Linear Algebra.
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Lecture 41
Orthogonal Complements.

So, we have discussed the orthogonality of vectors and many properties of orthogonal vectors. Let us next discuss what it means to say that two vectors subspaces of a given inner product space are orthogonal.

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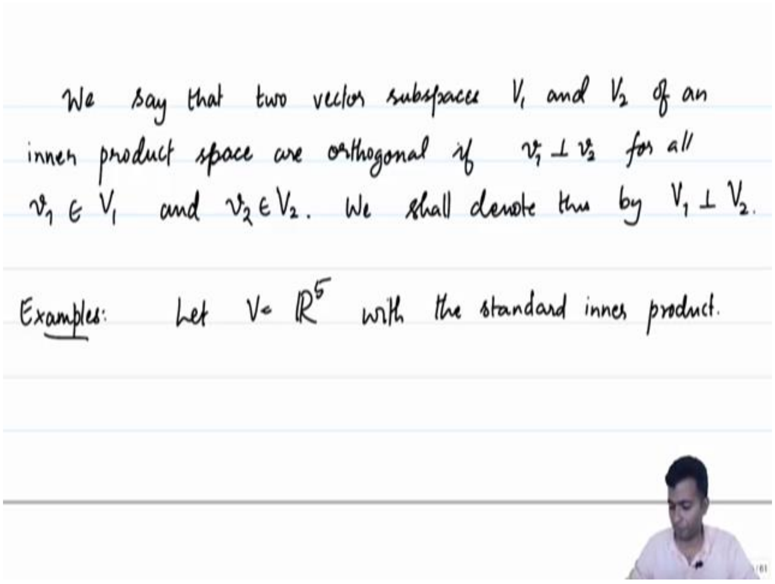

So, let us begin by a definition, we say that two vector subspaces V1 and V2 of an inner product space are orthogonal if v1 is orthogonal to v2 for all v1 in V1 and v2 in capital V2. So, if you take any arbitrary element in capital V1 and any arbitrary element in capital V2, and if they happen to be orthogonal to each other in every such instance, then we say that the subspace v1 is orthogonal to the subspace v2.

So, we shall denote this by capital V1 orthogonal to capital V2. So, let us look at some examples. Again, it is in an inner product space. So, let us look at a say R, let V be equal to say R5, V the vector space with the standard inner product, so, with the standard inner product, the dot product.

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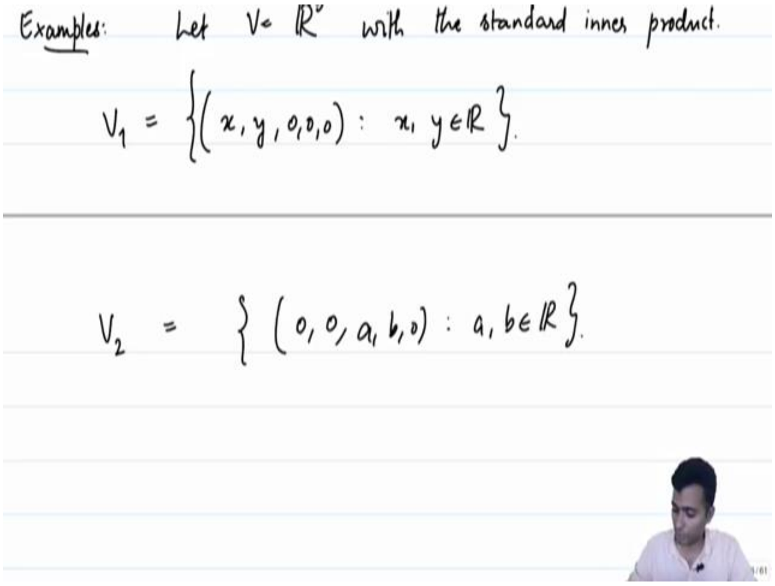

Now, let us consider two vector subspaces here. So, let us look at V1 to be equal to x, so the set of all x comma, y comma 0, 0, 0, so this is an R5, so x, y in R. So, this is a 2-dimensional subspace v1, which is generated by E1 and E2. Now let us look at V2, which is generated by 0, 0, z comma a, maybe I should use a comma, b comma, 0, where a and b are in reals. Then V2 is also a subspace, it is very easy to check that.

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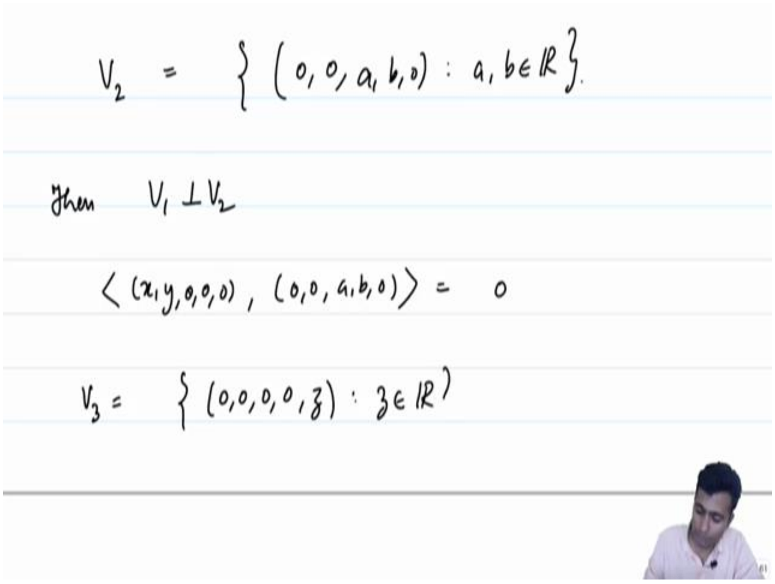

Then V1 is orthogonal to V2 and why is that the case, because let us look at the inner product of any 2 such elements. This is going to be x, y 0, 0, 0 and 0, 0, a, b, 0, which is equal to x times 0 is 0, y times 0 is 0, and then 0 times a is 0, 0 times b is 0 and 0 times 0 is 0, if you add it, you get back 0. So yes, any two vectors, you take in say V1 and V2 respectively, their inner product is 0, they are orthogonal to each other. Therefore, the subspace V1 is orthogonal to V2. You will look at v3 which is 0, 0, 0, 0, z, where z is in R.

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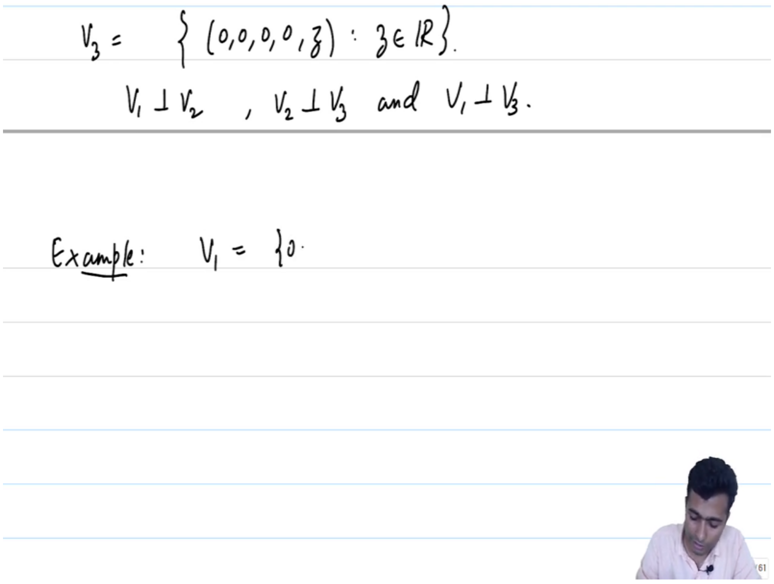

Then you can check that V1 is orthogonal to V2, V2 is orthogonal to V3 and V1 is orthogonal to V3. So, all these are orthogonal subspaces. So, another example is to look at what is the, so, let us look at one more example. Let V1 be the 0 subspace.

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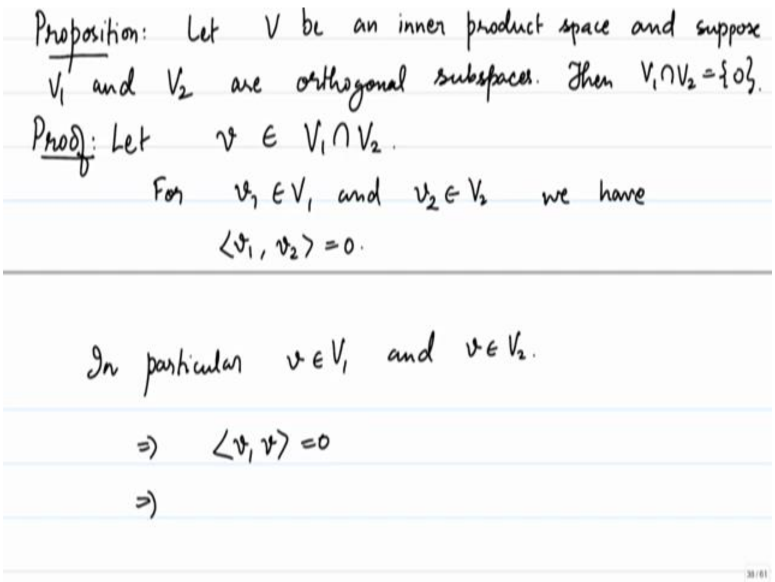

For v1 in V and V1, and v2 in capital V2, we have the inner product of v1 and v2 to be equal to 0. But in particular, v is in capital V1 and v is in capital V2 as well because it is in the intersection and therefore, taking V1 to be v and V2 to be v we have inner product of v with itself is equal to 0.

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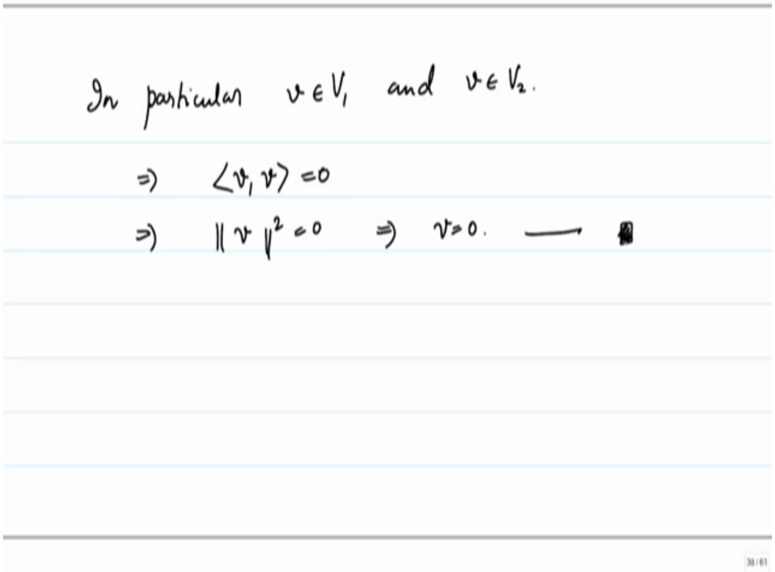

Which just implies that the length of v square is equal to 0, which implies length of V0 and therefore v is equal to 0, that completes our proof. So, basically if there are 2 subspaces, which are orthogonal, their intersection has to be necessarily just the, it has to be just the 0 element, 0 vector. Note that 0 will always be there, it belongs to every vector subspace and therefore the intersection will always have 0, that is the only vector that will be in the intersection.

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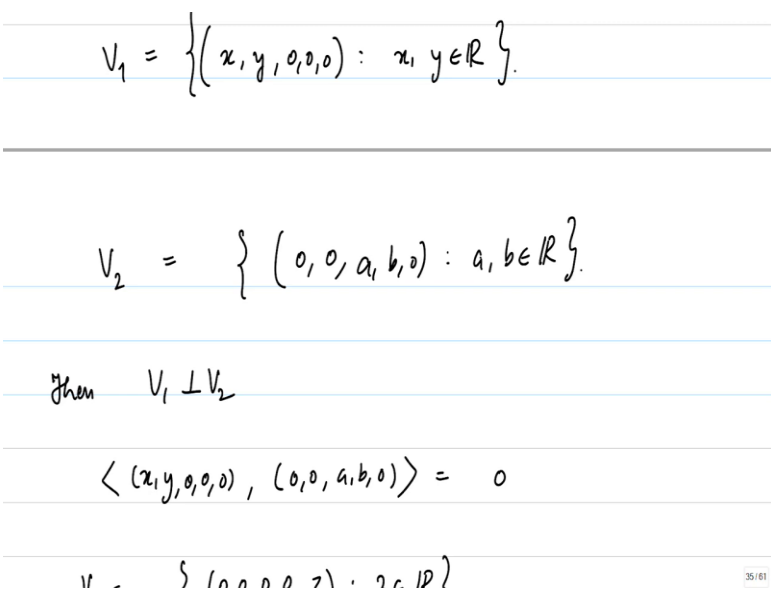

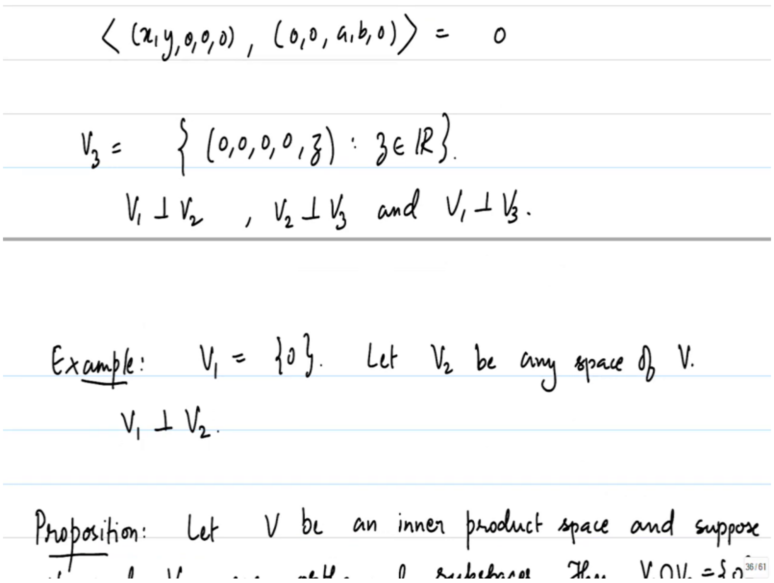

So, in the previous example, in one of the previous examples, here, when we looked at the subspaces v2 and subspace v3, we observed that both happened to be orthogonal to V1, V1 is orthogonal to V2 and V1 is orthogonal to V3 as well. So, there is nothing unique that we can say about orthogonality here. However, we will now define the orthogonal complement, which happens to be the largest subspace which is orthogonal to our given vector space, subspace.

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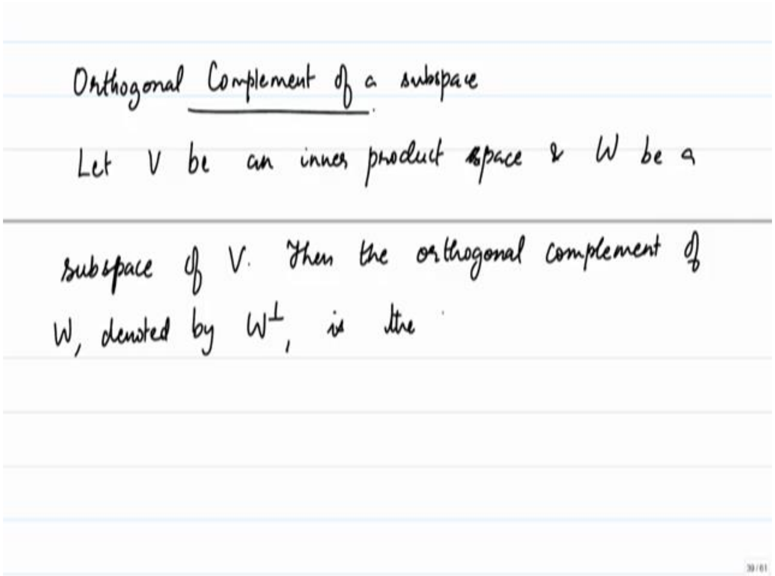

So, let us define what is the orthogonal complement. Orthogonal complement of a, of a subspace. So, let V be an inner product space, I will slowly stop writing this and W be a subspace of V, we have just taken some arbitrary subspace of V. Then the orthogonal complement of W, it is denoted as W orthogonal, orthogonal complement of W is denoted by W orthogonal, is the set.

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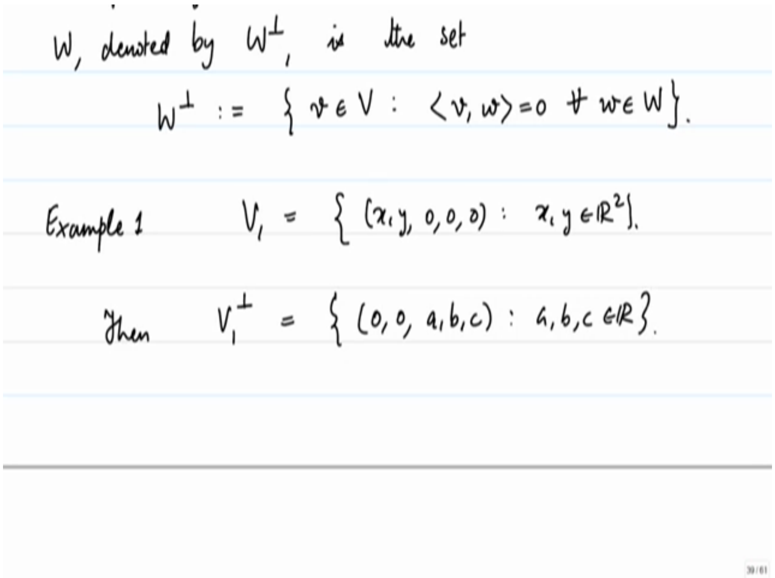

W orthogonal defined to be the set of all v in capital V, such that this is equal to 0 for all w in capital W. So, you look at those vectors which are orthogonal to every vector in capital W. That is called the orthogonal complement of w. So, let us get back to the first example. So, let example 1. So, what was V1 here? V1 was the set of all x, y, 0, 0, 0 such that x comma y belongs to R2. Then V1 orthogonal, my claim is that this is nothing but all those vectors which are of this type. So, let us see if that is the case.

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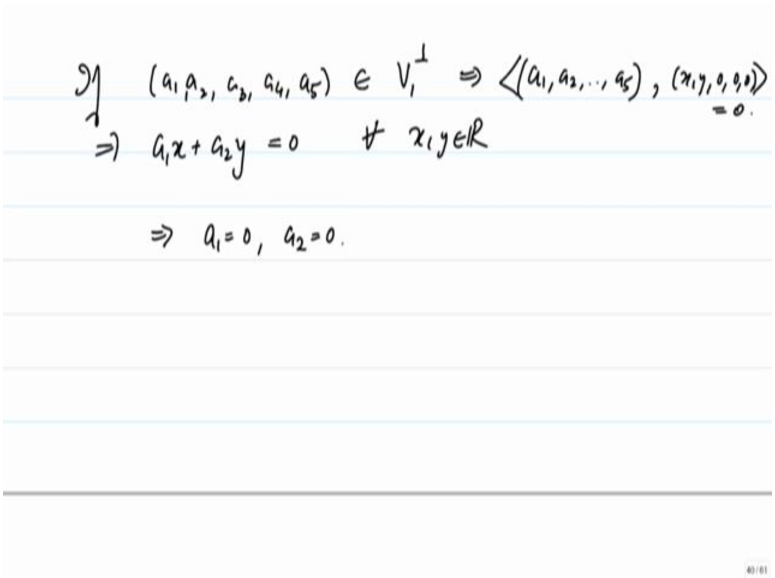

So, if a1, a2, a3, a4, a5 belongs to V1 orthogonal. What does that mean? This means that a1 x plus a2 y is equal to 0 for all x comma y belonging to R. In particular, this would force a1 to be equal to 0 and a2 to be equal to 0 there is no other restriction however for this to happen. For any values of a3, a4, a5, the inner product will turn out to be 0. Why is this the case, this implies that a1 to a5 with x, y and then 0, 0, 0, this is equal to 0. That implies a1 x plus a2 y equal to 0.

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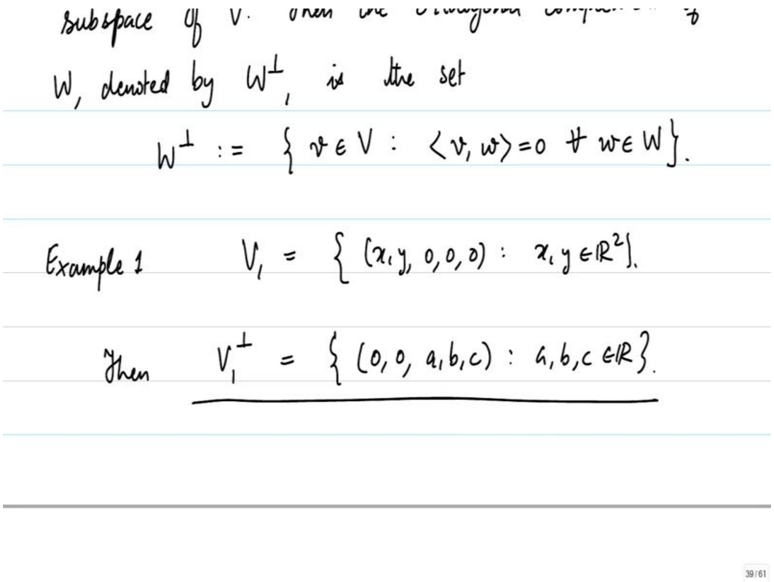

So, this is clearly the case when this happened. So, V1 orthogonal is this particular set and if you notice, we will come to that. Let us look at what will be the orthogonal complement of the 0 subspace.

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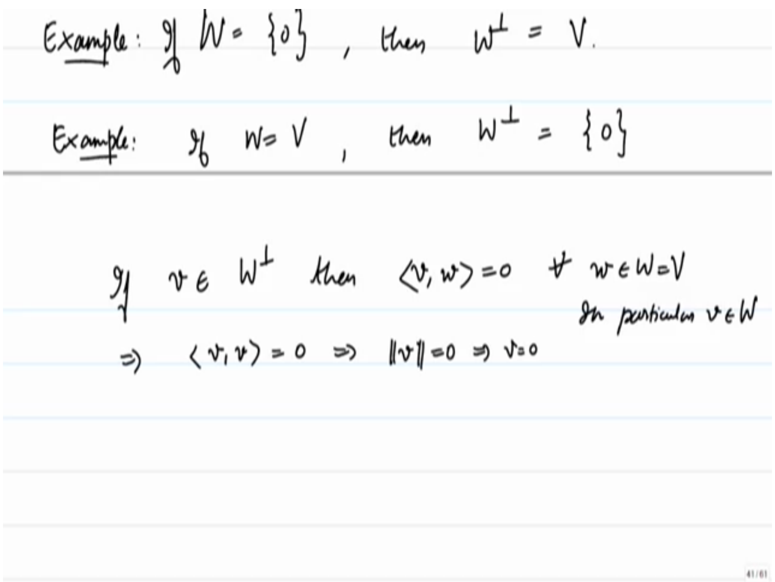

So, another example is that if W is the 0 subspace, then this is true in every inner product space, then the orthogonal complement of W is equal to V. That is quite straightforward because you take any vector V, the inner product of V with 0 is 0. So therefore, this is certainly contained in the orthogonal complement, but then we have already consumed every vector because it is already equal to V, therefore it has to be equal to V.

Another example would be to check that if W is equal to V, the entire vector space, then W orthogonal, we should sit back and think about what this is going to be equal to. We will come back to it a few minutes later. Or maybe you should pause and think about it and then see that the answer is the 0 vector space. The reason for that is, see what are the vectors in the orthogonal complement of W, this will be those vectors which are orthogonal to every vector in capital W.

So, if v belong to belongs to W orthogonal then v and w is equal to 0 for all w in capital W. But our capital W is equal to V. So, in particular, small v also belongs to W because small v belongs to capital V and therefore, inner product of v with itself is equal to 0. But that implies the length is equal to 0, which implies that the vector is to be necessarily equal to 0 or the positivity will tell us that v is equal to 0.

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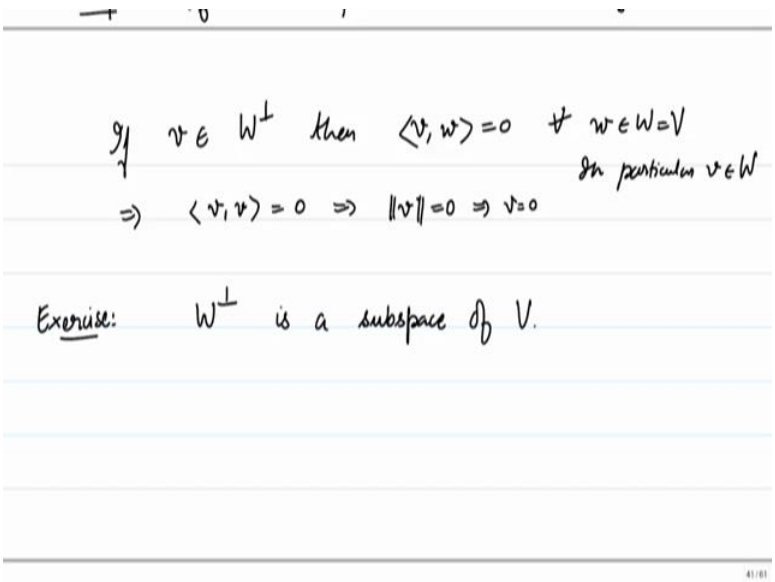

So, let us get back to this example this example, using indicator of what type of the, one more thing, one more thing to notice that exercise for you. The orthogonal complement of a given subspace is a subspace and hence an inner product space of V. So, this comes from one of the previous results which we have proved, wherein we showed that if v is orthogonal to w1, w2, up to wk, then V is also orthogonal to every linear combination.

You can use that to prove that the orthogonal complement of W is also a subspace. I will leave that as an exercise for you. Let us next prove a result which tells us that the orthogonal complement is in some sense the largest space which satisfies this property. So, let us state a proposition here.

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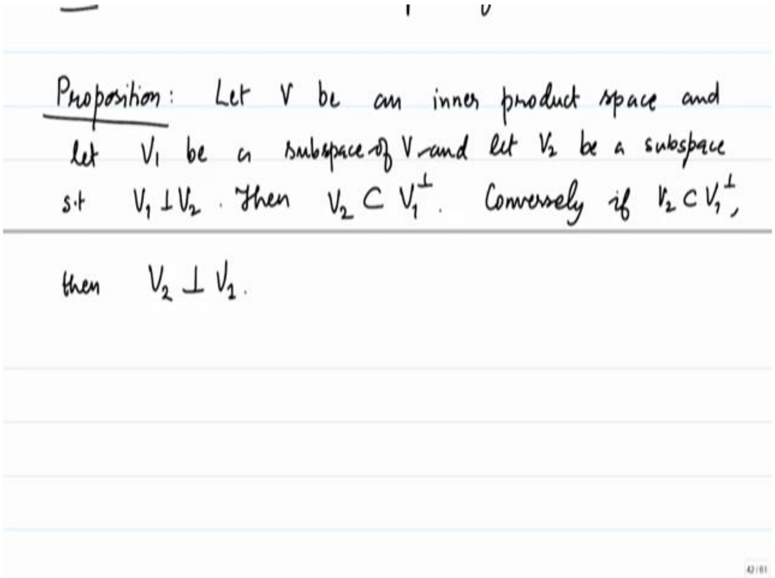

So, let V be an inner product space and let V1 be a subspace of V, suppose V2 is another subspace of V which is orthogonal to V1. So, and let V2 be a subspace such that V1 and V2 are orthogonal. Then V2 is contained in V1 orthogonal, the orthogonal compliment of V1. Conversely, if V2 is contained in the orthogonal complement of V1, then V2 is orthogonal to V1. So that is what it means to say that, in some sense, it is the largest subspace which is orthogonal to our given subspace V1.

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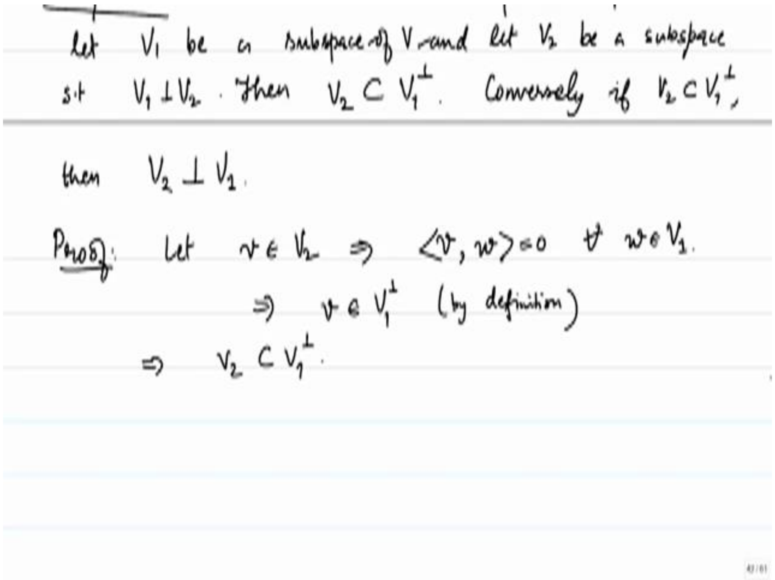

So, let us give a proof of this proposition. It is actually quite straightforward. So, let us assume that. So, let us try to prove that if V2 is orthogonal to V1, then it is contained in the orthogonal complement. So, let v be a vector in V2, but this implies that v inner product with, let me use the, v inner product with w is equal to 0 for all w in V1.

But if you just go back to the definition of what it means for a vector to be in the orthogonal complement, it means that it should be orthogonal to every vector in V1 and this gives that v should be in the orthogonal complement by the very definition, by definition, by definition of the orthogonal complement, and this implies that V2 is contained in the orthogonal complement of W. How about the other way?

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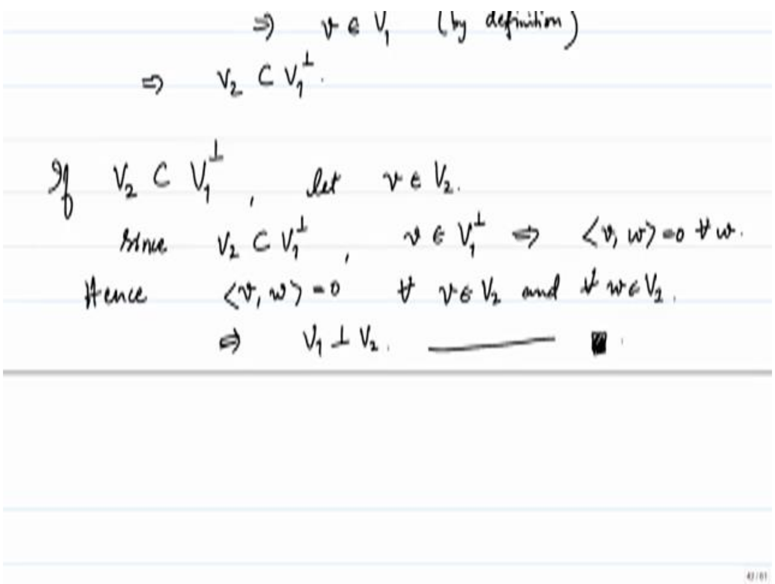

If V2 is contained in the orthogonal complement of V1, I meant orthogonal complement of V1 earlier as well even though I said orthogonal complement of W. So, if V2 is contained in the orthogonal complement of V1, then let v be in capital V2. So, we would like to show that.

So, our goal is to show that V2 is orthogonal to V1. So that means that we would like to show that the inner product of v with any vector w in V1 is 0. So let us pick an arbitrary vector v, but because v is contained in V1 orthogonal, V2 is contained in V1 orthogonal, v belongs to V1 orthogonal and by definition, this implies that the inner product of v with w is equal to 0 for all W and our choice of v was arbitrary, this gives that inner product of v with w is equal to 0.

So hence, inner product of v with w is equal to 0 for all v in V2 and all w in V1. This implies that V1 is orthogonal to V2. So yes, we have proved that the orthogonal complement is the largest subspace, which is orthogonal to our given subspace. It is sometimes very useful to obtain concretely what the orthogonal complement of a given subspace is and our next proposition tells us that it is many times possible to get hold of one explicitly.

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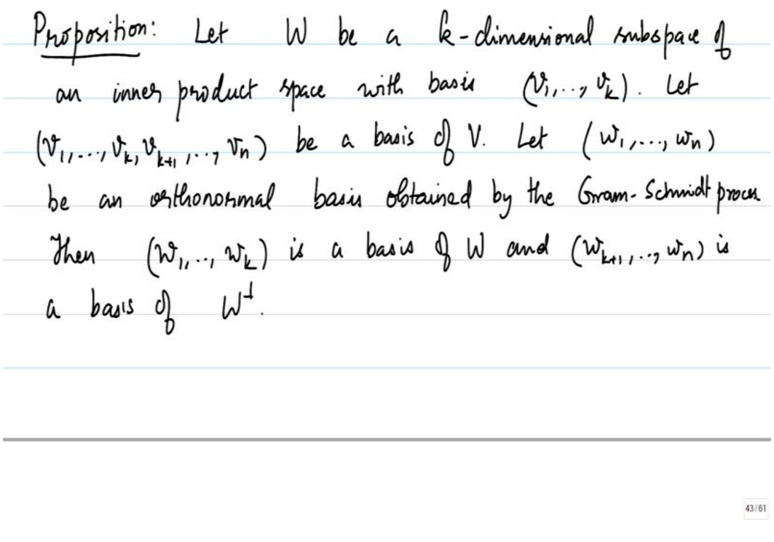

So, proposition. So, let W be a k-dimensional subspace, let W be a k-dimensional subspace of an inner product space with basis v1 to vk. Let us extend it to a basis of v, let v1 to vk, vk plus 1 to vn be a basis of capital V. So, we are, again we are considering inner product spaces which have, which has a finite dimension. Now, let us apply the Gram Schmidt process, orthonormalize it to obtain a new basis. Let w1 to wn be an orthonormal basis obtained by the

Gram Schmidt process. Then w1 to wk is a basis of W and the remaining vectors are a basis of W orthogonal, and wk plus 1 to wn is a basis of W orthogonal.

So, what this says is that you start off with a vector space, the procedure is also explicitly given in the proposition. You start off with a vector space you look at a basis of our given subspace, extend it to a basis of V, apply a Gram Schmidt process and orthonormalize it to get an orthonormal basis, w1 to wn.

This proposition tells us that the first k of the orthonormal basis that we get, will be a basis of a subspace W and the remaining vectors wk plus 1 to wn will turn out to be a basis of the orthogonal complement of W. So, let us give a proof of this.

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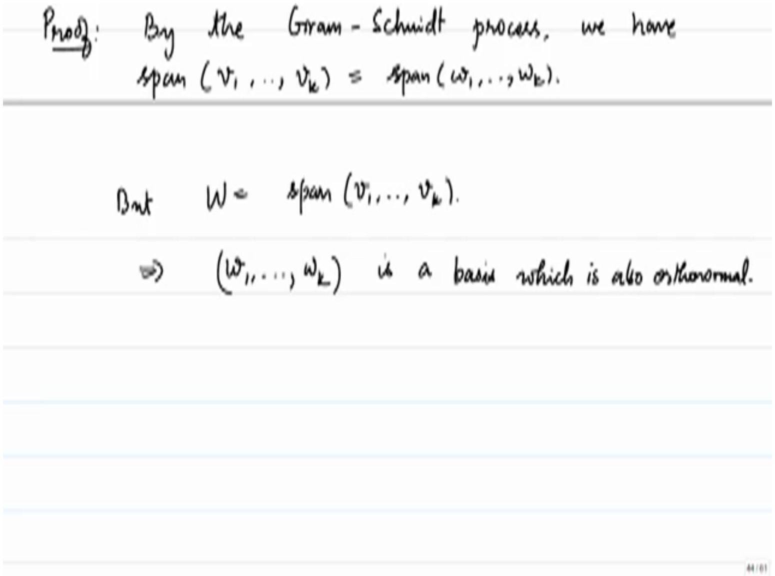

So, by the very Gram Schmidt process span of w1 to wk is equal to span of v1 to vk. By the Gram Schmidt process, we have span of v1 to vk is equal to the span of w1 to wk. But what does that mean, what is span of v1 to vk? We started off with v1 to vk as being a basis of W, but W is equal to the span of v1 to vk, that means that w1 to wk is an orthonormal. The orthonormal part comes later, is a basis, it is a spanning set of W which is k-dimensional and any k, any spanning set of size k in a k-dimensional vector space would necessarily be a basis. So, this is a basis which is also orthonormal. Which is also orthonormal.

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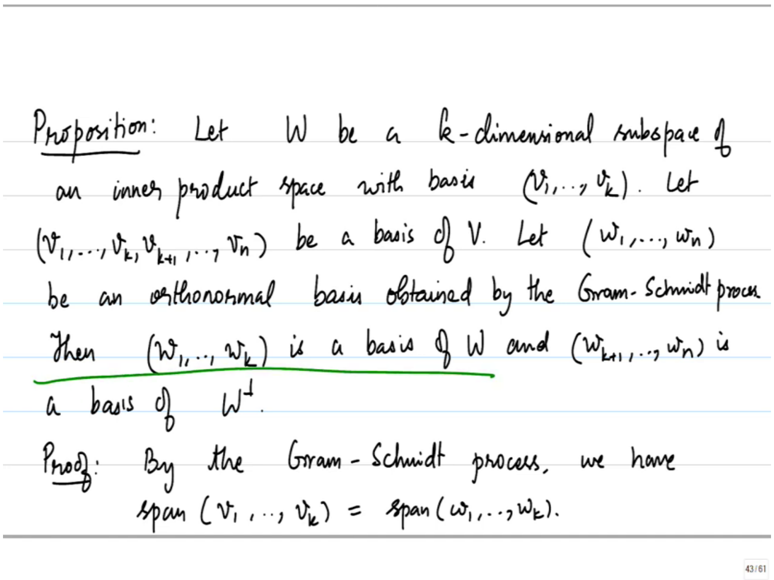

So, we have proved the first part of our proposition which said that w1 to wk which I am underlining, this is a basis of W that much has been already established. Now we know that w1 to wn is an orthonormal basis of V, because it is obtained by a Gram Schmidt process.

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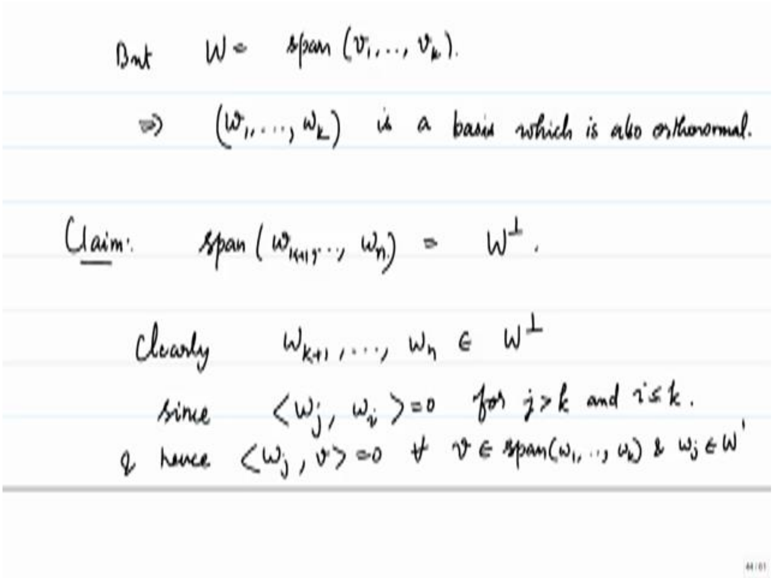

So, let us now prove that the span of, so claim, let me write it down explicitly what we are proving. We will prove that span of w1 to wk, or sorry, wk plus 1 to wn, this is equal to the orthogonal complement of W. So, let us have a look at the containment in this direction. So, let us prove that if any vector is in the span of wk plus 1 to wn, then it is in the orthogonal complement as well.

So clearly, wk plus 1 to wn, each of these vectors belong to the orthogonal complement of W because they are, after all it is, since wj and wi is equal to 0 for all i not equal to j. So, in particular for j greater than k, and for i less than k plus 1, the inner product should necessarily be 0 and therefore, it will be orthogonal to every linear combination of w1, w2 up to wk and hence wj comma v is equal to 0 for all v in the span of v1 to vk.

And so, let me write it like this, for j greater than k and i less than or equal to k and therefore, this tells us that for v in the span of not v1 to vk, w1 to wk, we will be wrong if we write, of course, it will be right because the span is the same, but the reason why we are able to conclude this is because v is in the span of w1 to wk and this basically means that wj belongs to W orthogonal, that is what we have concluded just now. wj belongs to the orthogonal complement of W, but then orthogonal complement of W is a subspace

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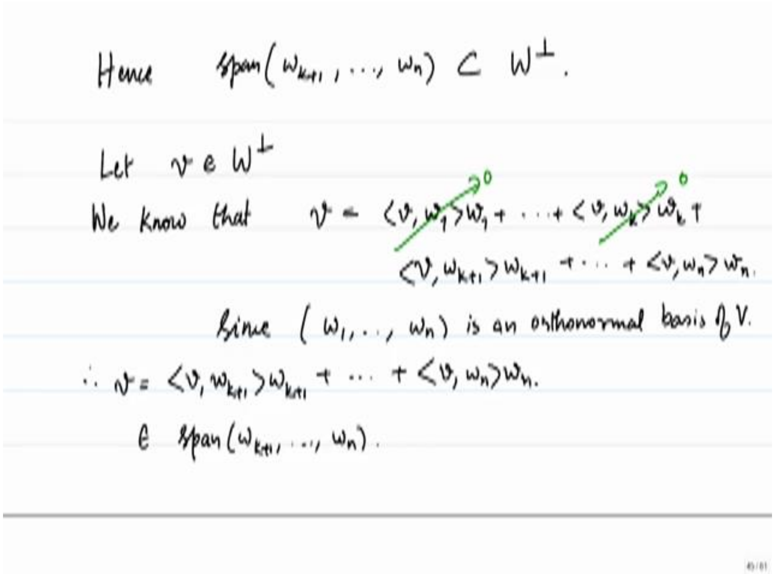

And hence span of wk plus 1. So, this is for all j greater than k. Span of k plus 1 to kn, span of wk plus 1 wn is contained in the orthogonal complement of W. Let us prove the other way. So, let v be a vector in the orthogonal complement of W, we would like to show that this is in the span of wk plus 1 to wn. So, what do we know about writing down a vector v as the linear combination of vectors in an orthonormal basis.

We know that, we know the explicit formula. So, we know that v is equal to, recall that w1 to wn is an orthonormal basis. So, this is equal to v inner product with w1 times w1 plus v inner product w. So, let me write it like this v inner product wk times wk plus v inner product wk plus 1 times wk plus 1 plus up to v inner product wn times wn, because w1, w2, wk, wk plus

1 up to wn is an orthonormal basis of V. Since w1 to wn is an orthonormal basis of V, so any vector can be written like this.

But then we started off with the assumption that our v is in the orthogonal complement of W. So, every vector in capital W will be orthogonal to v. So, in particular, this is going to be 0, everything up to this is going to be 0, because it is orthogonal and therefore, our v will be just the inner product of v with wk plus 1 times wk plus 1 plus… inner product of v with wn times wn, which belongs to the span of wk plus 1 to wn and therefore, we took an arbitrary vector in w orthogonal and we showed that it is in span of wk plus 1 to wn.

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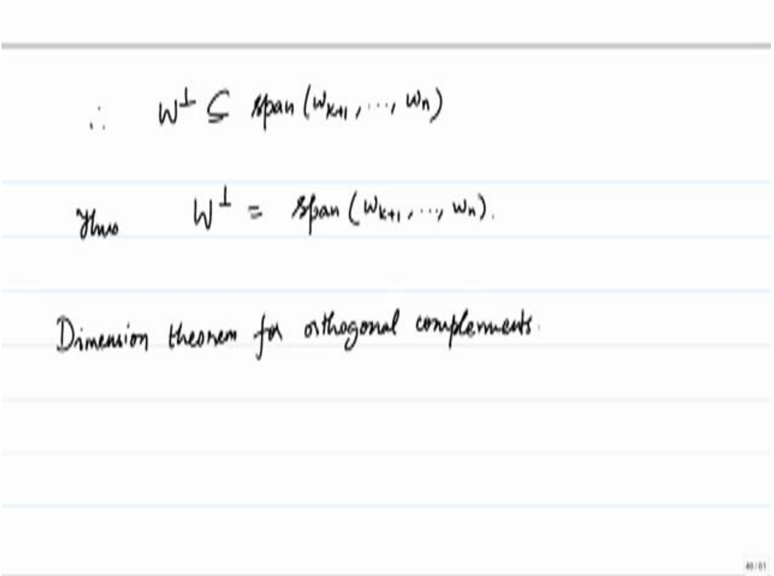

Therefore, orthogonal complement of W is contained in the span of wk plus 1 to wn and therefore, we have shown both sides containment. Thus, orthogonal complement of W is equal to the span of wk plus 1 to wn. So, this is not just giving us an explicit basis for the orthogonal complement, it also tells us a dimension theorem. So, for the orthogonal complement, for orthogonal complements. What it tells us is that the dimension of the orthogonal complement is n minus k.

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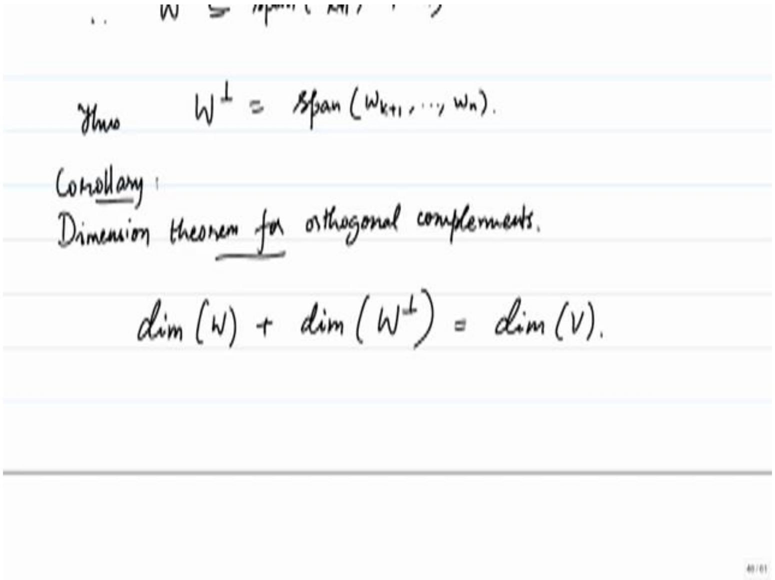

So, in other words, dimension of w plus the dimension of the orthogonal complement of W is equal to the dimension of V. So this we obtain as a corollary. Let us now look at a few examples.

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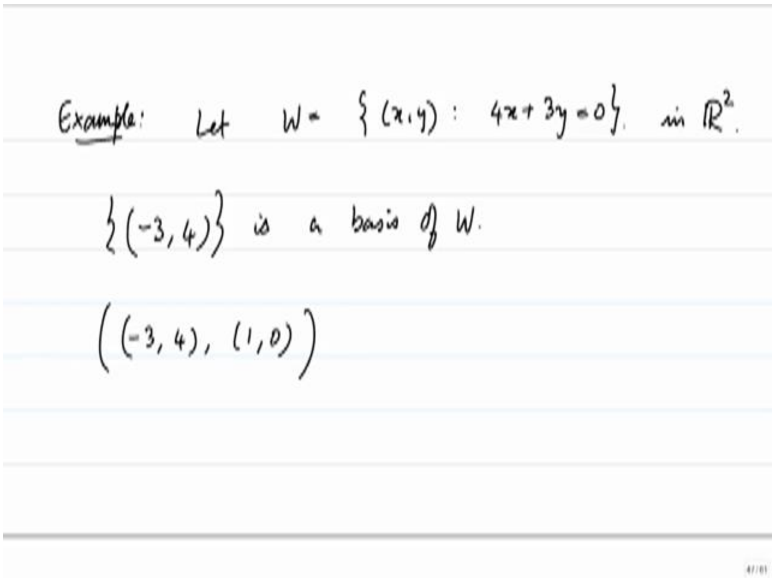

So, let us look at an example in R2. So, let W be equal to the set of all x, y, such that 4x plus 3y is equal to 0. So, this is going to be a one-dimensional subspace in R2 and we can get hold of some basis, so let us see. Minus 3, 4 will form a basis of this set, is a basis of W. Now let us try to get hold of what our orthogonal complement of W is explicitly. So let us complete it into a basis. So, let minus 3, 4, and say 1, 0, these are linearly independent and any set of size

2, which are linearly independent, which is linearly independent will be a basis. So this is in particular a basis of R2.

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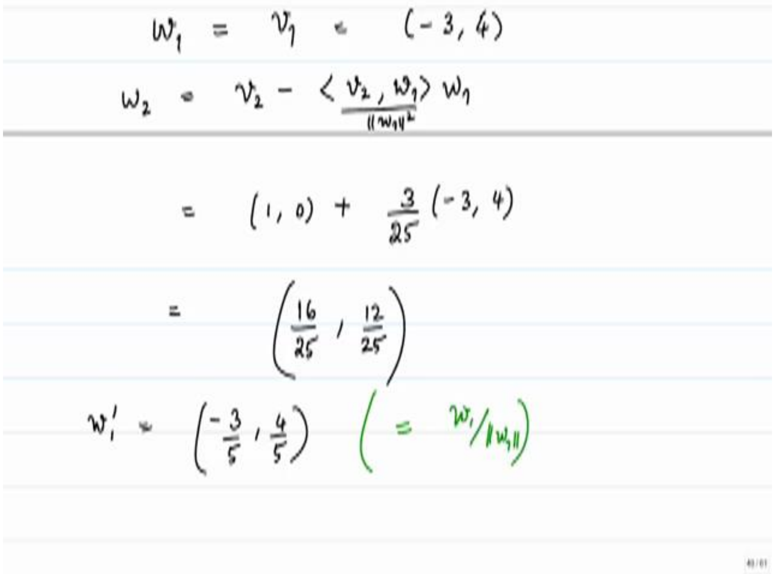

Now let us orthonormalize it, what do we get, we get v1, so let us now calculate w1, w1 is just v1, which is equal to minus of 3 comma 4. What was w2, w2 was v2 minus inner product of w1 with, sorry, w2, v2 minus inner product of v2 with w1 by the length of w1 square times w1. So, this is just going to be 1, 0 minus inner product of v2 with w1, which is minus 3, so this is going to be plus 3 by 25 times minus 3 comma 4.

Which is just going to be equal to minus of 8 by 25, I am sorry, that is wrong. It is going to be 25 minus 9, which is 16 by 25, and this is going to be 12 by 25. So, let us now normalize this. So w1 prime will just be minus of 3 by 5 and 4 by 5. After, so this is just, let me write it down in green. So, this is just equal to w1 by the length of w1 and how about the second one? That is just going to be not so nice looking number.

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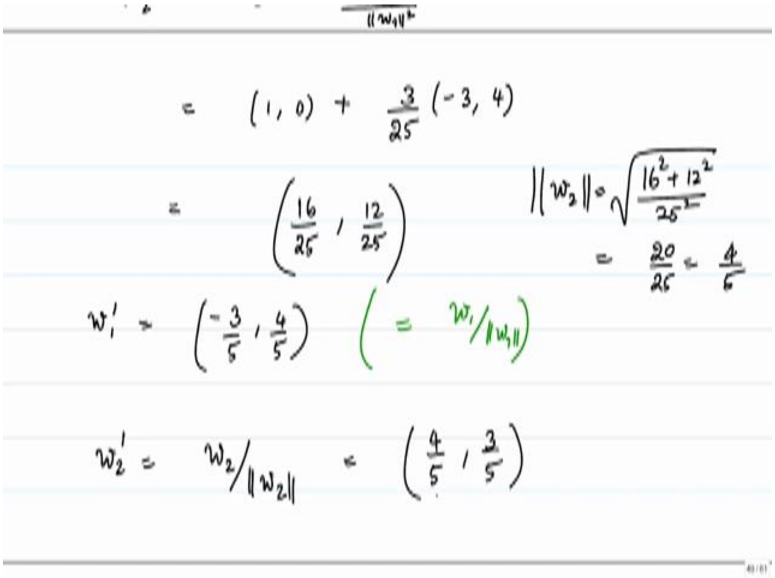

Let us see, let me calculate that. 16 square 256, 144, 300, 400, 20 by 25, 20 by 25 is just, 20 by 5 is just equal to 4. So, what is the, so let us next calculate w2 prime, which is w2 by the length of w2. Let us see what was w2, w2 was let us recall, w2 is right here. So, what is length of w2? This is just square root of 16 square plus 12 square by 25 square, which is equal to 20 by 25, which is equal to 4 by 5.

So, we divide by 4 by 5, which is going to give you 4 by 5 here, and this is going to give you 3 by 5. So, we get, w1 prime is minus 3 by 5, 4 by 5 and let us see minus 12 plus 12, yes, so this is certainly orthonormal and it is an orthonormal basis.

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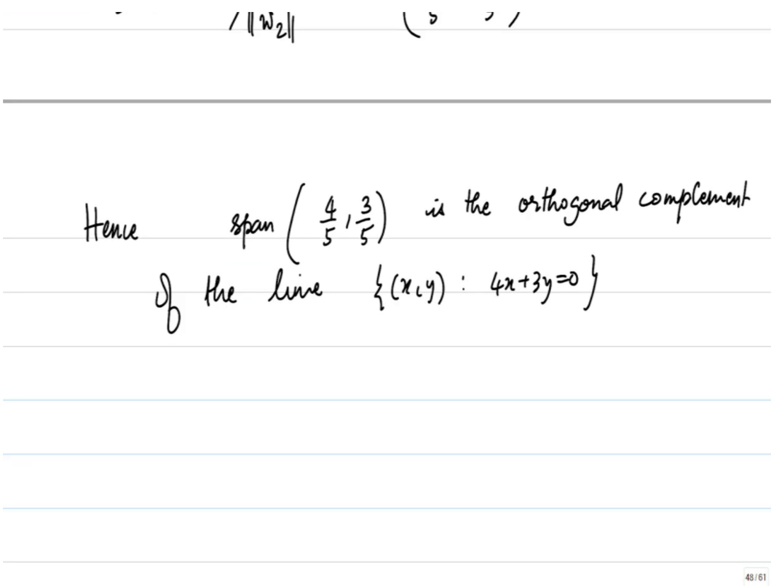

So hence, the span of this vector 4 by 5 comma, 3 by 5 is the orthogonal complement of the line W, or x comma y says that 4x plus 3y is equal to 0, I hope I have written the same line above. Yes. So, we have explicitly computed what the orthogonal complement is. So, this is quite straightforward. One could have really done it by a straightforward simple calculation as well, once we have understood what the orthogonal complement is. But anyway, now we have a technique to go about getting hold of the orthogonal complement, why not.

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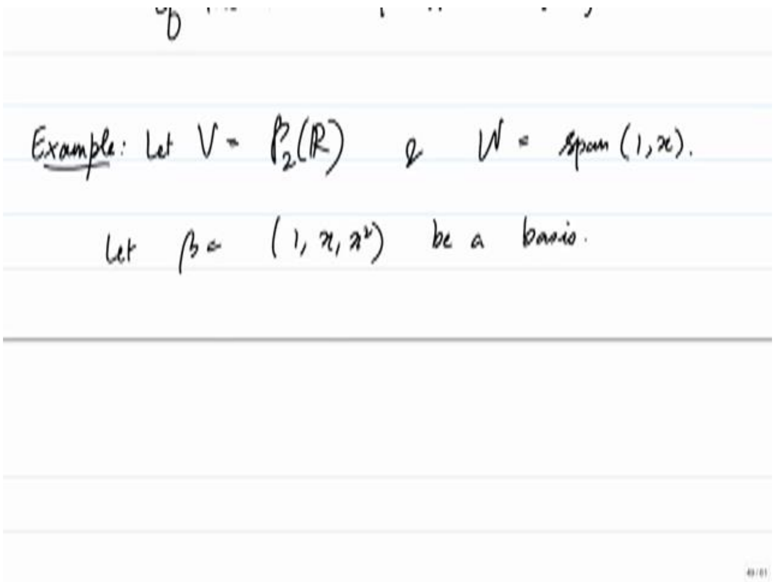

So, let us look at one more example to illustrate the same. So, let us consider that V be equal to P2 of R and let us consider the subspace W, which is equal to the span of 1 and x. So, we would like to look at what the orthogonal complement of W is going to be. So, let us complete this into a basis. So, let beta be equal to 1, x, x square, be a basis. So, we will orthonormalize it and the first two vectors that we get will turn out to be a basis of W and the third vector will be a basis of the orthogonal complement of W. So, what is the inner product that we are working with?

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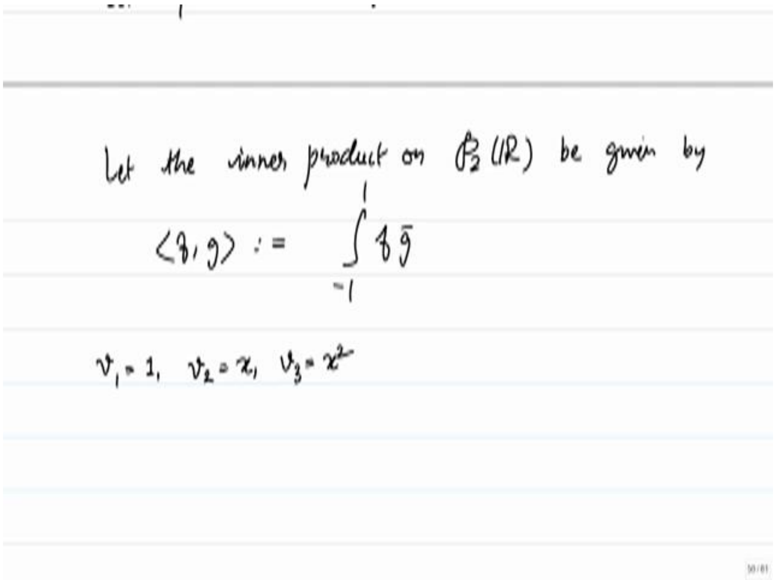

So, consider, let the inner product on P2 of R be given by inner product of f comma g, 2 polynomials of degree less than or equal to 2 is defined to be minus 1 to 1 f times g bar. So, let us see what happens to our orthonormalization here. So, recall that v1 is equal to 1, v2 is equal to x and v3 is equal to x square. Let us see what happens to Gram Schmidt orthonormalization here.

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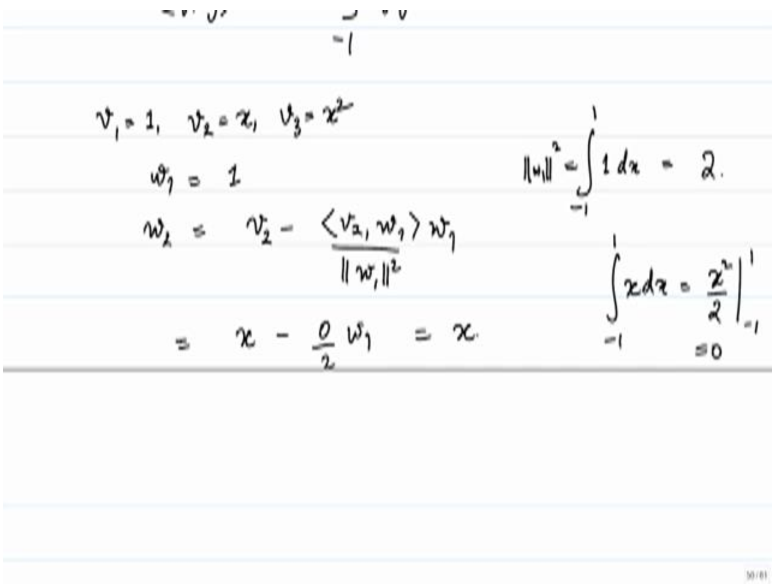

So, the first one, v1 is just equal to 1, how about v2, v2 is v2 minus inner product of, sorry, this is w1, w2 is v2 minus the inner product of v2 with w1 by the length of w1 square times w1. What is w1 here? So w1 is 1. So, what is the length of w1? So, the inner product of 1 times 1 bar is again 1 from minus 1 to 1 dx. Which is equal to 2, if you carefully see what the answer is, this is 2.

So, this is what our square of the length of 1 is going to be, or rather w1 square, let me put it that. So, this is just going to be equal to inner product of v2 is just x minus what is the inner product of x dx from minus 1 to 1. This is the inner product of x and 1, which is just going to be equal to x square by 2 evaluated from minus 1 to 1, which is equal to 0. So, this is 0 by 2 times w1 but that is okay, that is just equal to x. So, w2 is just x.

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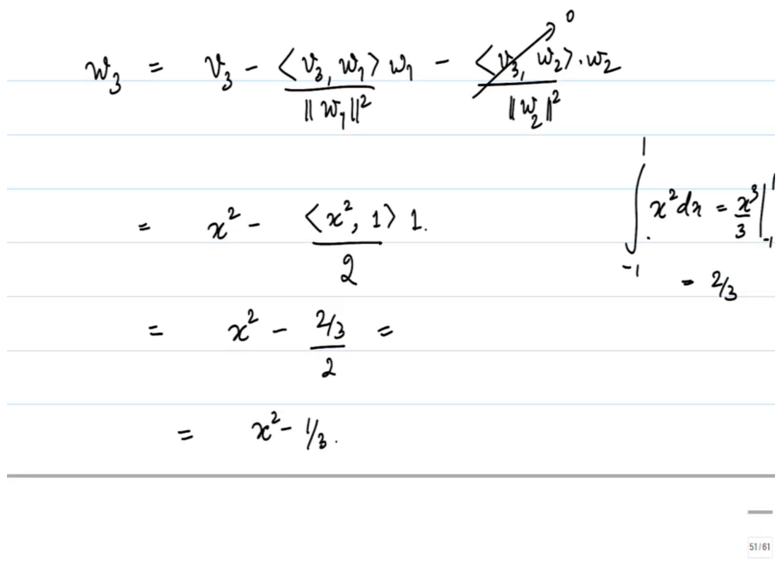

How about w3? W3 will just be v3 minus the inner product of v3 comma w1 times the length of w1 square times w1 minus v3 comma w2 by w1 square, w2 square rather times w2. So I would like to quickly note that this quantity will be 0, because if you look at x square and inner product of that with x, that will be the integral of x cube and by a similar argument, this will be 0. So, we just have to worry about this, it is just going to take a minute, so this is just going to be x square minus the inner product of x square with 1 by norm of w1 square was 2, if you recall n times 1.

So, what is the inner product of x square with 1, this is just going to be x squared dx from minus 1 to 1, which is x cubed by 3 evaluated from minus 1 to 1, which is 1 by 3 minus minus 1 by 3 which is 2 by 3. This is going to be equal to x square minus 2 by 3 by 2, which is equal to x square minus 1 by 3.

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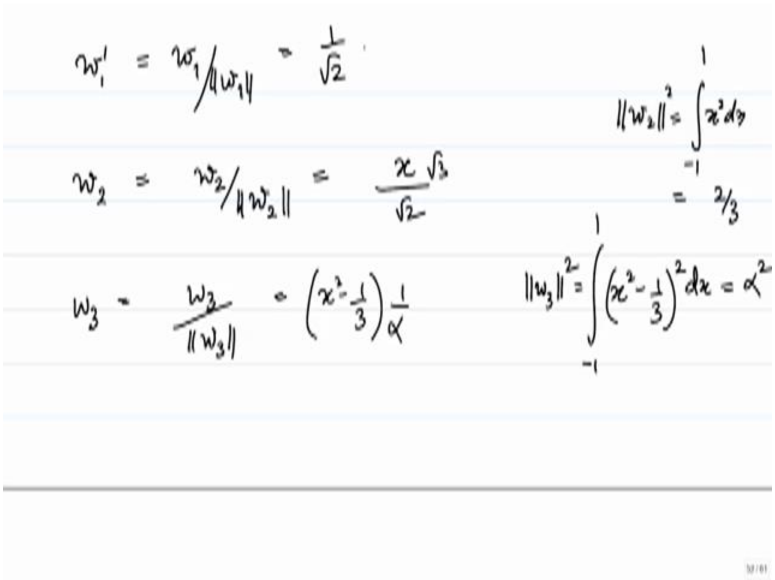

So, yes, so we have w1 prime, which is w1 by the length of w1, this is just equal to 1 by square root of 2. W2, which is equal to w2 by the length of w2 is going to be equal to x by what is going to be the norm of w2. The square of this is just integral of x square dx from minus 1 to 1 which is going to be 2 by 3 which we have already seen, because x cube by 3 on my and this is going to be root 3 by root 2 and how about w3, this is just going to be w3 by norm of w3, which will be x square minus 1 by 3 times some constant, let me just call it which I do not want to my.

So, w3 square, I will just write down what the formula is and leave it at that without calculating it. This is just going to be x square minus 1 by 3 the whole square dx. Let me call this something like alpha or rather root of this is equal to yeah, let me call it alpha square, where alpha is a positive number. So, this is going to be 1 by alpha times that. So, what is our w3, w3 will be a basis for the orthogonal complement of W.

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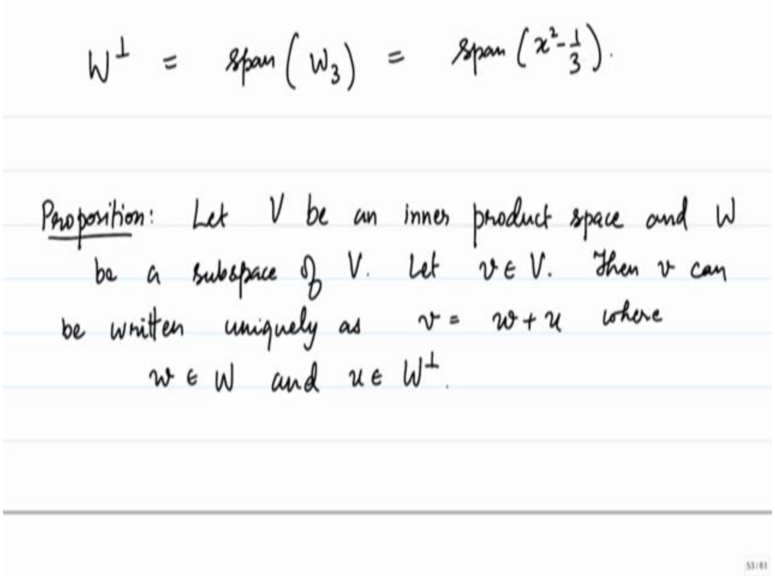

So, the orthogonal complement is the span of w3 or w3 prime, whichever you want. They are after all going to generate the same basis, sorry, same subspace this is just going to be span of w3 easier. So, this is going to be span of x square minus 1 by 3. Let us next take a first step towards defining what is the projection. So, in order to do that as a corollary to this statement, which we have just proved, let me write down a proposition.

The proposition says that if let V be an inner product space and W be a subspace of V. Suppose, v is some vector, let v be an element of capital V, then we can write v in a unique manner as a sum of a vector in W and a vector in W orthogonal or the orthogonal complement of W, then v can be written uniquely as v is equal to w plus u, where w belongs to capital W, and u is in the orthogonal complement of W.

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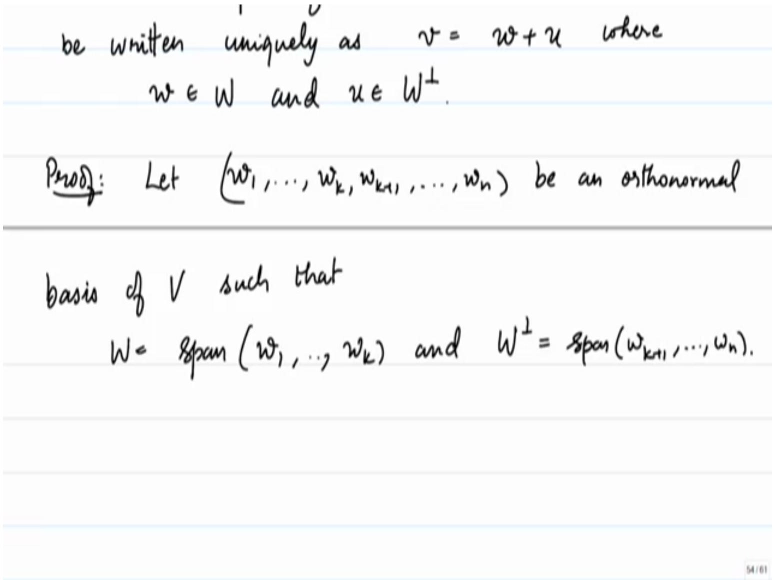

So, let us give a proof of this statement. So, given v, given w and the orthogonal complement of W we have now shown that their x is orthonormal basis. So, the previous theorem said that their x is a basis w1, w2 up to wn of V such that it is an orthonormal basis, the first k of them forms the basis of W and the k plus 1 to nth vectors and the ordered basis forms a basis of orthogonal complement of W.

So, let me just write that down. Let w1 to wk, wk plus 1 to wn be an orthonormal basis of capital V such that W is the span of w1 to wk and the orthogonal complement of W is the span of wk plus 1 to wn.

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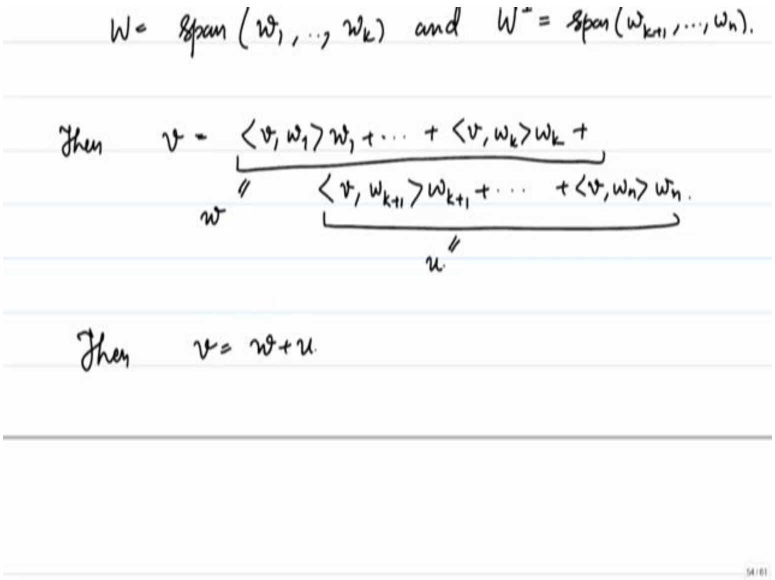

And the good thing about orthonormal vectors is that, then v is inner product of v times, with w1 times w1 plus up to inner product of v with wk times wk plus inner product of v with wk plus 1 times wk plus 1 square up to inner product of v with wn times wn. So, let us do one thing. Let us call the first thing here as being equal to small w and the thing here as being equal to small u. Then v is equal to w plus u.

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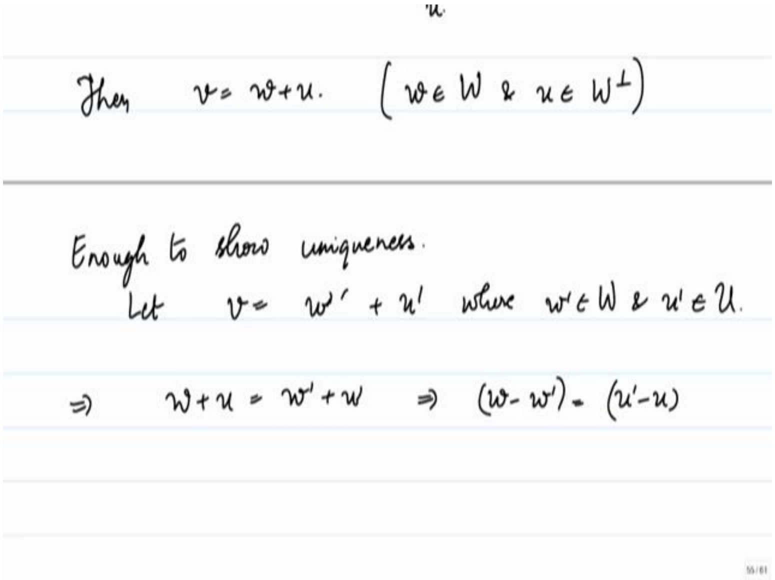

Then w is clearly w is in capital W and the u is in the orthogonal complement of capital W, because w is in the span of w1 to wk, which is a basis of capital W and u is in the span of wk plus 1 to wn which is a span of the orthogonal complement of W. So, hence, given any vector we can write it as a sum of two vectors w and u, where w is in capital W and u is in the orthogonal complement of W. So, we just have to now show that this expression is unique.

So, enough to show uniqueness. Suppose we have 2 such expressions. So, let v be equal to w prime plus u prime, where w prime is a vector in capital W and u prime is a vector in capital U. That means w plus u is equal to w prime plus u prime, or this implies that w minus w prime is equal to u prime minus u. But what is written on the left-hand side here that is a vector in W and what is written on the right-hand side is in the orthogonal complement of W.

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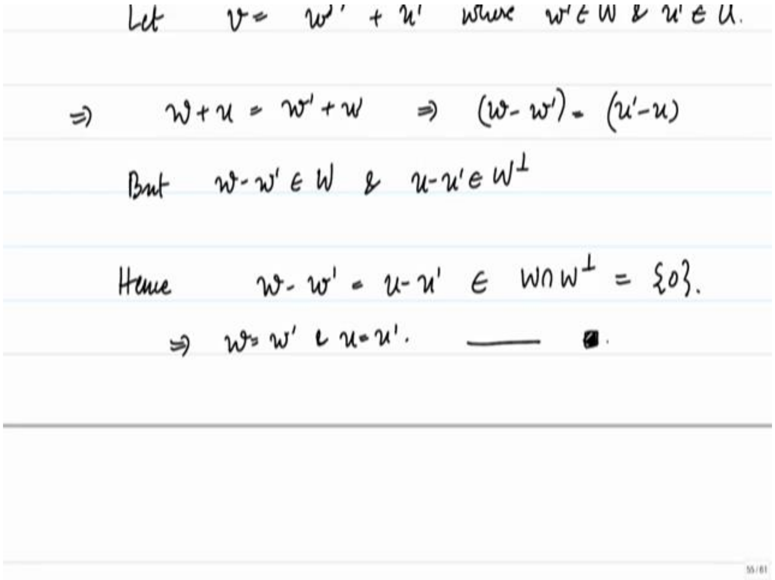

w minus w prime belongs to capital W and u minus u prime belongs to the orthogonal complement of W and we know that 2 orthogonal subspaces, they intersect only in the 0 vector. Hence w minus w prime which is equal to u minus u prime belongs to W intersected with the orthogonal complement of W, which is equal to the 0 vector and hence, w is equal to w prime, u is equal to the u prime and hence we have proved uniqueness as well. So, we call the vector small w. So, let me keep that in picture.

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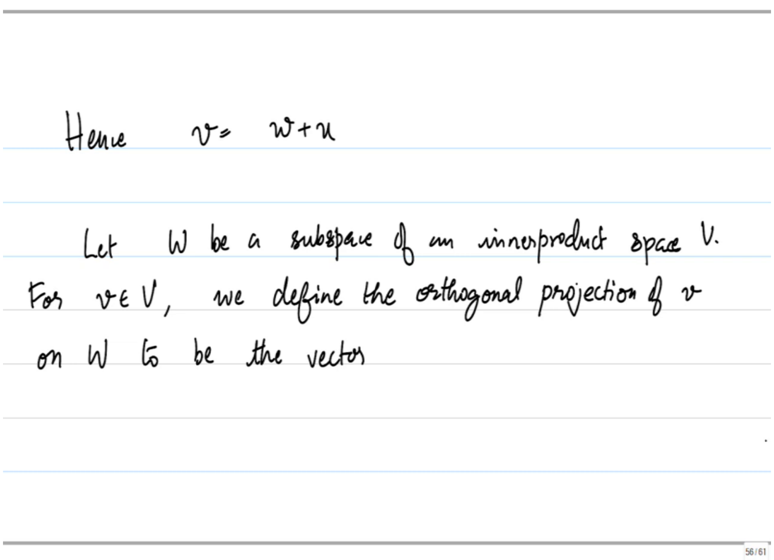

This vector small w, hence v is equal to w plus u, the w is in capital W. We say that W is the orthogonal projection or rather projection of v onto capital W. So, definition we, so, let W be subspace of an inner product space V. For a vector v in capital V, we define the orthogonal projection of small v on to W to be the vector small w above, to be the vector.

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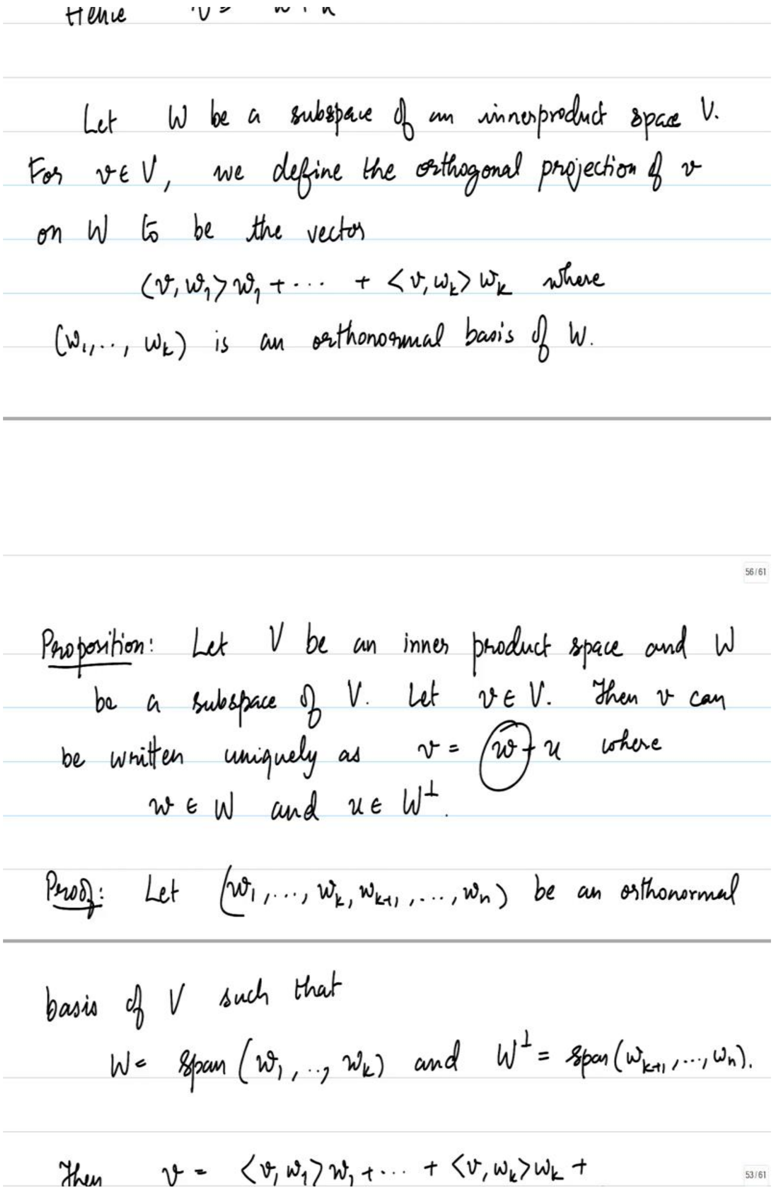

And what was small w? It was v, w1 times w1 plus v, wk times wk, where w1 to wk is an orthonormal basis of capital W and this definition is well-defined. because we just showed that the manner in which we write v as u plus, w plus u is unique, irrespective of what basis you take, the w that we get here, that is going to be the same.

So, this is a well-defined definition. So, we define the orthogonal projection of V onto a subspace W to be the vector inner product of v with w1 times w1 plus up to inner product of v with wn times wn, inner product of v with wk times wk, where v1 to vk is an orthonormal basis of capital V. So, let us look at maybe an example in this example rather. So, the simpler one, let us take the simpler one.

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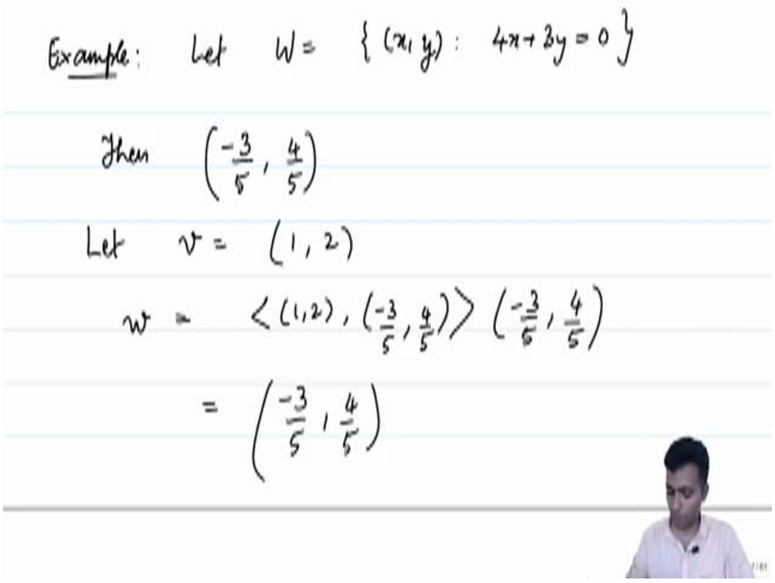

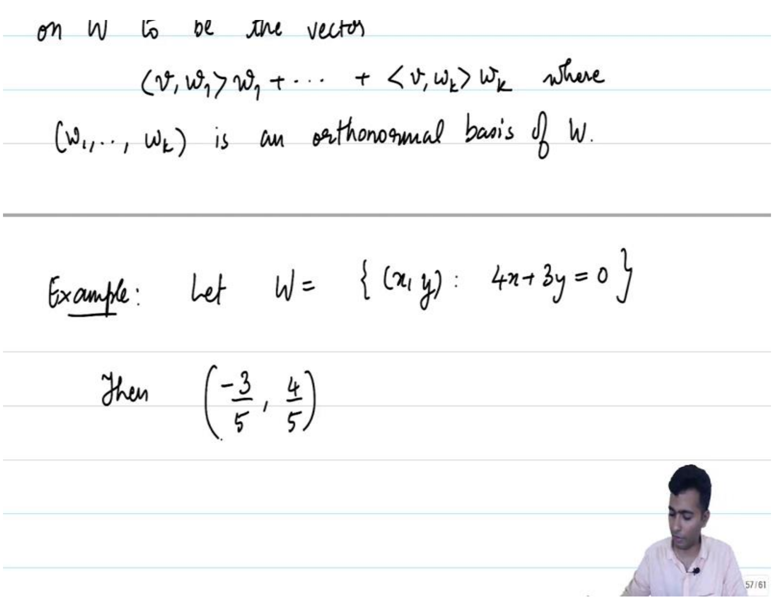

Let us look at the orthogonal projection of some vector on to say W. So, example. Let W be equal to the same example as above, x comma y is such that 4x plus 3y is equal to 0 and what is basis of W, then a basis was given by minus of 3 by 4, sorry, 3 by 5, and 4 by 5. I guess this is what we had calculated earlier. Let me just check. Yes, minus of 3 by 5 and 4 by 5, w1 prime, that is an orthonormal basis of our given subspace and therefore, thus, so let v be some vector. Let us pick some arbitrary vector.

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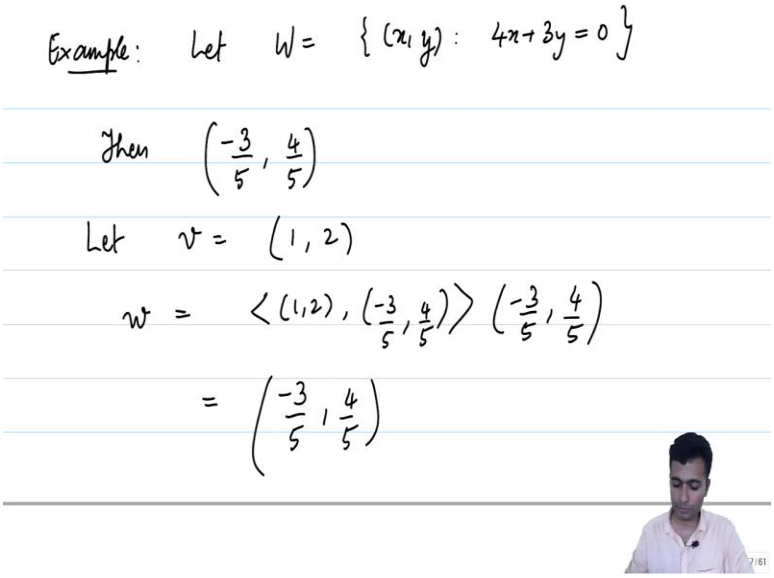

So, let v be equal to say 5 comma 6. Let us, take a small number calculation will become difficult otherwise, let us take it so to be 1 comma 2. Then the projection of v will just turn out to be equal to small w, which is equal to inner product of 1 comma 2, with minus of 3 by 5, 4 by 5 times this vector and what is this, this is minus of 3 plus 8, which is 5 and this is just going to be minus of 3 by 5, 4 by 5.

Incidentally, this basis vector itself turns out to be the projection on to our w. So, this projection is quite special in some sense. In some sense, it captures that vector in the subspace W, which is closest to this given vector. Let me write it down in a proposition to make it precise.

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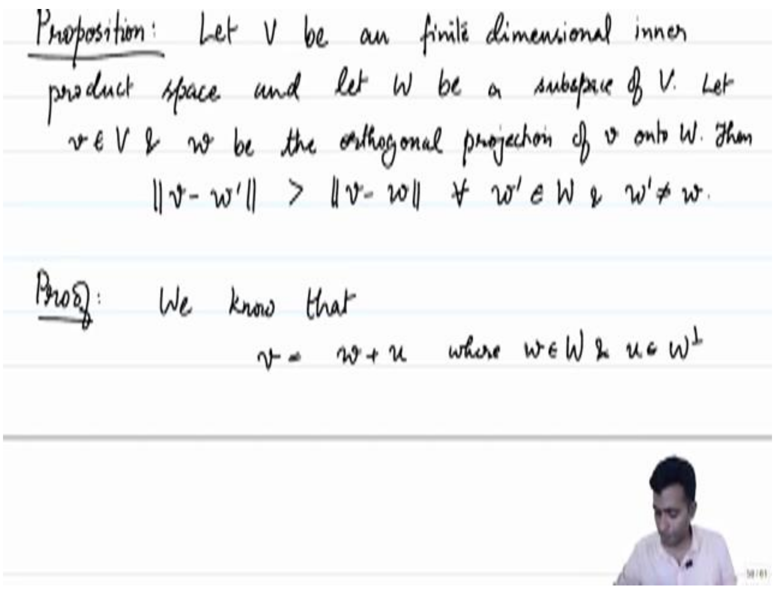

So, let V be an inner product space, a finite dimensional inner product space, this is important and let w be a subspace of capital V and let v be a small, let small v be vector in capital V, then and also, so let small w be the orthogonal projection of v on to W and small w be the orthogonal projection of v on to capital W, we just defined what that is.

The proposition tells us that this is the vector in capital W, which is closest. Then the distance of v minus w prime is greater than the distance of v minus w for all w prime in capital W and W prime not equal to small w. So, for any other vector in capital W, the distance is strictly greater than the distance of V to W. So, let us give a proof of this. So, this is the closest approximation in that subspace that we have of this particular vector v.

So, let us look at, so what do we have, what do we know about the vector v? We know that v can be written as w plus u, where w is in capital W and u is in the orthogonal complement of capital W. We know that v is equal to w plus u, where w is equal, is in a vector, is a vector in capital W and u is a vector in the orthogonal complement of capital W.

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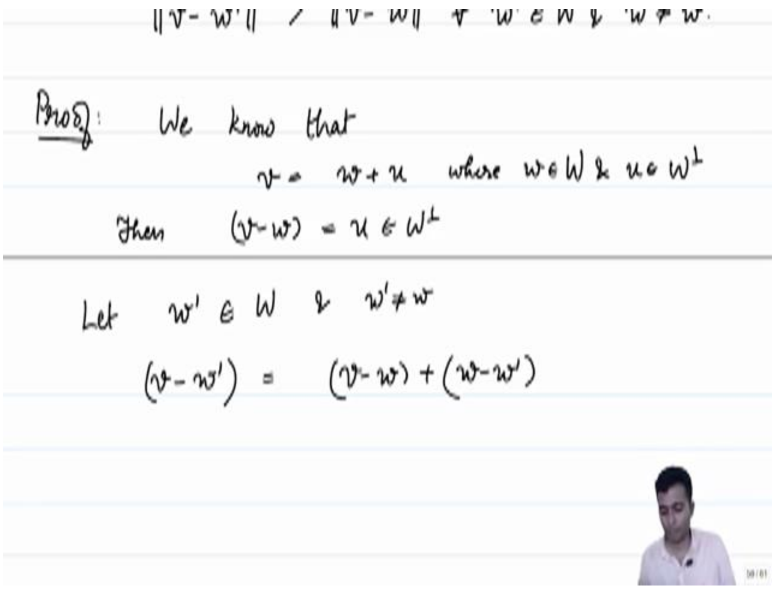

Then, v minus w, which is equal to u is in orthogonal complement of W. So let w prime be in capital W, and such that w prime is not equal to capital W. So, let us take some other vector in w, in capital W, we will call it W prime and let us look at what is v minus w prime. This is just v minus w plus w minus w prime.

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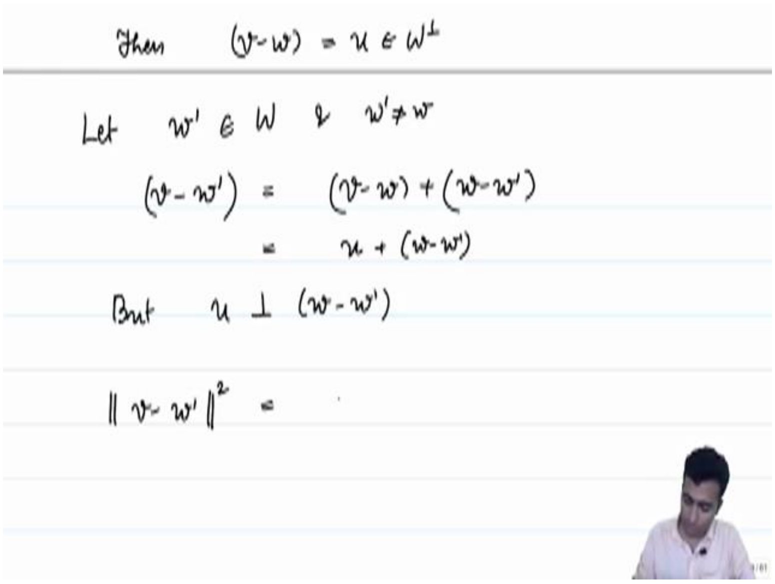

But notice that v minus w, we have just written this as u, let me show it to you. So, if you notice here, v minus w is equal to u, which is in the orthogonal complement of W, and therefore, this is just u plus some vector w minus w prime which belongs to capital W.

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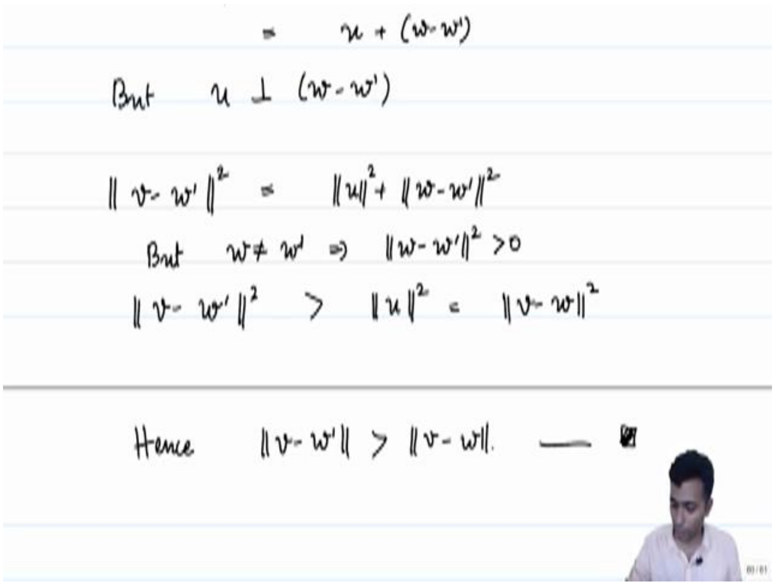

But u is orthogonal to w minus w prime because u is in the orthogonal complement of W and w minus w prime is a vector in capital W. So, let us now apply. Let us look at what is v minus w prime, the length of that, square of the length of w prime minus v or the square of the length of v minus w prime. We will use Pythagoras theorem to conclude that this is equal to the length of u square plus the length of w minus w prime square.

But then w is not equal to w prime, but w is not equal to w prime implies that the length of, square of the length of w minus w prime should be a strictly positive number. Therefore, this number to the left has to be certainly greater than norm of, the length of u square. But what is that, that is just equal to v minus w square. Hence, now, the square root of, square of positive numbers so, therefore, v minus w prime is greater than the length of v minus w, that is precisely what we have set out to prove. So, let us look at the example from before and try to ask the following question.

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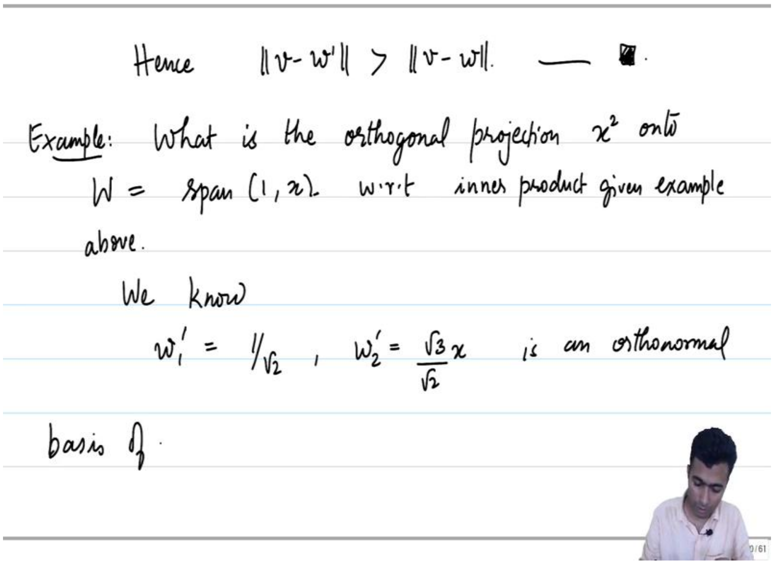

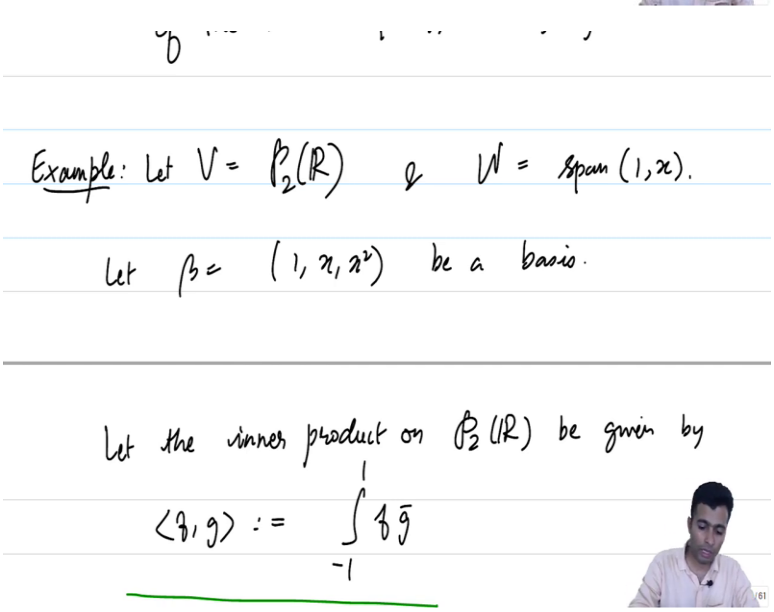

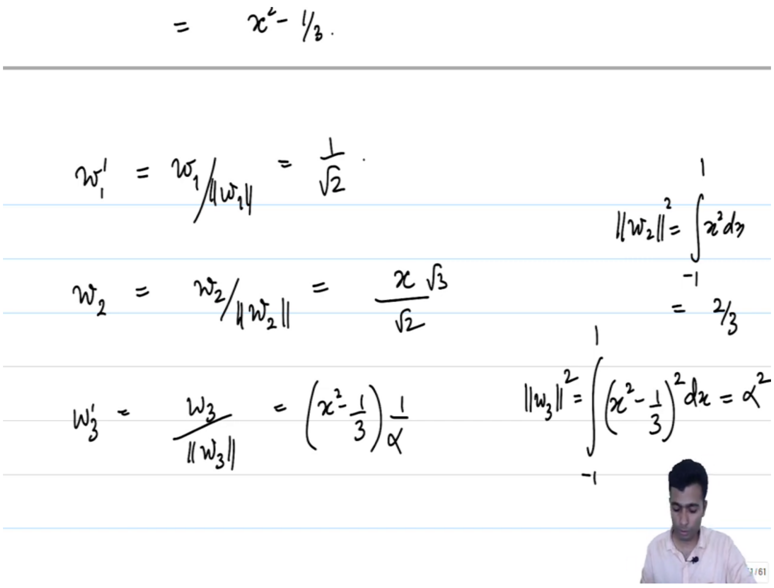

So, example we had considered, so what could be the best possible linear approximation. So, what is the best linear approximation, so nearest, what is the, let me just use the terms which we have just defined. What is the orthogonal projection, orthogonal projection of a degree 2 polynomial, let us pick some easy degree 2 polynomial x square onto a linear approximation, so polynomials of degree less than or equal to 1.

On to W which is the span of 1 comma x, but we have done all the, so with respect to the inner product given in example 2, example above, I do not remember the numbers, so let me just write example above and I will take you back to the example to show you exactly what we had done.

So yes, this is the example. If you notice, we did the same. Span of 1 comma x, I am underlining in green the inner product and we now know exactly what to do, we would like to find the approximation, the projection, orthogonal projection of x square onto the subspace generated by w1 and w2, w1 prime and w2 prime. So, let us see what w1 prime and w2 prime are. w1 prime is just 1 by root 2, and w2 prime is just x times root 3 by root 2. So, this, so w1 prime is 1 by root 2 and w2 prime is root 3 by root 2 times x. So, this is the, we know that this is an orthonormal basis of W.

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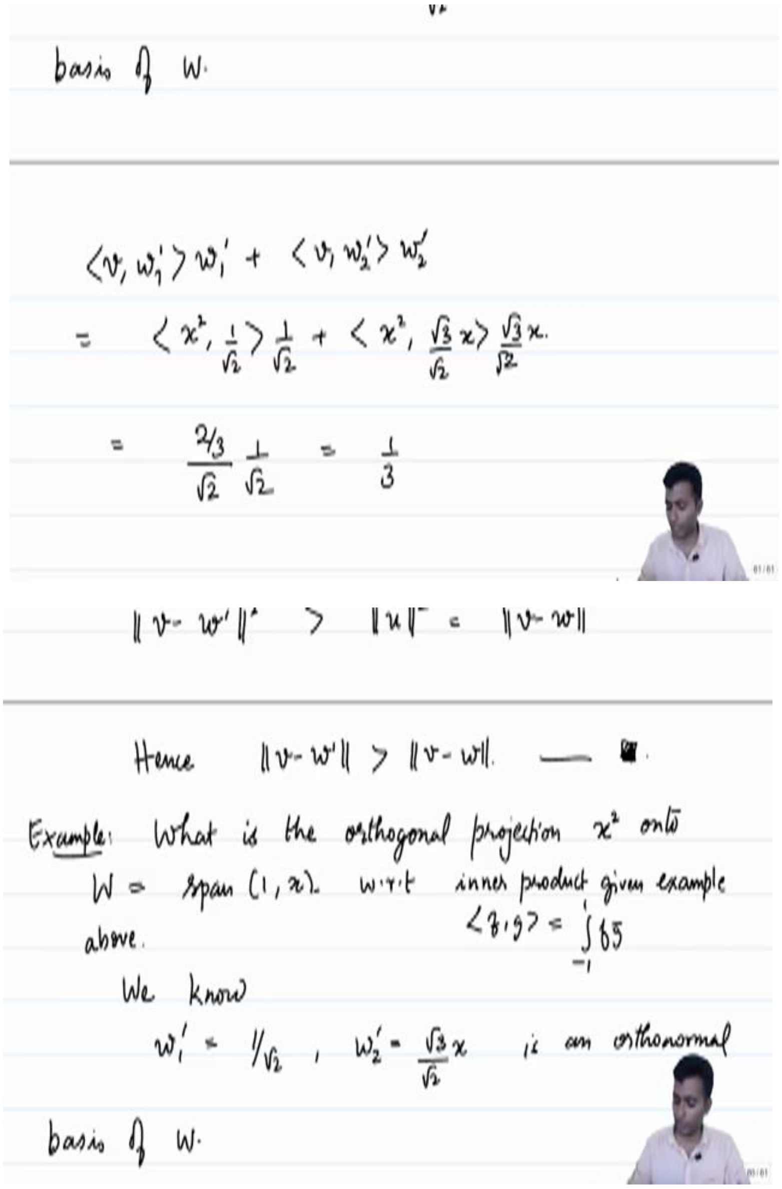

And we can check that, then the orthogonal projection will just turn out to be v inner product with w1 prime times w1 prime plus v inner product with w2 prime times w2 prime, which is just going to be the inner product of x square with 1 by root 2 times 1 by root 2 plus inner product of x square times root 3 by root 2 times x root 3 by root 2 times x. So, I have already done the calculations here, this is going to be 2 by 3 by root 2 times 1 by root 2 and the next term vanishes and this is just going to be hence equal to 1 by 3.

So, it turns out that the best possible approximation, let me not use that word again, the orthogonal projection of x square onto the subspace generated by 1 comma x, so the best linear polynomial, the closest linear polynomial is nothing but the constant polynomial given by 1 by 3. Of course, this very much depends on the inner product that we had picked.

So, I have not written it here, recall that the inner product was inner product of f comma g is integral minus 1 to 1 f g bar, f of x g bar of x dx. If we had changed the inner product, we would have got a different answer, because the length changes. Of course, the vector which is closest should depend on the length, so it changes with respect to what the inner product is. Nevertheless, we have a very explicit way of getting hold of that particular vector in a given subspace which is closest to our given vector. Let me stop here.