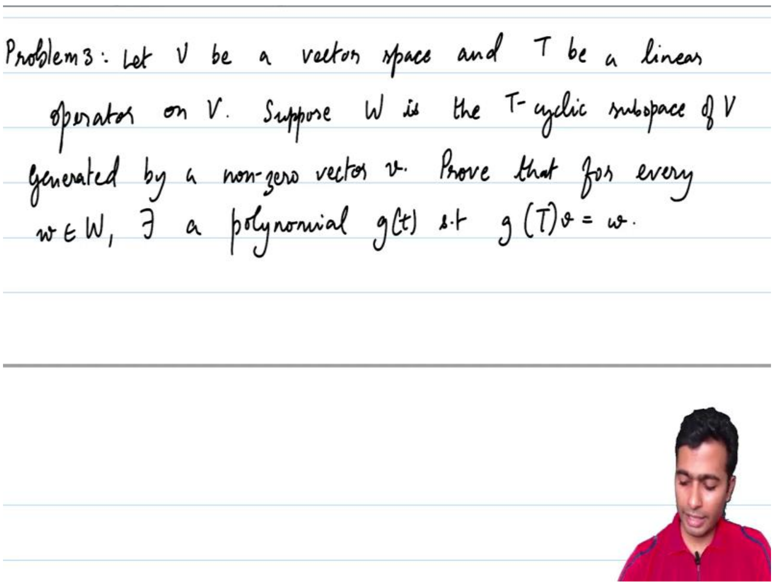

The next problem is regarding the cyclic subspace, cyclic subgroup generator. So, the next problem is regarding the T cyclic subspace generated by a vector in a vector space capital V. So, let me write down the problem.

So, let V be a vector space and T be a linear operator, this is I think problem 3, linear operator on V. Suppose, W is the T cyclic subspace of V generated by some non-zero vector by a non-zero vector V, then prove that x is polynomial g such that g of T for every w in capital W, there exist a polynomial g such that g of T V is equal to W, so that is the problem. Prove that for every w in capital W, there exist a polynomial g such g of t, such that g of T v is equal to w.

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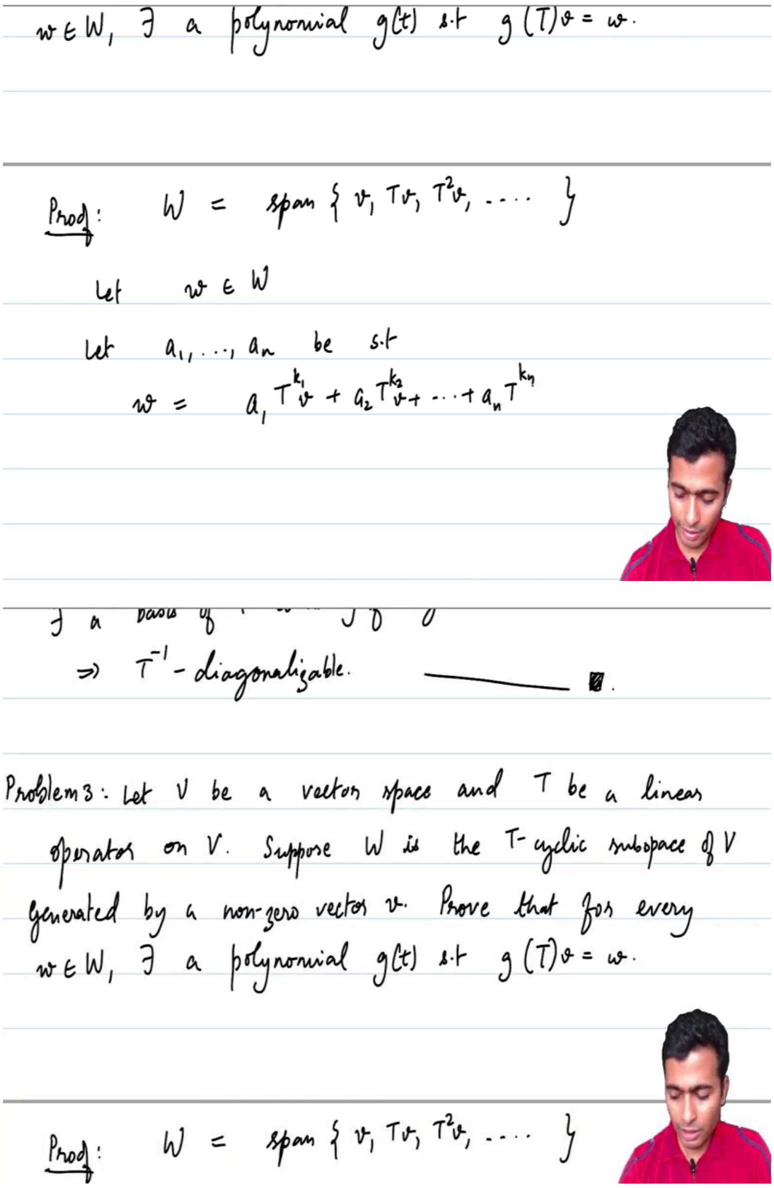

So we have done some, the techniques or the idea that will be used to proof this particular statement. So, let me just show you the statement once more we have already seen something similar ones, but it is good to keep these kind of arguments in mind. So, let us just go over what to do to get hold of some such g, g over here.

So, what is the definition of a T cyclic subspace generated by a vector? So, let me give it as a proof. So, W is the T cyclic subspace of v generated by v that means that this is the span of the vectors v, Tv, T square v, and so on. T cube v and so what does T to the power kv that is T acting on T acting, so T acting on v k times, so T of T of T of T of v, k times this is acting that is what T to the power k v s and w is the span of the vectors v, T v, T square v, T cube and so on.

So, notice that we have not imposed the requirement that V should be a finite dimensional vector space that requirement is not there, so it is possible that w is infinite dimensional, okay. So, let w be in capital W that means, it is in the span of v Tv and so on. So, that means let a1 to an be such that w is equal to a1 T k1 v plus a2 T k2 v plus an T kn v.

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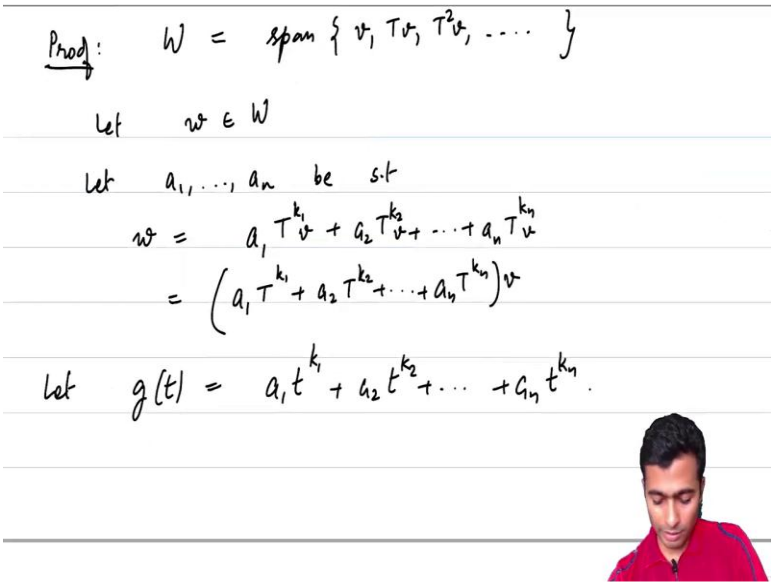

But this is the same as a1 T k1 v so a1 k Tk 1 plus a2 T k2 plus an Tkn, this acting on v. So, let us define g of t to be equal to a 1 times t to the power k1 plus a2 times t to the power k2 plus an times t to the power kn.

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Problem 3: Let V be a vector space and T be a linear operator on V. Suppose W is the T-cyclic subspace of V generated by a non-zero vector v. Prove that for every $w \in W$, $\exists$ a polynomial $g(t)$ s.t $g(T)v = w$. Moreover, if W is finite dimensional, then g can be picked such that $\deg(g) \leq \dim(W)$.

Proof: $W = \operatorname{span}\{v, Tv, T^2v, \dots\}$

Let $w \in W$

When then all the conditions in the hypothesis are satisfied by g and yes, here x is a polynomial g such that g of T v is equal to w. Let me just add a small portion to this. Moreover, if W is finite dimensional, then g can be picked. That is also actually is simple from what we had just done such that degree of g is less than or equal to the dimension of W.

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Let $g(t) = a_1 t^{k_1} + a_2 t^{k_2} + \dots + a_n t^{k_n}$.

Suppose $\dim(W) = m+1$

Claim: $T^k v \in \operatorname{span}\{v, Tv, \dots, T^m v\}$. $\forall\, k \geq m+1$.

Claim: $T^k v \in \operatorname{span}\{v, Tv, \ldots, T^m v\}$. $\forall\ k \geq m+1$.

$\Rightarrow W = \operatorname{span}\{v, Tv, \ldots, T^m v\}$.

$\Rightarrow w \in \operatorname{span}\{\quad '' \quad\}$.

$\Rightarrow \exists\ b_0, \ldots, b_m$ s.t

$w = b_0 v + b_1 Tv + \cdots + b_m T^m v$

define $g(t) = b_0 + b_1 t + \cdots + b_m t^m$

Okay, so let us see what this means. What this means is the following suppose, dimension of V is, dimension W is such m. So, my first claim is that T to the power k v belongs to the span of V, T let me put it as m plus 1, so that I have, I do not have to worry about keeping this m minus 1 at the top I can now write say Tmv for all so this is my first claim, for all k greater than or equal to m plus 1, this is my first claim.

So, the proof we have already seen this proof in one of the lecture. So, I will not go into it, it is a proof by induction, you check that for k equal to m plus 1 this is getting satisfied and then by the strong induction hypothesis you can write that any T to the power k v is in the span of this, but the moment this happens what this implies is that W is the span of V, Tv upto T to the power m v, because any T to the power kv for k greater than m will mean the span already.

So, the span of V, Tv, T square V and so on will be in this, exactly equal to this in fact. But that implies that w is in the span of all these vectors, which implies that there exists a1 to this case, let me now use b1, so that there is no confusion b1 to bm such that w is equal to v or rather let me use b0 to, b0 v plus b1 Tv plus upto bm Tmv, now define g of t to be equal to b0 plus b1 t plus upto bm t to the power m.

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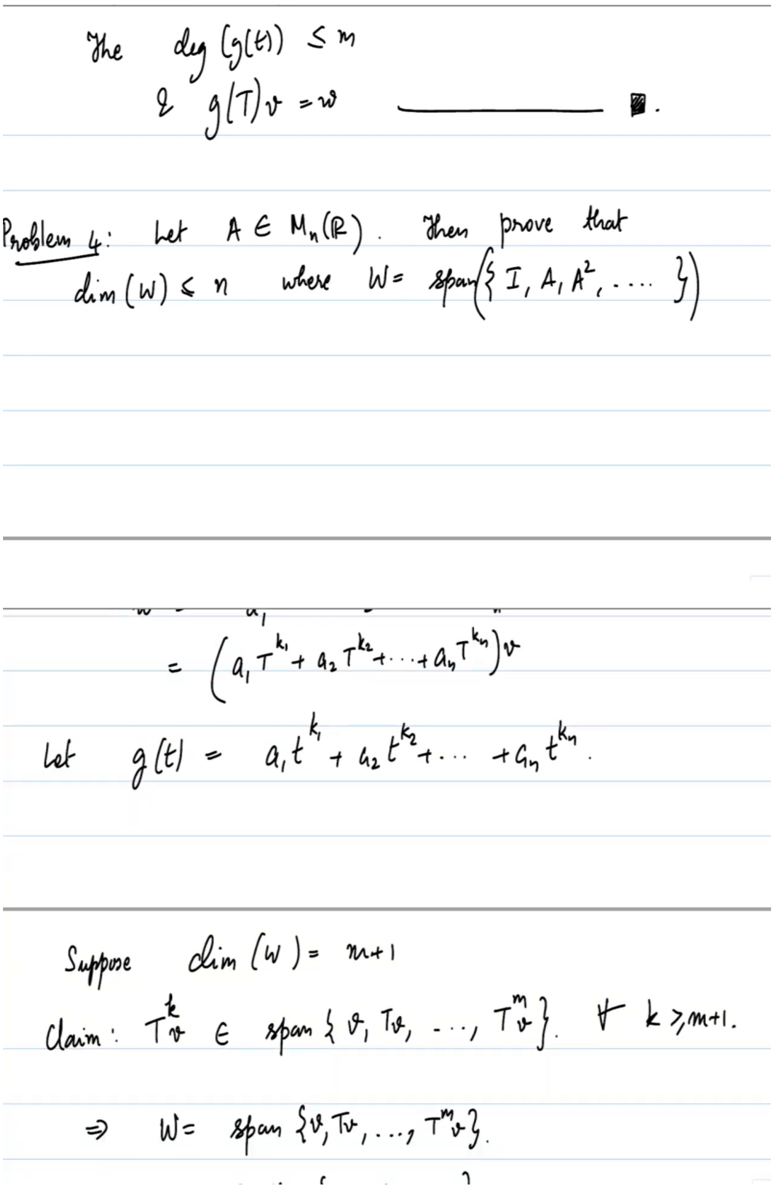

And clearly, g of t has degree less than or equal to m and g of T v is equal to w. So, if w is finite dimensional, we can even pick g to satisfy some nice properties. In the next problem we will discuss an application of the Cayley Hamilton theorem. So, let me write down the problem for you. So, this is going to be problem 4. So, let A be an M cross, M cross n matrix illustrated as Mn R let A be and M cross n matrix with real entries.

Then prove that dimension of the vector subspace W of Mn of R is less than or equal to n, where W is the span of the vectors I, A, A square, and so on, this is span of the collection I, A, A square, and so on. And then we will be able to say that the dimension of W is less than or equal to n.

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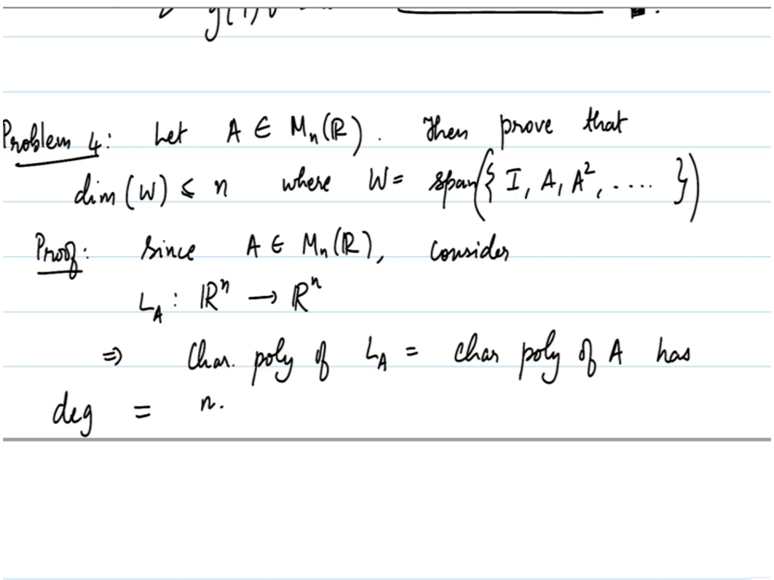

And that is quite straightforward. So, let us see if we have the power of the Cayley Hamilton theorem, then this is quite nice. If you have the power of the Cayley Hamilton theorem, this is quite straightforward. Notice that an n cross n matrix can be thought of as a linear transformation from Rn to itself.

Since A belongs to Mn of R consider LA from Rn to itself, which is n dimensional space, Rn is an n dimensional space. This implies that the characteristic polynomial of LA which is the characteristic polynomial of A has degree less than or exactly in fact equal to the dimension of Rn, which is equal to n, that is a degree of the characteristic polynomial dimension of V.

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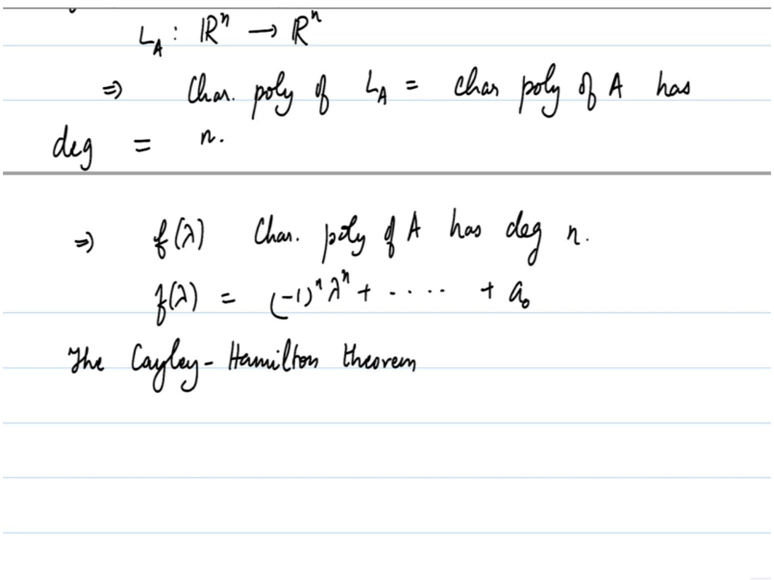

So, that means f of lambda which is the characteristic polynomial of A has degree n, so let us see, f of lambda is equal to minus 1 to the power lambda minus 1 to the power n times lambda to the power n plus so on, the lower order terms plus a0, it is better not to be the determinant. Anyway, we are not interested in that, but nevertheless, by the Cayley Hamilton theorem, the matrix A satisfies this polynomial.

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$$f(\lambda) = (-1)^n \lambda^n + \cdots + a_0$$

The Cayley-Hamilton theorem

$$f(L_A) \equiv 0$$

$$\Rightarrow (-1)^n L_A^n + \cdots + a_0 I \equiv 0$$

$$\Rightarrow (-1)^n A^n + a_{n-1} A^{n-1} + \cdots + a_0 I \equiv 0.$$

$$\Rightarrow A^n \in \text{span}(I, A, A^2, \ldots, A^{n-1}).$$

The Cayley Hamilton theorem, we are shown a version of the Cayley Hamilton theorem for the linear operators, but it is going to be the same, so it is going to imply the same f of LA is equal to the 0 vector.

So identically equal to 0, that is what it means. But what is the meaning of f of LA, this implies that, well equal to 0 implies that minus 1 to the power n times LA acting on itself n times plus so on up to a0 I is the 0 linear operator. That is what it means, and by going down to the basis this implies that minus 1 to the power n times A to the power n plus so on. Let me just write the second term an minus 1, a to the power n minus 1 plus a0 I is the 0 vector. This implies that A to the power n belongs to the span of I, A, A square, upto A to the power n minus 1.

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$$\Rightarrow \quad A^n \in \text{span}(I, A, A^2, \ldots, A^{n-1}).$$

$$\Rightarrow \quad A^k \in \text{span}(I, A, A^2, \ldots, A^{n-1}) \quad \forall\ k \geq n.$$

$$\Rightarrow \quad W \subseteq \text{span}(I, A, A^2, \ldots, A^{n-1})$$

Hence $\quad W = \text{span}(I, A, A^2, \ldots, A^{n-1})$

But that implies that implies that A to the power k belongs to the span of I, A, A square, upto A to the power n minus 1 for all k greater than or equal to n, which implies that W is contained in the span of I, A, A square, upto A to the power n minus 1. But what is W? W is a span of I, A, A square and so on.

And therefore, the right hand side in particular is contained in W, this implies hence, by this observation and the fact that span of I, A, A square, upto A to the power n minus 1 is contained in W, we get W is equal to the span of I, A, A square, upto A to the power n minus 1.

(Refer Slide Time: 41:31)

Hence $\quad W = \text{span}(I, A, A^2, \ldots, A^{n-1})$

Since W has a spanning set of size n, we can conclude that

$\dim(W) \leq n$. ——— $\blacksquare$

What do we have? We now have a spanning set which has size n and what can we say about the dimension, the dimension will always be less than or equal to the size of a spanning set. Since, W has a spanning set of size n, we can conclude that dimension of W is less than or equal to n. Hence, we have to proved the result.

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<u>Problem 5</u>: Let V be an inner product space & $u, v \in V$. Then prove that $\langle u, v \rangle = 0$ iff $\|u\| \leq \|u + av\| \; \forall \; a \in \mathbb{F}$.

In the next problem we will discuss some properties in an inner product space, this is going to be problem 5, so let V be an inner product space, so let capital V be an inner product space and when u, comma v the vectors in capital V then the inner product of u, comma v, then prove that, this is what we have to show, prove that inner product of u, comma v is equal to 0 if and only if norm of u is less than or equal to the norm of u plus a times v for all a in f.

So notice that I am using the word F. So, this is true even in the case when V is a complex inner product space. So, but the statement tells us that, tells us is that if you have that u and v are orthogonal, then norm of u is less than or equal to norm of u plus av for all a in F. And further if norm of u is less than or equal to norm of u plus av for all a in F, then u and v are orthogonal to each other, both sides are being demanded.

(Refer Slide Time: 43:39)

$u, v \in V$. Then prove that $\langle u, v \rangle = 0$ iff

$\|u\| \leq \|u + av\| \quad \forall\, a \in \mathbb{F}$.

<u>Proof:</u> If $\langle u, v \rangle = 0$,

$$\|u + av\|^2 = \|u\|^2 + \underbrace{|a| \|v\|^2}_{(>0)} \quad \forall\, a \in F$$

$$\Rightarrow \quad \|u\|^2 \leq \|u + av\|^2 \quad \forall\, a \in \mathbb{F}$$

$$\Rightarrow \quad \|u\| \leq \|u + av\| \quad \forall\, a \in F.$$

So let us prove, let us prove the result, one side should be easy. So, let us see, if inner product of u, comma v is equal to 0, then what is it that we know about the norm of u plus av the whole square. We know that by Pythagoras Theorem this is equal to well I was using the word norm but what I mean is length, this is going to be length of u square plus mod of a times length of v square, notice that this is a positive quantity the mod of a a times length of v square is a positive quantity.

So, this implies that u square is less than or equal to the length of, length of u square is less than or equal to the length of u plus av the whole square by taking square root the length of u is less than or equal to length of u plus av. Now, this is true for all a in F and therefore, this is also true for all a in F, this is also true for all a in F. So, we have proved one side of the result which was easy. We have shown that if u, comma v, the inner product is 0 if u and v are orthogonal, then the quality, inquality satisfied.

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$$\Rightarrow \quad \|u\| \leq \|u+av\| \quad \forall \quad a \in F.$$

Conversely, let $\|u\| \leq \|u+av\| \ \forall \ a \in F$.

$$\|u+av\|^2 = \langle u+av, u+av \rangle$$

$$= \|u\|^2 + a\langle v,u\rangle + \bar{a}\langle u,v\rangle + |a|^2\|v\|^2$$

$$\|u+av\|^2 - \|u\|^2 = 2\,Re\left(\bar{a}\langle u,v\rangle\right) + |a|^2\|v\|^2$$

$$\|u+av\|^2 - \|u\|^2 \geq 0 \quad \forall \ a \in F$$

$$\Rightarrow$$

Let us assume okay, so the conversely let length of u be less than or equal to length of u plus av for all a in F. Let us now try and conclude that u plus av is, u and v are orthogonal. Okay, so what is the length of u plus av the whole square this is just the inner product of u plus av with u plus av and that is equal to the length of u square plus a times v, comma u plus a bar times u comma v plus mod a square times the length of v square.

So, let us look at what is the length of u plus av square minus the length of u square. That is equal to a times v, comma u, so let me just write this as 2 times the real part of a bar times u, comma v, why is that the case because a times the inner product of v plus u when added to its conjugate, that will give you 2 times the real part of one of the two, well, a times the inner product of v, comma u will have the same real part as the complex number a bar times u, comma v.

So, I am writing down everything under the assumption that it is possible that our vector spaces are complex inner product space. So, then this makes sense. So, plus mod a square times the length of v square, so this is to be satisfied by every scalar a. So, let us now carefully pick our scalar. So, this, so we want the quantity to the left to be greater than 0 this the whole square minus this is greater than or equal to 0 for all a in F implies.

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$$= \|u\|^2 + a\langle v,u\rangle + \bar{a}\langle u,v\rangle + |a|^2\|v\|^2$$

$$\|u+av\|^2 - \|u\|^2 = 2\,Re\left(\bar{a}\langle u,v\rangle\right) + |a|^2\|v\|^2$$

$$\|u+av\|^2 - \|u\|^2 \geqslant 0 \quad \forall\, a \in \mathbb{F}$$

$$\Rightarrow 2\,Re\left(\bar{a}\langle u,v\rangle\right) + |a|^2\|v\|^2 > 0 \longrightarrow (*)$$

If $v = 0$, the proof follows.

When $v \neq 0$, $a = \dfrac{\langle u,v\rangle}{\|v\|^2}$

Let us write down the right hand side, 2 times the real part of a bar times inner product of u, comma v plus mod a square times the length of v square is greater than 0. So, let us carefully pick a, pick a to be equal to, okay so let us see, if v is equal to 0, then there is nothing to prove, then u, comma v will have inner product equal to 0 and we will be done. So, the case when v is equal to 0, the solution is clear.

The proof follows. So, we may assume without loss of generality that v is not equal to 0, so when v is not equal to 0, let us pick a very special, this is after all satisfied, the star has satisfied for all a in F. So, let us pick a to be equal to the inner product of u with v by the length of v square.

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When $v \neq 0$, $a = -\dfrac{\langle u,v\rangle}{\|v\|^2}$

Then L.H.S of $(*)$ will be

$$2\,Re\left(-\frac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u,v\rangle\right) + \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2$$

$$= 2\,Re\left(-\frac{|\langle u,v\rangle|^2}{\|v\|^2}\right) + \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$= -\frac{2|\langle u,v\rangle|^2}{\|v\|^2} + \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$= 2\,Re\left(-\frac{|\langle u,v\rangle|}{\|v\|^2}\right) + \frac{|\langle u,v\rangle|}{\|v\|^2}$$

$$= -\frac{2|\langle u,v\rangle|^2}{\|v\|^2} + \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$= -\frac{2|\langle u,v\rangle|}{\|v\|^2}$$

$$\|u+av\|^2 - \|u\|^2 = 2\,Re\left(\bar{a}\langle u,v\rangle\right) + |a|^2\|v\|^2$$

$$\|u+av\|^2 - \|u\|^2 \geq 0 \quad \forall\, a \in \mathbb{F}$$

$$\Rightarrow 2\,Re\left(\bar{a}\langle u,v\rangle\right) + |a|^2\|v\|^2 > 0 \longrightarrow (*)$$

If $v = 0$, the proof follows.

When $v \neq 0$, $a = -\frac{\langle u,v\rangle}{\|v\|^2}$

Then L.H.S of $(*)$ will be

$$2\,Re\left(\frac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u,v\rangle\right) + \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2$$

Let us pick this particular scalar, and let us see what happens then star then L.H.S of star will be 2 times real part of a bar will be inner product of u, comma v bar by the length, length of v is a real number. So, this is just a real number times the inner product of u, comma v.

And how about mod a square, mod a square will be inner product of u, comma v the whole square by norm v square length v square, the whole square so this is going to be length v to the power 4 times length of v square, which is just going to be equal to 2 times the real part of absolute value of u, comma v the whole square by length of v square.

Why would it be the absolute value should check that if Z is equal to a plus Ib Z bar is equal to a minus Ib, and Z Z bar will be just a square plus b square, which is the square of the absolute value of Z. So, this is exactly what we will be getting plus after cancellations, the next term will just be equal to absolute value of inner product of u, v the whole square by norm, by the length of v square. But notice that we are looking at the real part of a real number. So, that has to be the number itself. So, this is equal to 2 times, let me be a little careful here.

So, let us do one thing, there was a, this will not work. So, what we will do is we will change the sign a bit, let us put a to be minus of this number, that is when the fun begins, then this will be minus of this, then this is going to be 2 times minus of this, which is 2 times minus of this, which is minus of 2 times the inner product, absolute value of inner product of u, comma v square by length of v square plus absolute value of u, comma v the whole square by the length of v square.

I will just explain why, what, what was going, what was going on. If we had picked, if the way we had picked earlier, we would have just ended up with some number which is greater than or equal to 0 and we would not have been able to conclude much, but what we have done is by changing the sign, now here, this is just going to be equal to minus of 2 times the absolute value of the inner product of u, comma v by the length of v square.

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$\|u+av\| - \|u\| = 2\,Re\left(a\langle u,v\rangle\right) + |a|\,\|v\|$

$\|u+av\|^2 - \|u\|^2 \geq 0 \quad \forall\, a \in F$

$\Rightarrow \quad 2\,Re\left(\bar{a}\langle u,v\rangle\right) + |a|^2\|v\|^2 > 0 \longrightarrow (*)$

If $v = 0$, the proof follows.

When $v \neq 0$, $a = -\dfrac{\langle u,v\rangle}{\|v\|^2}$

Then L.H.S of $(*)$ will be

$2\,Re\left(-\dfrac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u,v\rangle\right) + \dfrac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2$

And what do we know just go back to star, R here tells us that that quantity should not be necessarily greater than 0.

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$$= \quad -\frac{2|\langle u,v\rangle|}{\|v\|^2}$$

By $(*)$ $$-\frac{2|\langle u,v\rangle|}{\|v\|^2} \geqslant 0$$

$$\Rightarrow \quad -|\langle u,v\rangle| \geqslant 0$$

$$\Rightarrow \quad |\langle u,v\rangle| \leq 0$$

$$\Rightarrow \quad |\langle u,v\rangle| \leq 0$$

$$\Rightarrow \quad |\langle u,v\rangle| = 0$$

$$\Rightarrow \quad \langle u,v\rangle = 0 \quad \text{———} \quad \blacksquare$$

Star by star minus of 2 times absolute value of u, comma v by the length of v square should be greater than 0, we can safely multiply by length of v square, which is a positive quantity and conclude at, in fact, we can conclude that the absolute value of u, comma v is less than, this should be greater than or equal to 0, this should not be less than or equal to 0. I am sorry, this is greater than or equal to 0, by multiplying by minus 1 we get this is less than or equal to 0.

But absolute value of a complex number, this cannot be negative, this implies that the absolute value of u, comma v is equal to 0. But how can the absolute value be equal if it is if equal if and only if the complex number itself is equal to 0. And that is precisely what we had said out to proof.

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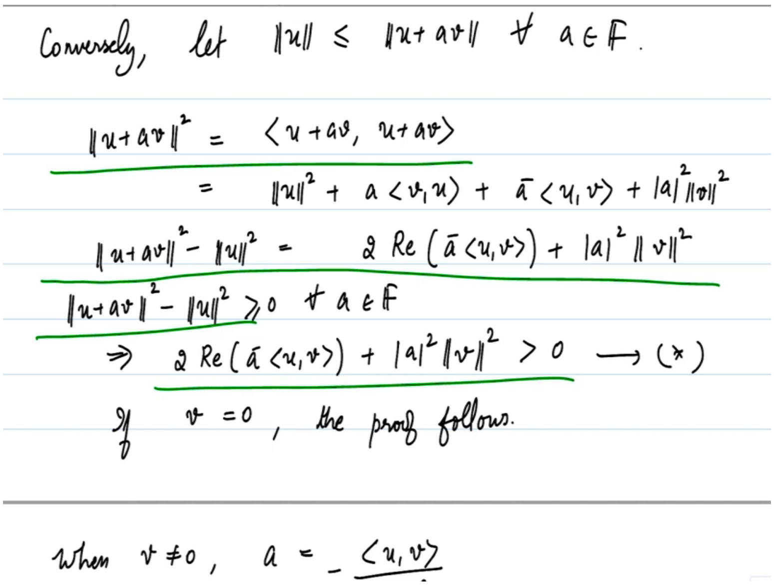

So, let us just quickly go back to see what we have done. What we have done is that we wrote down the expression for the length of u plus av square and from there, we managed to get hold of an equality of this sort. And we know that length of u is less than or equal to the length of u plus av in particular length of u square is less than or equal to the length of u plus av the whole square and therefore the length of u plus av the whole square minus the length of u square is greater than or equal to 0. And that is precisely what was used here. And we concluded at RHS will also hence be equal to, be greater than or equal to 0.

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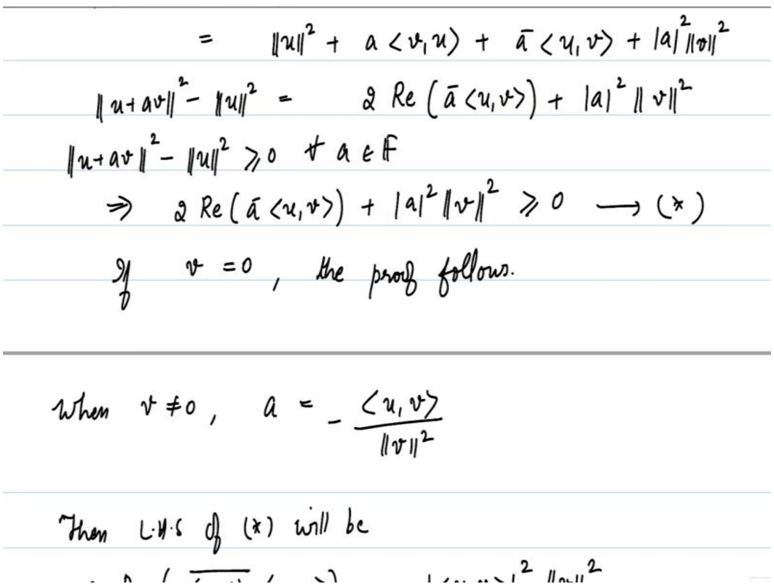

When $v \neq 0$, $a = -\frac{\langle u, v\rangle}{\|v\|^2}$

Then L.H.S of $(*)$ will be

$$2\,\mathrm{Re}\left(-\frac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u,v\rangle\right) + \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2$$

$$= 2\,\mathrm{Re}\left(-\frac{|\langle u,v\rangle|^2}{\|v\|^2}\right) + \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$= -\frac{2|\langle u,v\rangle|^2}{\|v\|^2} + \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$= -\frac{2|\langle u,v\rangle|}{\|v\|^2}$$

By $(*)$ $$-\frac{2|\langle u,v\rangle|}{\|v\|^2} \geq 0$$

$$\Rightarrow -|\langle u,v\rangle| \geq 0$$

$$\Rightarrow |\langle u,v\rangle| \leq 0$$

$$\Rightarrow |\langle u,v\rangle| = 0$$

$$\Rightarrow \langle u,v\rangle = 0 \quad \blacksquare$$

So I will just make a slight correction here. (())(54:20) the greater than or equal to this should be a greater than or equal to, from the above. And that is precisely, and then what did we do we picked our a very carefully when we is not equal to 0, we picked our a to be something like this. And then with that carefully chosen a, we ended up with the absolute value of the inner product of u, comma v being equal to 0, which (())(54:45) to be orthogonal to each other. Okay, that solves the fifth problem.

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Problem 6: Prove that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leqslant \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right).$$

$\forall \; a_i, b_i \in \mathbb{R}$ where $i = 1, 2, \ldots, n$.

The next problem is an application of the celebrated Cauchy Schwarz inequality. As you might know, as I told you earlier Cauchy Schwarz inequality is one of the most important theorems in the field and its power is immense. So, let us look at one problem to slightly indicate that it is actually quite a powerful theorem.

So, prove that summation aj bj the whole square is less than so this is j equal to 1 to n, this is less than or equal to summation j aj square, j is again from 1 to n times summation bj square by j, where j is equal to 1 to n, this is true for all ai, comma bi in R where i is equal to 1 to n. So, at first glance, it might not look like Cauchy Schwarz inequality might come into the picture, so this is just some inequality involving a few real numbers and the squares.

But we will very soon, very soon we will convert this problem into a problem which involves the relevant vector spaces as just to be guessed in Rn, we will use the standard inner product and the Cauchy Schwarz inequality in a smart way.

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$\forall \; a_i, b_i \in \mathbb{R}$ where $i = 1, 2, \ldots, n.$

__Proof__: Consider $\mathbb{R}^n$ with the standard inner product.

Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n$.

So, let us see how it can be done. So, this is a proof problem. So, let me give a proof, so consider Rn with the standard inner product and let us pick a1 to an and b1 to bn arbitrarily. So, let a1 to an, comma b1 to bn be vectors in Rn. So, let them be n tuples.

(Refer Slide Time: 57:14)

$\forall \; a_i, b_i \in \mathbb{R}$ where $i = 1, 2, \ldots, n.$

__Proof__: Consider $\mathbb{R}^n$ with the standard inner product.

Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n$.

By Cauchy-Schwarz inequality, we have

$$\left| \langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle \right| \leq \|(a_1, \ldots, a_n)\| \|(b_1, \ldots, b_n)\|$$

Now, let us try and recall what Cauchy Schwarz inequality says. By Cauchy Schwarz inequality, we have the inner product of a1 to an and absolute value of the inner product of a1 to an, comma b1 to bn. This inner product, this is less than or equal to the length of the vector a1 to an times the length of the vector b1 to bn, this is precisely what the Cauchy Schwarz inequality says, but we will do a slight tweaking here.

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Define $a_i' = \sqrt{i}\, a_i$

|||ly $b_i' = b_i / \sqrt{i}$

Apply C-S to $(a_1', \ldots, a_n')$ and $(b_1', \ldots, b_n')$

$$|\langle (a_1', \ldots, a_n'), (b_1', \ldots, b_n') \rangle| \leq \|(a_1', \ldots, a_n')\| \, \|(b_1', \ldots, b_n')\|$$

$\Rightarrow$

So, let us do one thing, let us now define ai prime to be equal to i times ai. Similarly, or maybe not, maybe square root of i times ai. Similarly, bi be equal to bi by square root of i. So, we will define now ai, bi prime, ai prime and bi prime in this manner. So, apply Cauchy Schwarz to these vectors, we will not apply Cauchy Schwarz to our a1 to an and b1 to bn, we will apply it to a1 prime to let me write it down.

Applying, let me shorten it to C-S for Cauchy Schwarz to a1 prime upto an prime and b1 prime upto bn prime. What will be the L.H.S, L.H.S will be the absolute value of the inner product of we will write it down why be in a hurry this and b1 prime to bn prime, this is less than or equal to the absolute value of or the length of a1 prime to an prime, times the length of b1 prime to bn prime.

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III^{ly} $b_i' = b_i/\sqrt{i}$

Apply C-S to $(a_1', \ldots, a_n')$ and $(b_1', \ldots, b_n')$

$$|\langle (a_1', \ldots, a_n'), (b_1', \ldots, b_n') \rangle|^2 \leqslant \|(a_1', \ldots, a_n')\|^2 \|(b_1', \ldots, b_n')\|^2$$

$$\Rightarrow |(a_1'b_1' + a_2'b_2' + \cdots + a_n'b_n')|^2 \leqslant$$

Problem 6: Prove that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leqslant \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right).$$

$\forall\ a_i, b_i \in \mathbb{R}$ where $i = 1, 2, \ldots, n$.

Proof: Consider $\mathbb{R}^n$ with the standard inner product.

Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n$.

By Cauchy-Schwarz inequality, we have

What is the L.H.S here? L.H.S is just a1 prime b1 prime plus a2 prime b2 prime plus upto an prime, bn prime, absolute value of this, this is less than or equal to. So, what is it that we have to show? We have to show that this square is less than or equal to this this. So, in particular this square is also less than all these are positive quantities. So, Cauchy Schwarz will in particular give us this.

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$$|\langle(a_1',\ldots,a_n'),(b_1',\ldots,b_n')\rangle|^2 \leq \|(a_1',\ldots,a_n')\|^2\|(b_1',\ldots,b_n')\|^2$$

$$\Rightarrow |(a_1'b_1'+a_2'b_2'+\cdots+a_n'b_n')|^2 \leq (a_1'^2+\cdots+a_n'^2)(b_1'^2+\cdots+b_n'^2)$$

So, the left hand side this square will be equal to what will be the length of a1, a2 upto an, a1 prime a2 prime upto an prime in this with respect to this inner product, it will just be equal to the sum of a1 prime square plus upto an prime square that will be the length of a1 prime upto an prime and this will b1 prime square plus up to bn prime square.

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$$|\langle(a_1,\ldots,a_n),(b_1,\ldots,b_n)\rangle| \leq \|(a_1,\ldots,a_n)\|\|(b_1,\ldots,b_n)\|$$

Define $\underline{a_i' = \sqrt{i}\,a_i}$

|||ly $\underline{b_i' = b_i/\sqrt{i}}$

Apply C-S to $(a_1',\ldots,a_n')$ and $(b_1',\ldots,b_n')$

$$|\langle(a_1',\ldots,a_n'),(b_1',\ldots,b_n')\rangle|^2 \leq \|(a_1',\ldots,a_n')\|^2\|(b_1',\ldots,b_n')\|^2$$

$$\Rightarrow |(a_1'b_1'+a_2'b_2'+\cdots+a_n'b_n')|^2 \leq (a_1'^2+\cdots+a_n'^2)(b_1'^2+\cdots+b_n'^2)$$

Now, let us go back to see, what our ai prime and bi prime, ai prime was square root of i times ai and bi prime was bi by square root of i. So, ai prime times bi prime is just ai times bi.

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$$|\langle (a_1,\dots,a_n),(b_1,\dots,b_n)\rangle| \leq \|(a_1,\dots,a_n)\|\,\|(b_1,\dots,b_n)\|$$

$$\Rightarrow |(a_1'b_1' + a_2'b_2' + \dots + a_n'b_n')|^2 \leq (a_1'^2 + \dots + a_n'^2)(b_1'^2 + \dots + b_n'^2)$$

$$\Rightarrow (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \leq \left((1a_1)^2 + (\sqrt{2}a_2)^2 + \dots + (\sqrt{n}a_n)^2\right)\left(\frac{b_1^2}{1} + \frac{b_2^2}{2} + \dots + \frac{b_n^2}{n}\right)$$

$$\leq \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right) \qquad \blacksquare$$

So, that is what we will use to write the left hand side here. And I can throw out the absolute value because this square will anyway be a positive number. So, this is going to be a1 b1 plus a2 b2 plus upto an bn the whole square, this is equal to this by the way, and what about this, this is just going to be equal to 1 times a1, the whole square, no square root of 1 times, which is 1 plus square root of 2 times a to the whole square plus square root of n times an the whole square times.

Similarly, b1 by 1 which is b1 square plus, so let me just write it down here by 1 plus b2 square by 2, and so on plus bn square by square root of n the whole square which is n this is less than or equal to the R.H.S is now just summation, well, let me write it as j aj square times summation bj by bj square by j where j goes from 1 to n. And that is precisely what we had set up to proof. So, by picking our vectors very carefully, Cauchy Schwarz inequality is giving us some remarkable inequality that we have seen here.

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Problem 7: Let V be a complex inner product space. Prove that

$$\langle u, v\rangle = \frac{\left(\|u+v\|^2 - \|u-v\|^2\right) + i\left(\|u+iv\|^2 - \|u-iv\|^2\right)}{4}$$

So, the final problem and this problem session involves an identity that is true in a complex inner product space. So, let us look at what the problem is. Problem 7. So, let V be a complex inner product space. Prove that the inner product of u, comma v, this is equal to the length of u plus v the whole square minus the length of u minus v the whole square plus i times the length of u plus i times v the whole square minus the length of u minus i times v the whole square whole divided by 4. Okay, so let us see what this is, so let us look at what the R.H.S is going to be here like.

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that

$$\langle u, v\rangle = \frac{\left(\|u+v\|^2 - \|u-v\|^2\right) + i\left(\|u+iv\|^2 - \|u-iv\|^2\right)}{4}$$

$$\left(\|u+v\|^2 - \|u-v\|^2\right) + i\left(\|u+iv\|^2 - \|u-iv\|^2\right) =$$

$$\left(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle\right) + i\left(\langle u+iv, u+iv\rangle - \langle u-iv, u-iv\rangle\right)$$

So, let us look at what is the length of u plus v the whole square minus the length of u minus v the whole square plus i times the length of u plus iv the whole square minus, this is just going to be a huge computational problem involving i and uv and i. But nevertheless, it is beneficial to look at it because it will help us get familiarized to using inner products.

So, let us see this will be what is the length of u plus v the whole square that is the inner product of u plus v with itself. And how about the next term, this is u minus v with itself plus i times the inner product of u plus iv with itself minus u minus iv, inner product of that with u minus iv.

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$$\big(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle\big) + i\big(\langle u+iv, u+iv\rangle - \langle u-iv, u-iv\rangle\big)$$

$$= \Big(\big(\|u\|^2+\|v\|^2+\langle u,v\rangle+\langle v,u\rangle\big) - \big(\|u\|^2+\|v\|^2 - \langle u,v\rangle - \langle v,u\rangle\big)$$

$$+ i\big(\langle u,u\rangle + \bar{i}\langle u,v\rangle + i\langle v,u\rangle + i\bar{i}\langle v,v\rangle\big) - \big(\langle u,u\rangle - i\langle v,u\rangle - \bar{i}\langle u,v\rangle + i\bar{i}\langle v,v\rangle\big)\Big)$$

$$= 2\big(\langle u,v\rangle+\langle v,u\rangle\big) + i\Big(\big(\|u\|^2 - i\langle u,v\rangle + i\langle v,u\rangle + \|v\|^2\big) - \big(\|u\|^2 - i\langle v,u\rangle + i\langle u,v\rangle + \|v\|^2\big)\Big)$$

$$= 2\Big(2\,Re\big(\langle u,v\rangle\big) + i\,2\,\big(i\,(\underset{\overline{\langle u,v\rangle}}{\langle v,u\rangle} - \langle u,v\rangle)\big)\Big)$$

$$= 4\,Re\big(\langle u,v\rangle\big) + i^2\,2\,\big(2i\,Im\,(\langle v,u\rangle)\big)$$

$$= 4\big(Re(\langle u,v\rangle) - i\,Im\,(\overline{\langle u,v\rangle})\big).$$

$$= 4\big(Re\,(\langle u,v\rangle) + i\,Im\,(\langle u,v\rangle)\big).$$

$$= 4\,\langle u,v\rangle. \qquad \blacksquare$$

Now, let us expand it out, and let us keep track of all the properties of the inner product that we have seen. This is just going to be equal to I will be a little quick here to write it down, u, so I will just write it down the first time will be length of u the whole square plus length of v the whole square plus inner product of u, comma v plus inner product of v, comma u, that is the first term minus what will be the second term that will be, that will be the length of u square plus the length of v square minus 1 minus 1, 2 minus 1 will cancel off and minus the inner product of u, comma v minus the inner product of v, comma u, that is the first term.

How about the second term? i times let me be a little careful here I will write down things explicitly, and then we will write down the final expression this is going to be u, comma u plus u comma iv, which is going to be i bar times u, comma v, plus iv comma u, inner product of iv comma u, which is i times inner product of v, comma u plus the inner product of iv comma iv, which is mod of i square.

So, let me put it this way, i, i bar times v, comma v, that will be the first term, that will be the first term minus, how about the second term, second time will be very similar, u, comma u now minus of i times v, comma u minus of i bar times u comma v plus i i bar times v, comma v.

The second word, term is where all the i had to be taken care of carefully. So, the first term is easy. First time will u square, length of v square and length of v square, cancels off this is just going to be 2 times the inner product of u, comma v, plus the inner product of v, comma u. And now let us focus on the second term, the second term will be plus i times the length of u square plus i bar times, so i bar is just going to be minus i, you should check that minus of i times u, comma v plus i times v, comma u plus, i i bar is absolute value of i square or you can check directly that it is minus of i i square, which is 1.

So, this is just going to be plus length of v square. So, there is this first term and then the second term is again similarly, length of u square minus i times v, comma there is a bracket here. This just turns out to be a lot of bookkeeping, but, so, this will cancel off and this will be plus i times u, comma v. Nevertheless, let me do it and this is going to be what is u, v plus v, u, v, u is u, v bar. So, this is equal to u, comma v conjugate and what is a plus ib plus a minus ib is 2a, which is 2 times real part, this is equal to 2 times the real part of the inner product u, comma v. And there is a 2 outside, so 2 times 2 of this, is good.

And how about this? This is i times the situation is quite similar, this is just going to be equal to 2 times maybe I can take i also out and there will be a inner product of v, comma u minus u, comma v, okay. So, that is good, notice that this is just u, comma v bar the conjugate of v, comma u and what will happen if you subtract the conjugate from something, what you get is, so let me write it down this is 4 times real part of the inner product of u, comma v plus, so there is an i which is i square times 2 into the imaginary part of v, comma or maybe I should put it this way. Yeah, let us see, we will, we will come to it in a minute.

So, this is equal to there is a 2. And this is why is that the case because this is safe, this is a plus ib, and this is a minus ib should do the subtraction and see that a plus ib minus a minus ib is, oh there is an i, there is an i so should be careful, so **two** i times the imaginary part of inner product of v, comma i, so the answer when you subtract will be 2 ib, which is 2 times i into imaginary part of that. That is what I have written here. So, this is again 2 2 4 is there 4 times the real part of u, comma v minus so 4 has been taken out i times imaginary part of v, comma u. But v, comma, inner product of v, comma u is let me just rewrite this this is inner product of u, comma v bar.

But when you look at the imaginary part of Z and imaginary part of Z bar, what do you have imaginary part of Z bar is the minus of the imaginary part of Z. So, this is just going to be equal to 4 times real part of the inner product of u, comma v plus i times the imaginary part of u, comma v. That is the imaginary, real part of Z plus i times the imaginary part of Z, which is actually equal to 4 times the inner product of u, comma v, which is what we had set out to proof.

So, I would request you if you are not familiar with the operations of complex numbers, I will request you to go through these steps very carefully. And I ran over the reasoning without bothering to emphasize and but all of them are quite straightforward. You should however, sit back and check that all these what are the properties of the complex numbers that are being used to conclude the (())(72:28). All right, so let me stop here.