Linear Algebra
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Lecture 40
Gram-Schmidt Orthonormalization.

So, we have seen what it means for a collection of vectors to be orthonormal and we have also seen the power of having an orthonormal basis in our, given in our product space. In particular, we can write any vector as linear, we can always do this, we can always write every vector as a unique linear combination of the basis vectors. However, when we have an orthonormal basis, we know explicitly what that linear combination is, by looking at inner product of the vector with the basis vectors.

So, it is certainly desirable to have a basis which is orthonormal as well. So, let us now discuss technique to obtain an orthonormal basis, from a basis which is not orthonormal to begin with. So, the process is called Gram Schmidt Orthonormalization.

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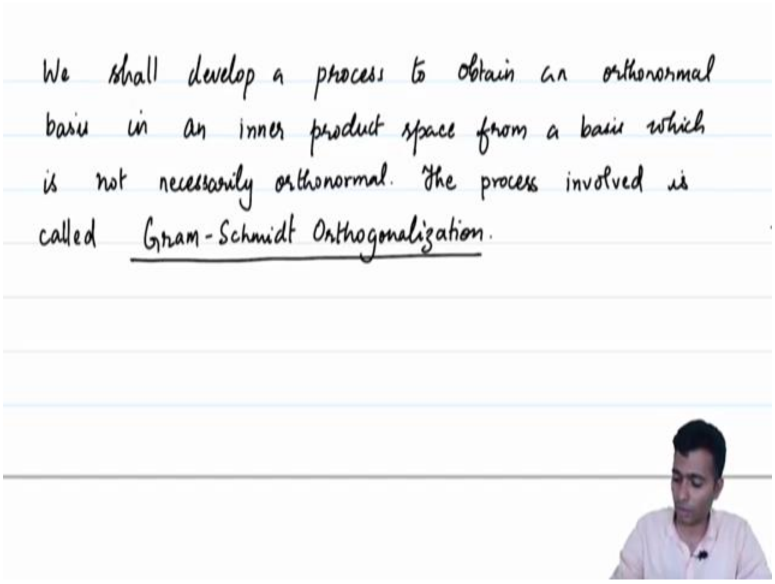

So, we shall discuss, shall develop a process to obtain an orthonormal basis in an inner product space, from a basis which is not necessarily orthonormal, from a basis which is not necessarily orthonormal. So, it is called Gram Schmidt, the process is called Gram Schmidt orthogonalization. The process involved, so it is not this process, this is only giving, going to give you a, an orthogonal set, process involved is called Gram Schmidt orthogonalization. So, let us begin by defining what it means for a vector to be a unit vector. So, we say that vector V is a unit vector, if it has length 1.

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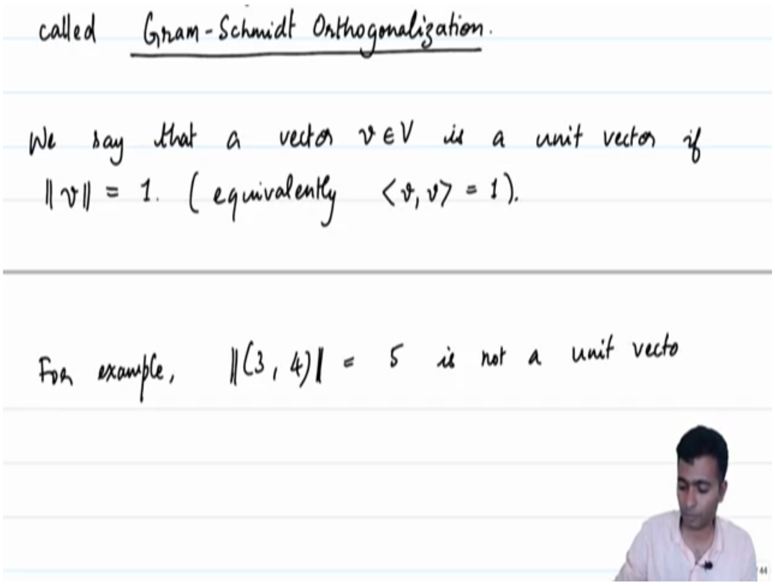

So, we say that vector, v in capital V is unit vector if the length of v is equal to 1, or equivalently the inner product of v with itself is equal to 1. So, for example, in the standard inner product, 3, 4 will have length equal to square root of 3 square plus 4 square, which is 5, this is not a unit vector.

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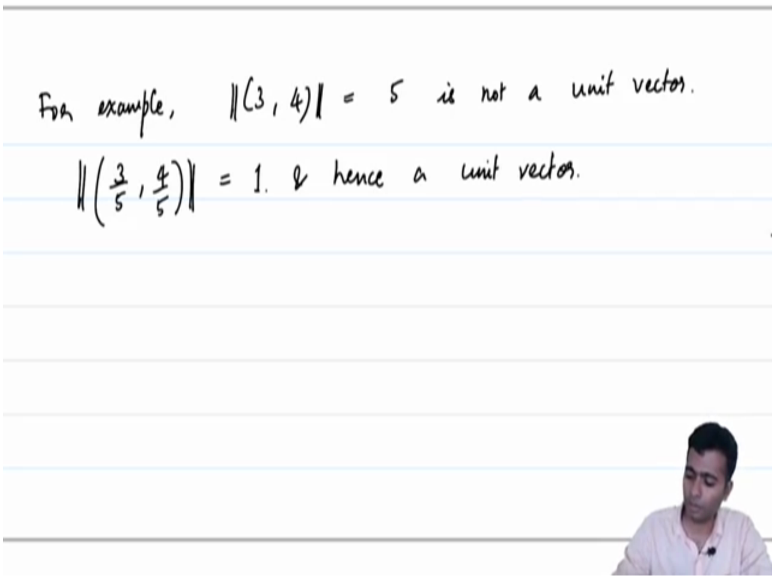

However, if you look at 3 by 5 and 4 by 5, you can check that this is equal to 1, this is a and hence a unit vector. All vectors in the orthonormal basis of Rn is a unit vector. So yes, so unit vectors just means the length of the given vector is 1. So, if we start off with a vector, so most vectors we come across in general will not be unit vectors. However, if you start off with a nonzero vector, we can normalize the given vector to obtain a unit vector.

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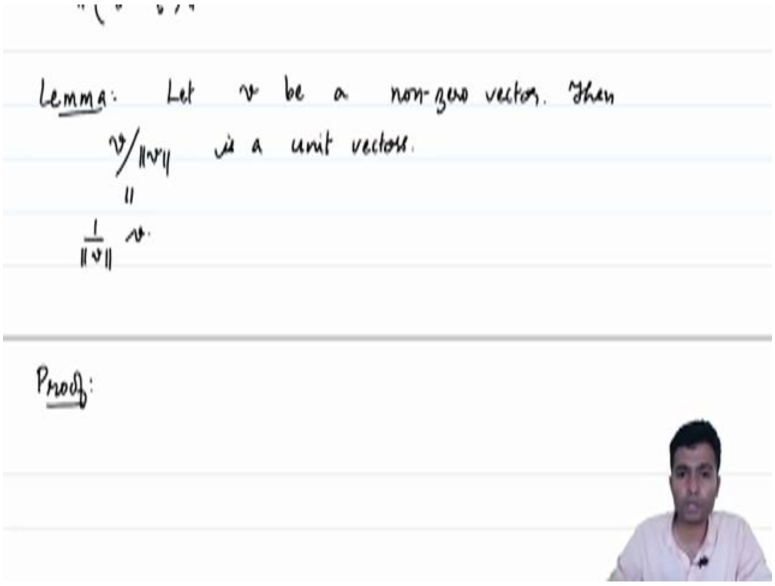

So, which I will just write down in the form of a lemma. So, let v be nonzero vector, so this is the P be the, v be a nonzero vector, then V by length of V is a unit vector. So, so why did we jump into discussing unit vector? So, let me give you the background on why we started doing this. So, the process of getting an orthonormal basis is in two steps. You start off with some arbitrary basis and by the Gram Schmidt orthogonalization, we obtain a collection of vectors which is orthogonal.

However, that is not enough to have a basis which is orthonormal, for a, for a basis to be an orthonormal basis, the vectors should also have unit length. The normalization is an easier process. So, once we have an orthogonal set of vectors, it is easy to obtain an orthonormal basis from that. So, that is precisely why we have started discussing unit vectors to begin with, we will first address the easier part of the problem, which is namely obtaining an orthonormal basis from an orthogonal basis.

So, to do that, let us develop or let us develop some theory to justify, why it can be done easily. So, let us start off with nonzero vector here and look at the vector given by v by length of v. So, what does it mean to say that, what does it mean to say v by length of v? So, this is just, the scalar multiple of, this is just the scalar multiple of 1 by the length of v to the vector v. So, you look at 1 by length of v. Notice that 1 by length of v is just a real number, real number is in particular a complex number as well.

So, you start off with any inner product space, whether it is a real inner product space or a complex inner product space. So, if you start with real inner product space, 1 by length of v is a real number in particular, it is a scalar. If you start off with a complex inner product space, the length is again a real number, which is also a complex number and hence, you can think of it as a complex number, which is a scalar.

So, the scalar multiplication of 1 by length of v with the vector v makes complete sense and the lemma states that this particular vector needs to have unit length. So, what do we do to prove that, the only thing to check is that the length of this vector v by length of v, that should also have length 1.

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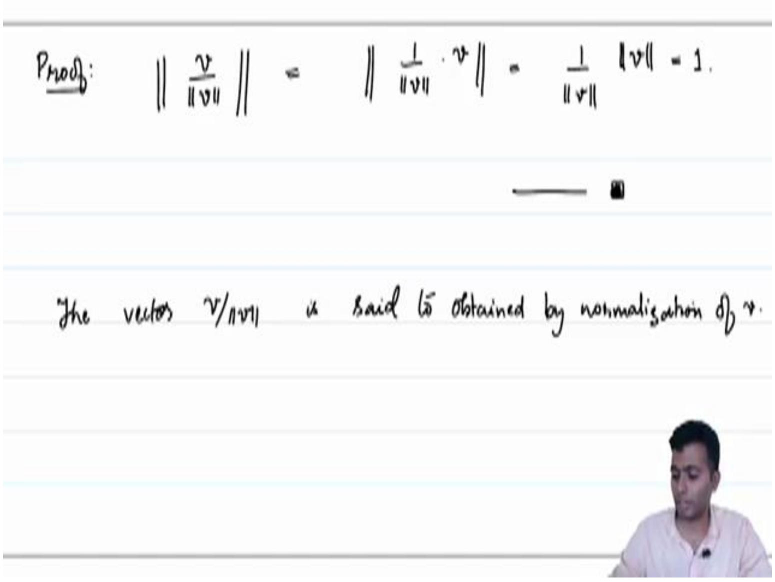

So, let us see what the length of that vector is. So, what is v by the length of v, this is what we would like to see. But what is this, as rightly noted, this is just the scalar product of 1 by length of v and the vector v and what do we know about the length of c times v, where c is a scalar. This is just going to be absolute value of 1 by the length of v, which is equal to 1 by length of v. So, I am just skipping one step and writing directly as this, which is equal to 1. So hence, there is nothing to prove any more or rather we have proved the statement.

I did not write anything down, but clearly all these equalities are fairly obvious. I just said what the reason for each of these equalities are. So, the vector v by length of v, which is now a unit vector, is said to be obtained by normalization of v. It is also called a normalized vector, normalization of V. So, now let us start with a basis.

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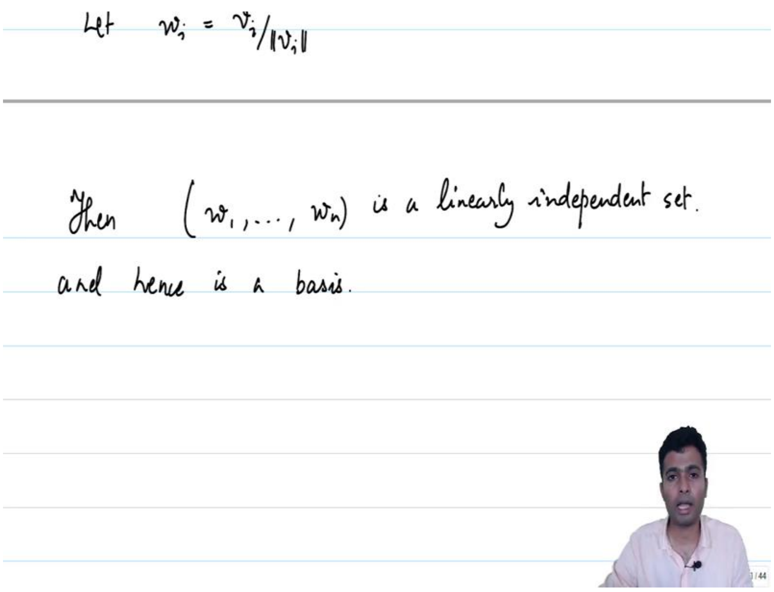

So, suppose, so suppose, now let us look at a basis of our given inner product space V and let us normalize each of the vectors in our basis and let us see what happens. So, suppose v1 to vn is basis. So, in particular of an inner product space V. So, in particular, none of these vectors v1, v2, up to vn can be 0, because it is a linearly independent set. So, let wi be obtained by normalizing vi. Then the vectors w1 to wn is linearly independent set.

Why is that the case? Because if it is a linearly independent set, if it is not a linearly independent set, we will be able to obtain a linear combination of w1, w2 up to wn, which is equal to 0 with not all coefficients equal to 0, that will give us a linear combination v1, v2 up to vn which is equal to the 0 vector and with not all coefficients being equal to 0 and hence, that will contradict the linear independence of the basis vectors v1 to vn.

So, this set w1, w2 up to wn is a linearly independent set and in a dimension n vector space, any linearly independent set of size n should be a basis and therefore, this is a basis but what is special about this basis? Every vector in this basis has unit length, it is normal, it is a unit vector.

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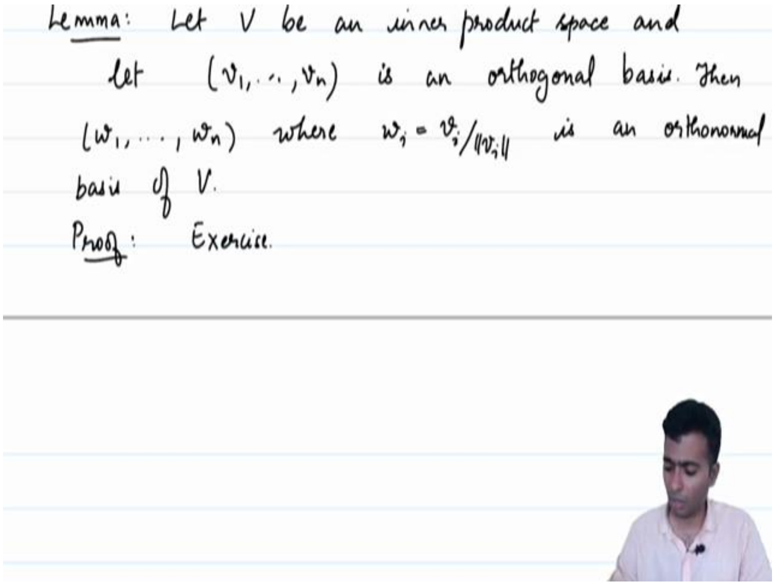

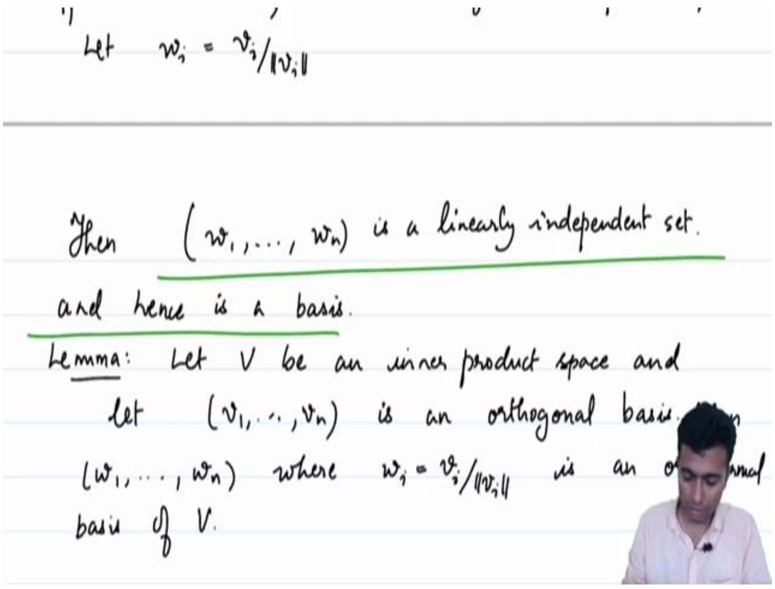

So, in particular, we have effectively proved this lemma. So, if v1 to vn is an orthogonal basis, what does that mean? That means that it is a collection of vectors which is orthogonal and a basis at the same time. Then w1 to wn obtained by normalizing, obtained by normalizing or let me just write it in this manner, where wi is equal to vi by the length of vi, it is obtained by normalizing vi.

This is an orthonormal basis, so orthonormal. So, I should have maybe put a context before this. So, let V be an inner product base and let v1 to vn be an orthogonal basis, then w1 to wn, where wi is vi by the length of vi is an orthonormal basis of V. So, the lemma has just been proven by this observation.

This is the, this observation which I have just underlined in green. So, yes, this will certainly be a basis and I leave it as an exercise for you to check that, so the proof is finally, just a small exercise to check that if v1, v2 up to vn is a collection which is orthogonal w1 to wn which is obtained by, which is obtained by normalizing vi will also be a collection which is orthogonal. So, I will leave that as an exercise for you.

You just have to use the linearity property of our inner product. So, by this whatever we have discussed, we can finally say the following. If we develop a method by which any basis, from any basis we can obtain an orthogonal basis from it, we just normalize the vectors, we get hold of an orthonormal basis. So, our goal has now been reduced to the following.

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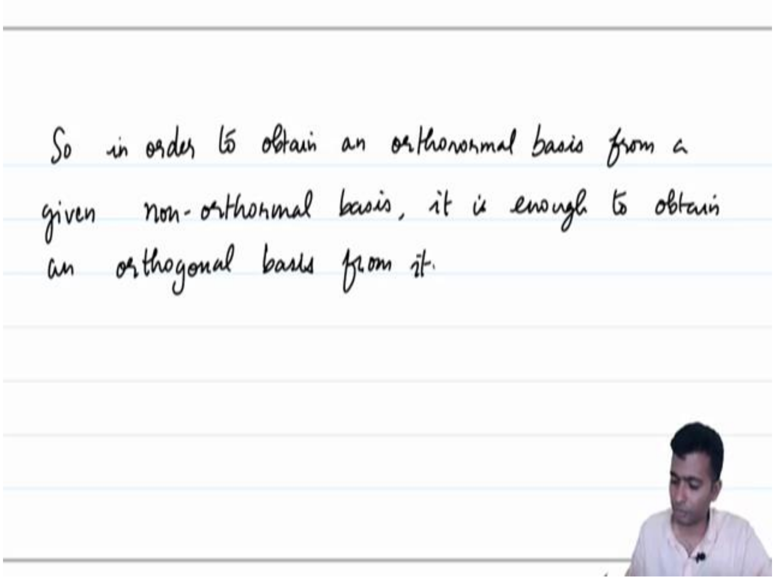

So, hence, so let me write it down. So, in order to obtain an orthonormal basis, an orthonormal basis from a given non-orthonormal basis, it is enough to obtain an orthogonal basis from it, an orthogonal basis from it and which we can orthonormalize, which we can normalize to obtain an orthonormal basis. So, let us now spend some time trying to develop a method. So, again so before we do that, let us just look at one or two examples.

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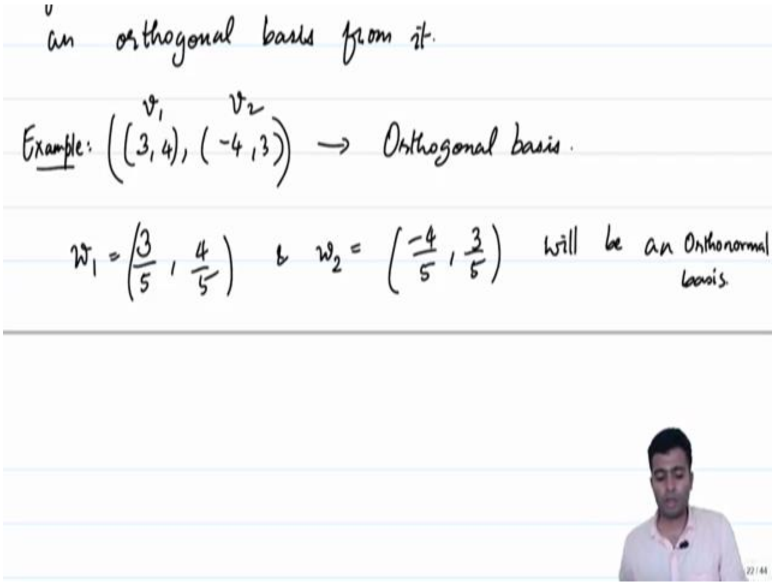

If you look at say 3, 4 and maybe minus 4, 3, this is an orthogonal, so example, this is an orthogonal basis, this is an orthogonal basis and we may convert this to unit vectors and we will get this 3 by 5, 4 by 5. So, if this is v1 and v2, w1 equal to this and w2 equal to minus of 4 by 5 and 3 by 5 will give you an orthonormal basis, an orthonormal basis. So, now let us try to develop some methods by which given a collection of vectors, you can convert them into vectors which are orthogonal to each other.

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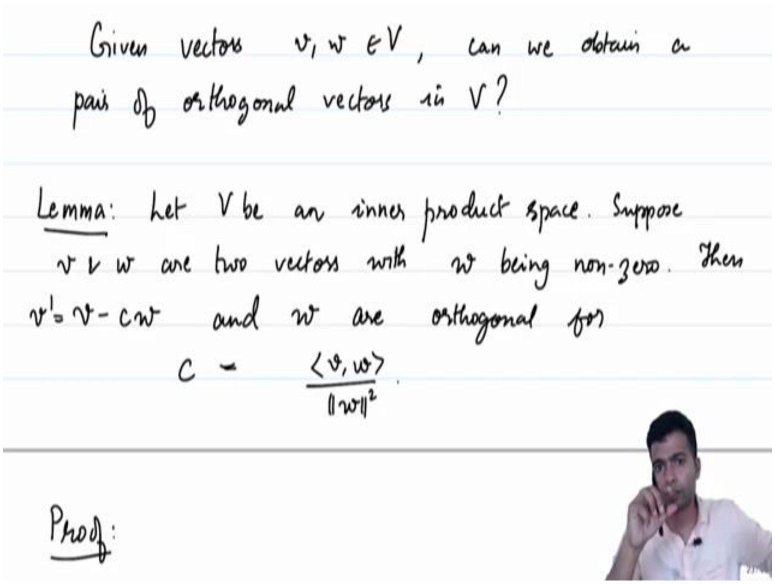

So, let us now develop, let us obtain. So, given vectors a v and w, we would like to ask the following question. In V again, it is an inner product space, so I will stop writing that v is an inner product space. All these results can be said over real inner product spaces or complex inner product spaces.

So, v is an inner product space, all through in this lecture, v is always going to be an inner product. So, given two vectors, can we obtain a pair of orthogonal vectors from this, a pair of orthogonal vectors in V? This is the question we would like to answer.

So, let V be an inner product space, I had said that I will not write it down, but that is okay, inner product space and suppose v and w are two vectors, suppose v and w are two vectors with w being nonzero, then v minus c times w and we are, so w prime is equal to this and w are orthogonal to each other, are orthogonal for c being equal to v, w inner a product by length of w square.

So, we can tweak our w, we can tweak our v by subtracting something from that. That will turn out to be a vector which is orthogonal to v. I made a mistake, it will be w and so let us call this v prime and w, they will be orthogonal to each other.

Let us give a proof of this. So, let us just run through the statement once more. What it tells us is that, if you start off with two vectors v and w, one vector can be tweaked to obtain another vector, which will be orthogonal to the other one. It might not be orthogonal to begin with.

However, you can tweak it a bit to obtain a new vector which is orthogonal to the other one and we will see that the process in which it is being done is quite good. We will come to that in a minute, but let us just prove this lemma first. So, to check everything, we just have to, fact that w is a nonzero vector allows us to conclude that the length of w is nonzero, therefore, we can divide by the square of the length of w.

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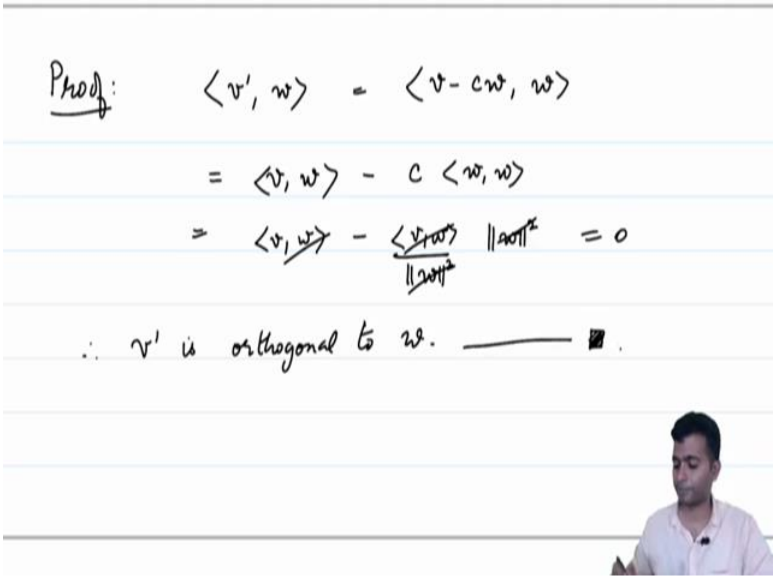

So, what is the inner product of v prime and w? This is nothing but inner product of v minus c w with w, which is equal by the linearity, this is equal to inner product of v and w minus c times the inner product of w and w. What is c? If you recall this is nothing but, the inner product of v with w by the length of w square. So, this is inner product of v with w by the length of w square times the inner product of w with itself, which is the length of w square. So, this cancels, this also cancels, which is equal to 0. Therefore, v prime is orthogonal to w, that is what we had set out to prove.

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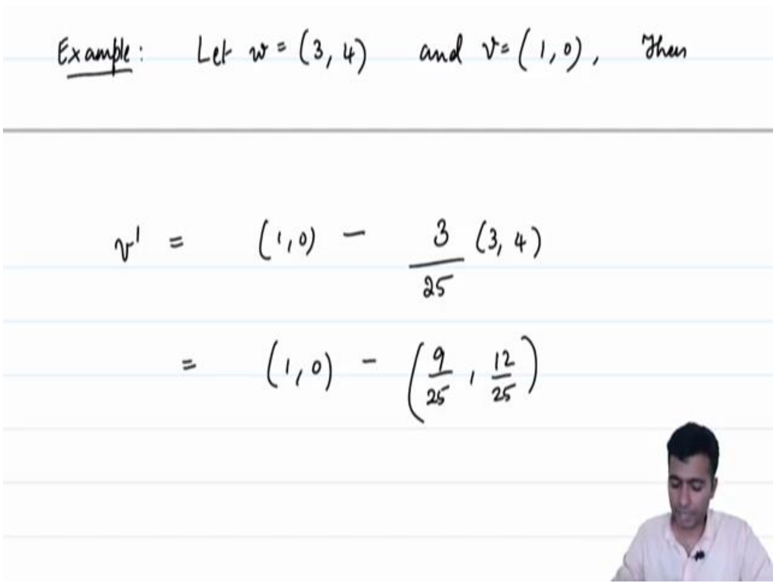

So, let us just see with one example what we have just done with v and w, they are the symbols. Let us take one concrete example and have a look at it. So, let us say, let us again pick 3 and 4, it is easy to compute the length here, the 3 and 4 and maybe 1, 0, be v and w.

Then or maybe this can be w, it is easy to compute the length here. So, let us compute v prime, v prime will just be equal to 1, 0 minus the inner product of v with w, which will just be 3 by the square of the length of w, actually it did not matter, times our 3, 4. This is 1, 0 minus 9 by 25, let us see 12 by 25. Yeah, we are doing it right.

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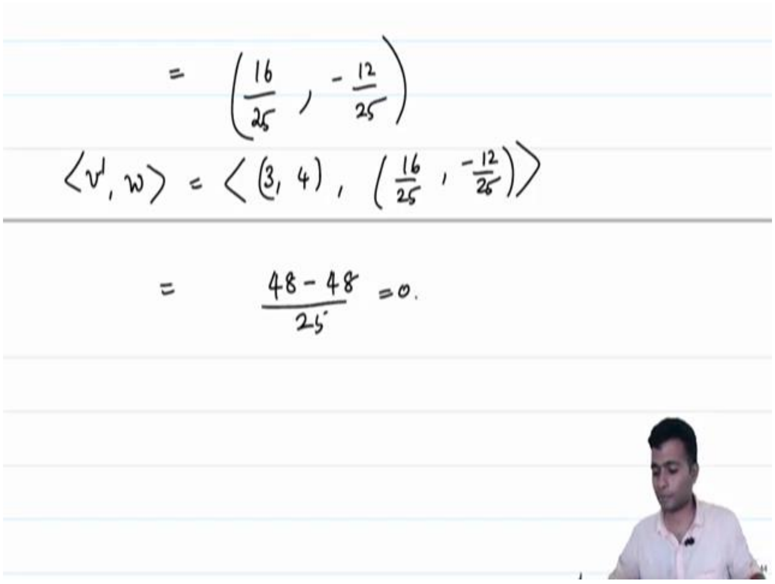

Let us see if my intuition has anything to do with that, it might be wrong, let us see. 25, 1 minus 9 by 25 will just be 16 by 25 and the other one is just minus of 12 by 20. So, let us see if v prime is orthogonal to w, then the inner product of v prime and w will just be inner product of 3, 4 and 16 by 25 minus of 12 by 25, which will just be equal to 48 minus 48 by 25, which is equal to 0. So, yes.

So, even though I was a bit skeptical about what we are doing, v prime which we have obtained by this process does turn out to be orthogonal to our w, which is 3, 4. So, we have seen what to do to, obtain orthogonal vectors given two vectors. We would like to now ask what we can say about something similar when finitely many of them are given. So, now let us recall our goal, our goal was to obtain an orthogonal basis from a given basis.

So, in order to do that, let us assume that after taking the first k vectors, we have already converted it into k orthogonal vectors and now we would like to take the k plus 1th vector and somehow convert it into a vector which is orthogonal to the test, which the ones which we have already obtained.

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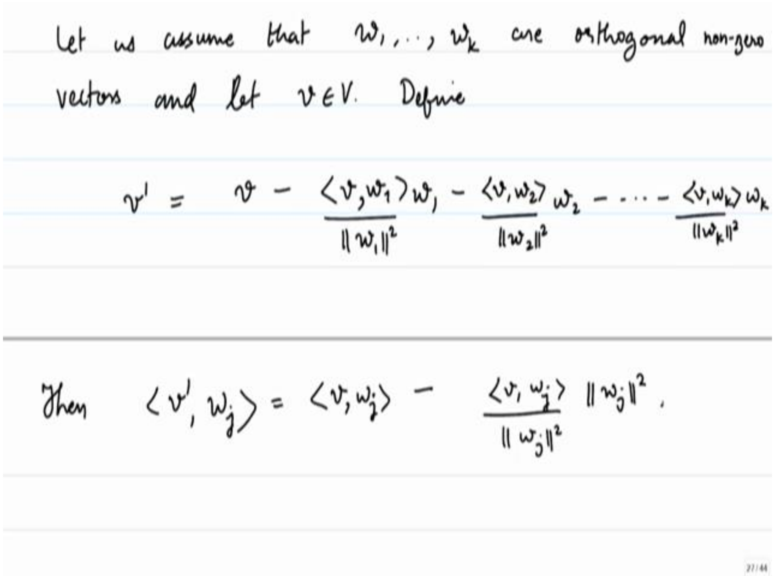

So, let us assume that w 1 to wk are obtained, are orthogonal vectors and we would like to now and let v be a vector in V, in capital V, the inner product space. Let us now develop or let us now give an identity or a new vector rather which will be orthogonal to each of w1, w 2, up to wk. So, define v prime to be equal to, so orthogonal and nonzero.

So, v prime now be equal to v minus the inner product of v with, this is not v1, this is v comma, v, w1 by the length of w1 square times w 1 minus v inner product with w2 by the length of w2 square times w2 minus up to v with wk by the length of wk square times wk.

Suppose, we define our v prime in this fashion. So, this is again going to tell something similar. Then what is the inner product of v prime with a wj, v prime and wj if you take the inner product, v prime has the expression as given above and each of the wjs are orthogonal to each other w1, w2 upto wj, wk are there, it is a collection of orthogonal vectors.

So, this will just be equal to by the linearity property, v, wj minus everything else cancels out other than v comma wj by the length of wj square times the length of wj square. Because the other terms vanish and because v, because wj and wk will be orthogonal, wj and wi is orthogonal for all i from 1 to n, which is not equal to j.

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$$\|w_j\|^2$$

$$= 0.$$

The vector v' is orthogonal to $w_1, \ldots, w$

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pair of orthogonal vectors in V?

Lemma: Let V be an inner product space. Suppose v & w are two vectors with w being non-zero. Then $v' = v - cw$ and w are orthogonal for

$$c = \frac{\langle v, w\rangle}{\|w\|^2}.$$

Proof:

$$\langle v', w\rangle = \langle v - cw, w\rangle$$

$$= \langle v, w\rangle - c\langle w, w\rangle$$

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$$v' = v - \frac{\langle v, w_1\rangle w_1}{\|w_1\|^2} - \frac{\langle v, w_2\rangle w_2}{\|w_2\|^2} - \cdots - \frac{\langle v, w_k\rangle w_k}{\|w_k\|^2}$$

Then $\langle v', w_j\rangle = \langle v, w_j\rangle - \frac{\langle v, w_j\rangle}{\|w_j\|^2}\|w_j\|^2.$

$$= 0.$$

The vector v' is orthogonal to $w_1, \ldots, w_k$.

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The linearity tells us this and this is also equal to 0. So, effectively we have proved that let me write down the statement. Should I write it down as a lemma, maybe not, the vector v prime is orthogonal to w1 to wk. So, let us just go back and check what we have done. The first thing we did was take two vectors, defined a new vector v prime in a very specific manner and we proved that the new vector v prime is orthogonal to w.

Suppose, we started off with the first few vectors and manage to orthogonalize it, suppose we did get orthogonal vectors w1, w2, up to wk out of it, nonzero vectors out of it and suppose V is a vector which is given to us, we would now like to tweak our v and get hold of a new vector which is orthogonal to all the w1, w2, up to wk and we prescribed the formula, this particular expression which now again is being underlined with green is the prescription for a candidate, which will be orthogonal to the already existing vectors, the already orthogonal vectors w1, w2 up to wk.

So, in particular, if we had started off with a basis, if we had taken the first k vectors and orthogonalized it, we would have w1, w2 up to wk, we take the next basis element do this process, we get a k plus 1 element which will be orthogonal to w1, w2 up to wk. But there are a few things which we have to take care of, in order to say that this process works.

So, let me write down the statement of Gram Schmidt orthonormalization orthogonalization and go through the details to, so there are a few things which I was mentioning, just as I which I was about to mention, we have to note that v prime cannot be 0 for the process to continue, because if it is 0, the next step cannot be done.

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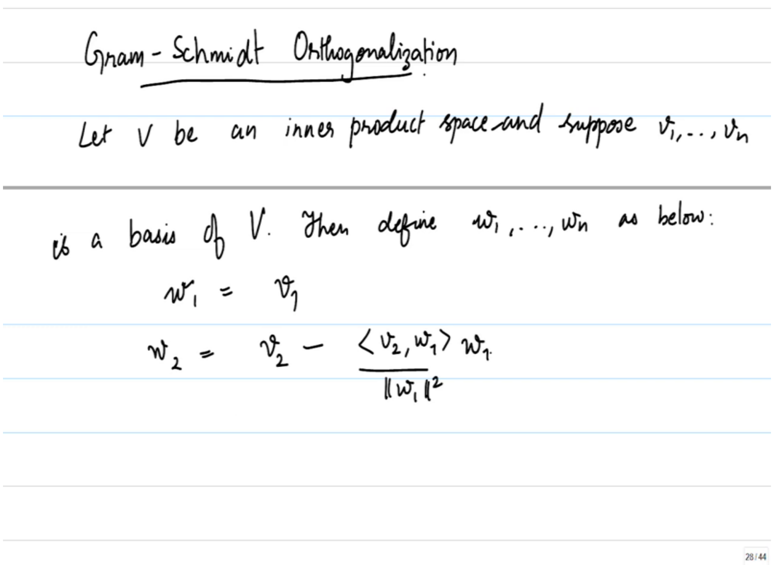

So, let us write down a statement for Gram Schmidt orthogonalization. So, let v be an inner product space, so let me write it because that is the formal statement and let v1, v2 up to vn be basis of V, suppose v1 to vn is a basis of capital V, then define, we already know how to define all these. Define w1 to wn as below, w1 is equal to v1, w2 is equal to v2 minus the inner product of v2 with w1 by norm of w1 square into w1

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$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$\vdots$$

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_n, w_2 \rangle}{\|w_2\|^2} w_2 - \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\|w_{n-1}\|^2} w_{n-1}$$

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And so on, up to inductively wn is equal to vn minus vn, w1 by norm of w1 square times w1 minus vn, w2 by the length of w2 square times w2 minus dot, dot, dot, dot. v n, wn minus 1, I should be a little more judicious with space, vn, wn minus 1 by the length of w n minus 1 square times w n minus 1. Suppose, we define w1 and then w2 and then w3 up to wn. So, the Gram Schmidt orthogonalization, you have done most of the hard work already.

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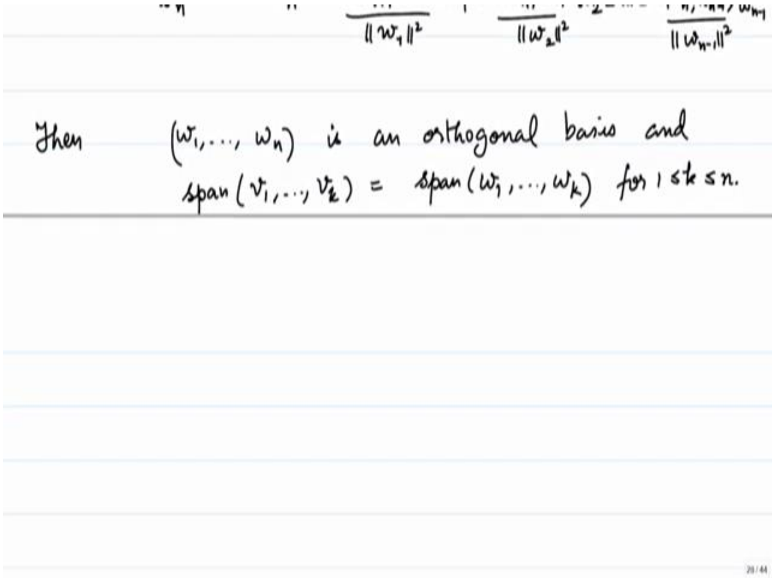

Then w, span of w1 to wn are orthogonal or it is an orthogonal basis, is a orthogonal basis and not just that, at every stage, span of v1 to vk is equal to span of w1 to wn. It is not just an orthogonal basis, at every stage the span of v1 to vk is the same as a span of w1 to wk, for 1 less than or equal to k less than or equal to n. So, this is what we were aiming for, is not it? We were trying to get hold of basis, which was orthogonal. So, let us give a proof of this.

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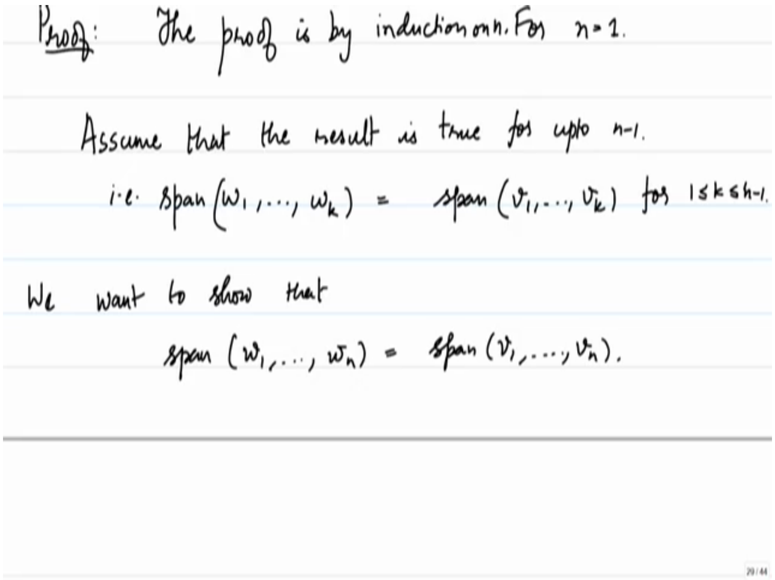

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$\vdots$$

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_n, w_2 \rangle}{\|w_2\|^2} w_2 - \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\|w_{n-1}\|^2} w_{n-1}$$

Then $(w_1, \ldots, w_n)$ is an orthogonal basis and $\text{span}(v_1, \ldots, v_k) = \text{span}(w_1, \ldots, w_k)$ for $1 \le k \le n$.

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We have, as noted, we have almost done all the hard work, we have, so the proof is by induction here and the base case is quite straightforward. For n is equal to 1, w1 is equal to v1 and there is not much to be done as can be noted, the span of w1 and the span of v1 is going to be the same and hence, this is the case. So, now let us assume that the theorem has been proved for up to n minus 1.

So, assume, so the theorem, the induction is on the dimension of n. So, proof is by induction on n, which is the dimension of our vector space. So, assume that the result is true for up to n minus 1. So, that means i.e. w 1 to wn minus 1, span of this, in fact, span of w1 to wk is equal to the span of v1 to vk for k, 1 less than or equal to k less than or equal to n minus. So, if we prove that for k equal to n also this is true, we are done. So, we want to show that span of w1 to wn is equal to the span of v1 to vn.

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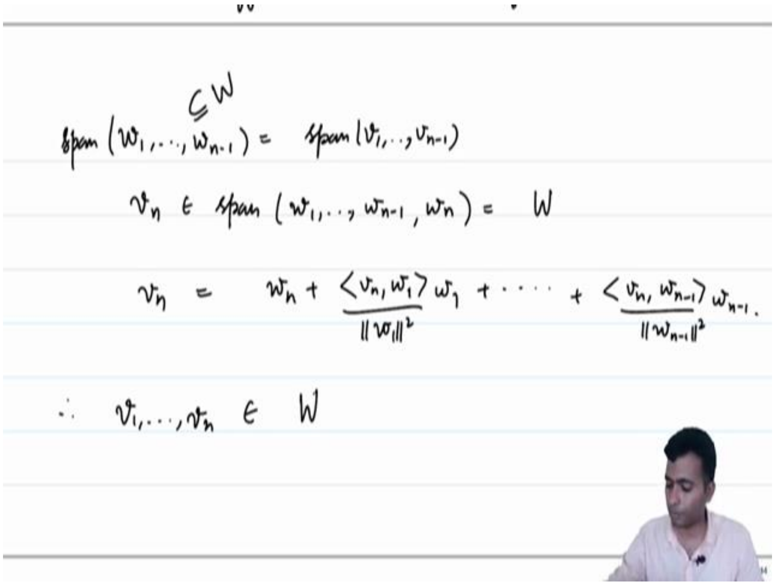

So, let us see what to do. So, let us give some names to these spaces. Let us call this as capital W, and let us call this as capital V. Then, what do we have? We have that w1 to span of w1 to wn minus 1 is the span of v1 to vn minus 1 by our induction hypothesis. This is equal to the span of, v1 to vn minus 1 and we also know that vn belongs to the span of w1 to wn minus 1 and wn. Why is this the case? Because if you recall our definition of wn, wn, let me write in fact by rewriting vn just turns out to be equal to wn plus inner product of vn comma w1 by the length of w1 square times w1 and so on.

This will be vn comma wn minus 1 by length of the wn minus 1 square times wn minus 1. So, yes vn belongs to the span of w1, w2 up to wn minus, up to wn or this is rather w and this is contained in, this is contained in w and therefore, v1 to vn certainly belongs to span of w1 to wn which is equal to W.

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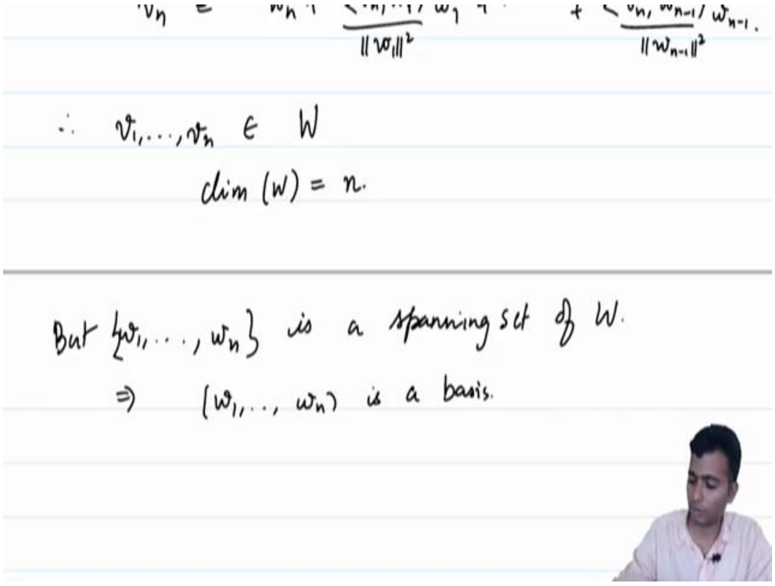

But then the said v1, v2 up to vn is linearly independent and therefore, dimension of W and therefore, what do we have therefore, dimension of W is greater than or equal to n, but of course, it is equal to n because it is a subspace of, v1 to vn is a basis. So, this is going to be a spanning set and therefore this is going to be equal to n. But we also know that w1 up to wn is the spanning set of W, because that is how we have defined capital W to be, it is a spanning set. It is the spanning set of w1, w2 up to wn. We have now a collection of n vectors, which is a spanning set in a dimension n space, therefore w1 to wn is a basis.

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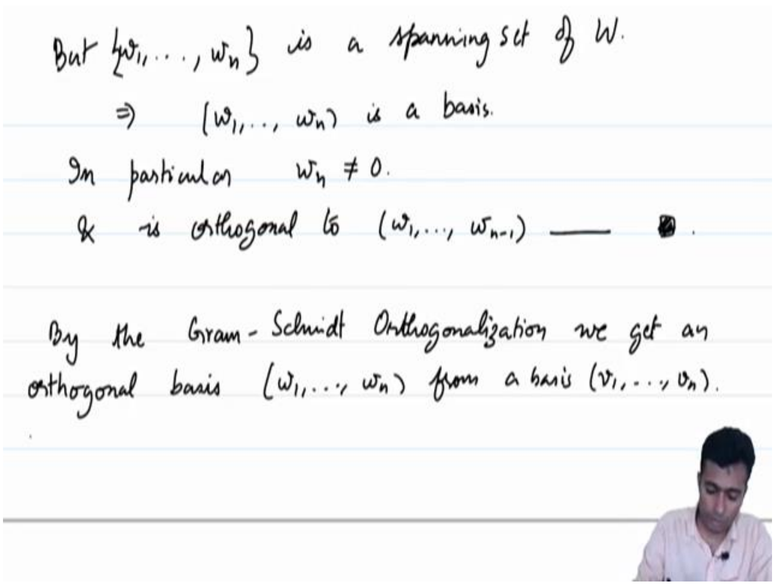

So, in particular, we cannot have wn to be 0, wn is not the 0 vector and moreover by construction and is orthogonal to w1 to wn minus 1, that completes the proof. Now, if we start off with a basis apply the Gram Schmidt orthogonalization, we obtain an orthogonal basis from the basis we started off with and now let us apply the normalization process to obtain a orthonormal basis from it. So, by the Gram Schmidt orthonormalization, orthogonalization we get an orthogonal basis say w1 to wn from a basis v1 to vn.

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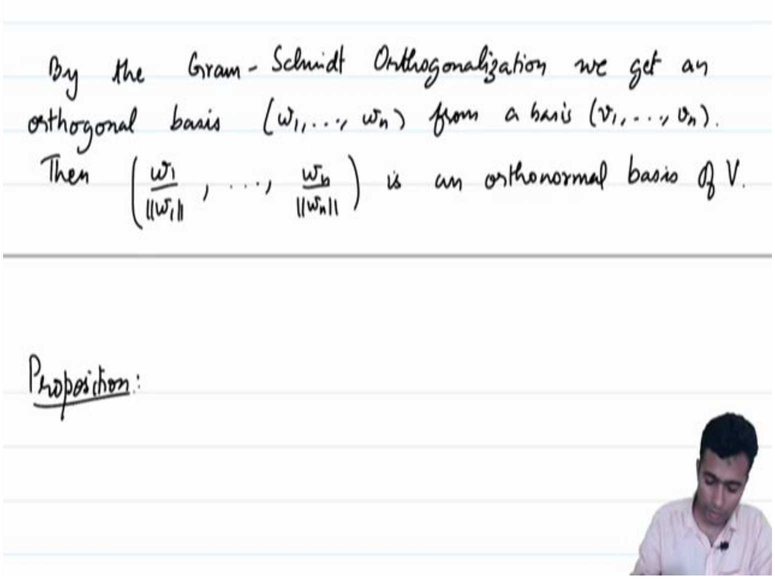

Now apply a normalization, then w1 by the length of w1 and so on up to wn by the length of wn is an orthonormal basis of the inner product space V.

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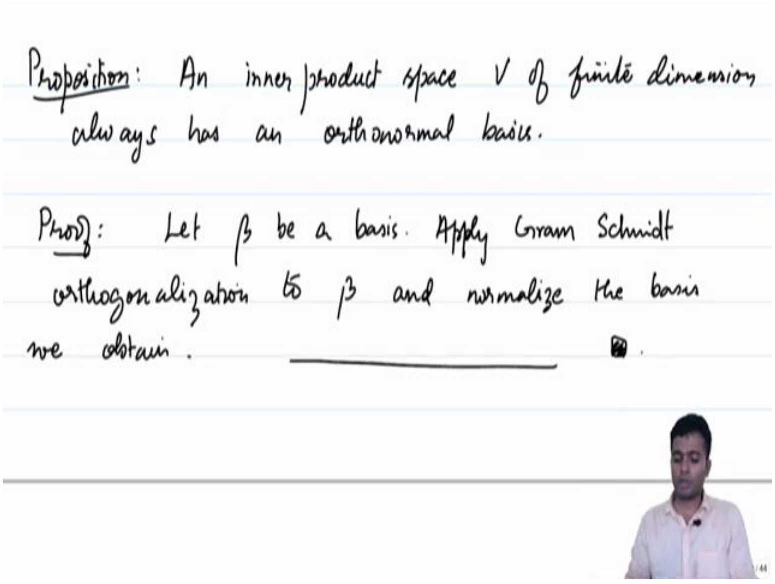

So, proposition which we have just proved is that you start off with an inner product space of dimension n, there always exists an orthonormal basis. So, an inner product space V of finite dimension always has an orthonormal basis. So, I will not write down the proof again, we just did that. We start with a basis or maybe let me just write it down.

So, let beta be a basis, apply orthogonalization, apply Gram Schmidt orthogonalization to beta and normalize the basis, normalize the basis we obtain to finitely obtain an orthonormal basis. So yes, we have given a proof of existence of an orthonormal basis in a finite dimensional inner product space.

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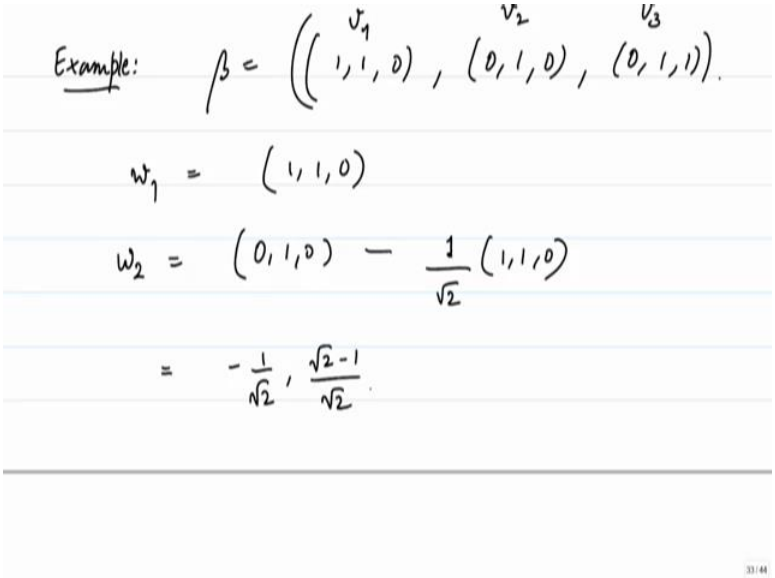

So, let us apply this to a simple example before we conclude, let us take some easy example of say beta being equal to 1, 1 and 1, 0 or maybe let us slightly make it more complicated 1, 1, 1, 1, 1, 0, 0, 1, 0 and 0, 1, 1, let these be our 3 vectors. This is our v1, this is our v2 and this is our v3. We know that this is a basis or we can check that this is a basis. So,, let us now get hold of our orthonormal basis from beta. So, let us apply Gram Schmidt orthonormalization to it. So, what is our w1? w1 will just turn out to be 1, 1, 0.

What is w2? This will be v2 which is 0, 1, 0 minus the inner product of, inner product of v2 with w1. So, this is going to be inner product of 0, 1, 0 with 1, 1, 0 which is 1 by the norm of w1 square which will be square root of 2 times 1, 1, 0. Maybe I should have put v1, v1 as v2 to make our (())(42:21) easy, but that is okay. This is going to be minus of 1 by square root of 2, 1 minus 1 by square root of 2 and 0.

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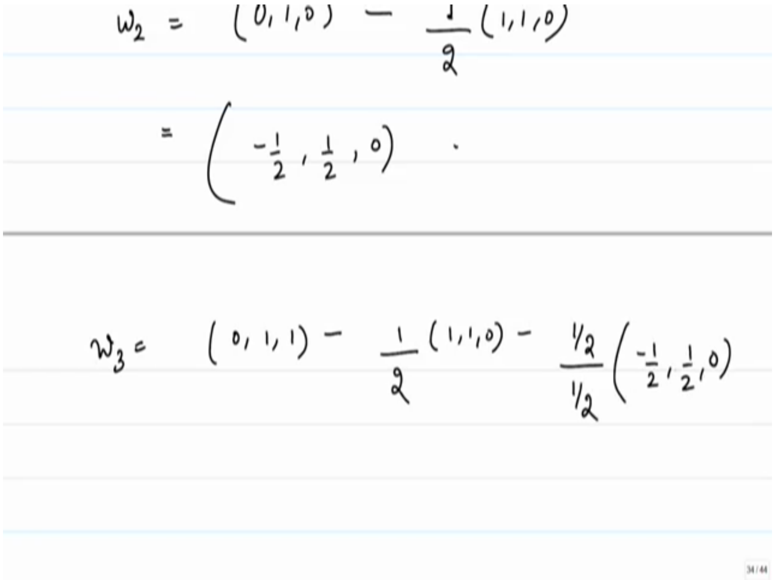

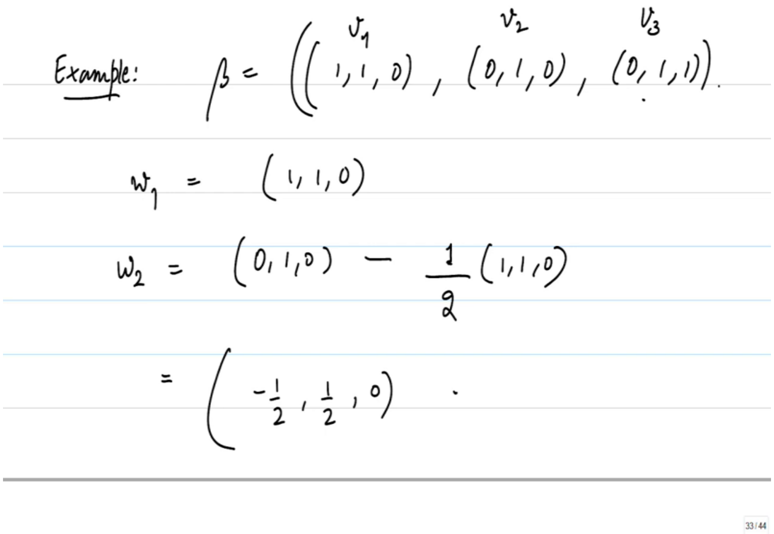

And then w 3 will just be equal to V 3, V 3 is 0, 1, 1. This is going to be 0, 1, 1 minus 0, 1, 1 inner product with, 0, 1, 1 inner product with w1 will just be equal to 1 again and the square root of 2 is the length of w1 times 1, 1, 0 and how about inner product with w2, w2 will be a bit more complicated. This will be inner product of v3, which this there is going to be 0, root 2 minus 1 by root 2. This is going to be root 2 minus 1 by root 2 by the distance is length of our, maybe I am making a mistake.

So, this, yes, I am making a mistake, because this cannot be root 2, this would have been 2. So, it is norm of w1 square. So, I have got our w2 wrong, so I am sorry, corrected. Maybe I should write down but that is okay. This is going to be minus half, half, 0. That is our w2. This is again 2 and this is going to be the inner product of v3 with w 2 which is half. This is just going to be half by norm of w2 square, which will be 1 by 2 square plus 1 by 2 square, which is 1 by 2 times minus 1 by 2, 1 by 2, 0. So, I hope I am making the correct calculations.

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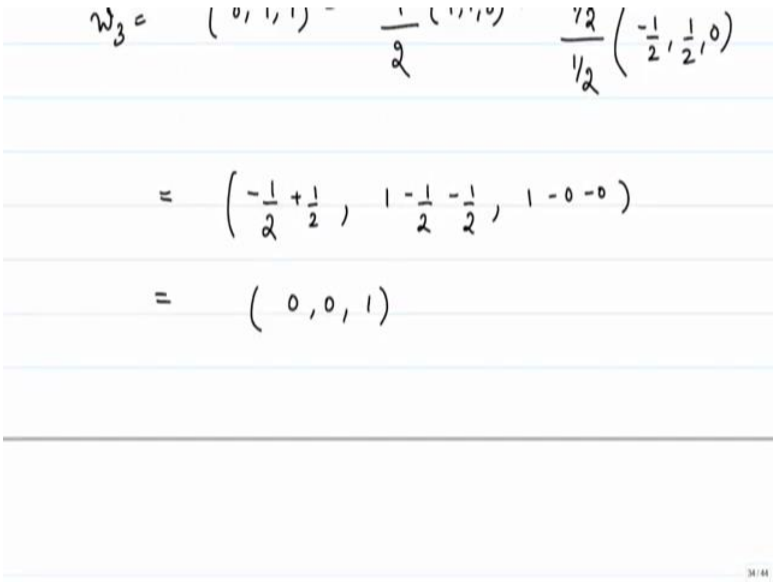

Maybe I should cross check it after I have written in this now, yes. So, this is going to be minus of 1 by 2 plus 1 by 2, which is 0 and 1 minus 1 by 2 minus 1 by 2 and the final one would be 1 minus 0 minus 0. So, we are just getting 0, 1, 0, that is kind of nice because now we have 3 vectors. So, let us see, it has to be wrong. There is something seriously wrong with this because w1 and w3, w1 and w2 certainly are orthogonal to each other.

If you notice w1 inner product with w2 is 0, but w3 is not orthogonal to w1 or w2, the process should ideally yield some w3 which is orthogonal to those. So, there is something wrong, let us see what went wrong. 0, 1, 1 minus half times 1, 1, 0 and the inner product here is what went wrong, 0 half that will be a half. Yes, there is a half and the square root of 1 by 2 square plus 1 by 2 square is 2 by 4, which is 1 by 2. So, this is 1 by 2, this is right, times minus half, half, 0.

So, minus half, plus half, this is 0, 1 minus half, minus half, that is 0. So, everything is right except that this should not be 0, 0, 1. Let us see now if it is right. Yes, it is orthogonal. So, it was just the last step which went wrong probably, I think everything else is right. So, let me write down whatever the formulas here.

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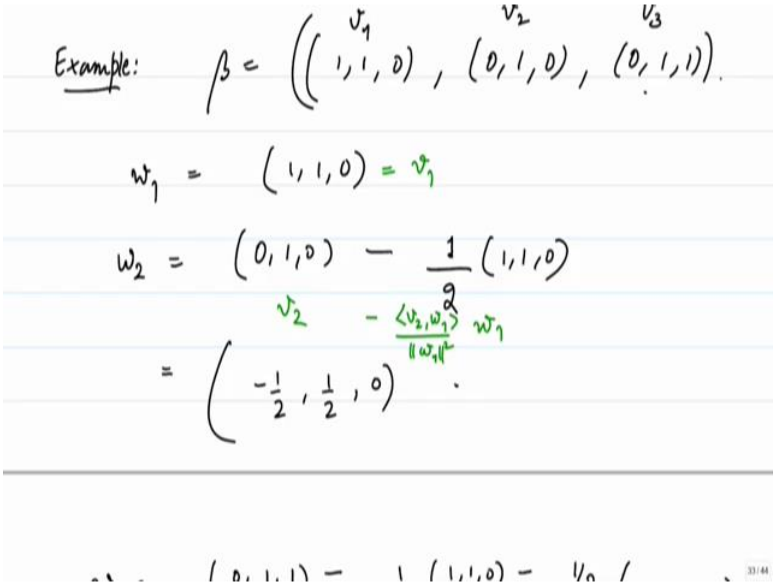

This is equal to v1, this is v2 minus inner product of v2 with w1 by w1 length square times w 1.

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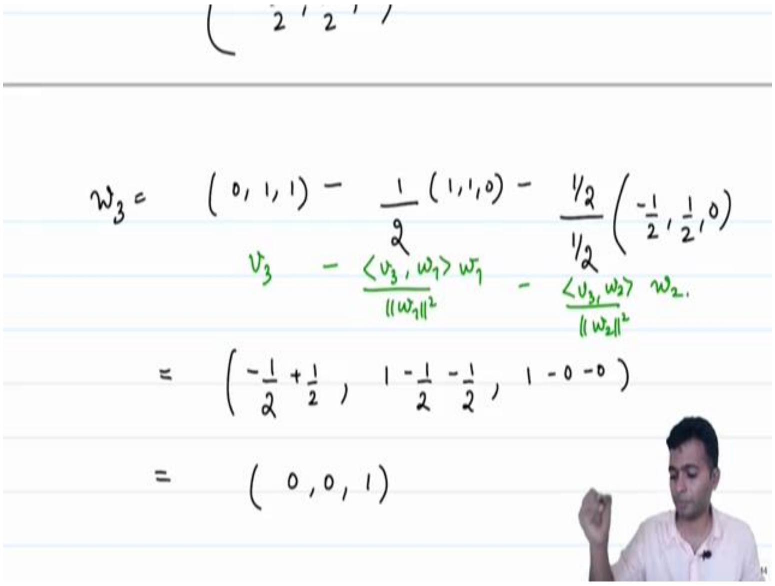

And let me write down what was here, this is our v3 minus inner product of v3 with w1 by w1 square times w1 and this one is inner product of v3 with w2 by length of w2 square times w2.

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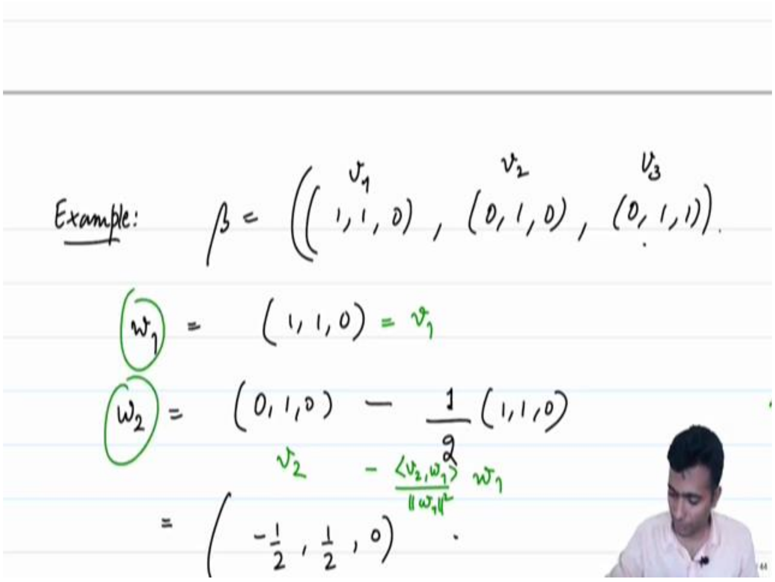

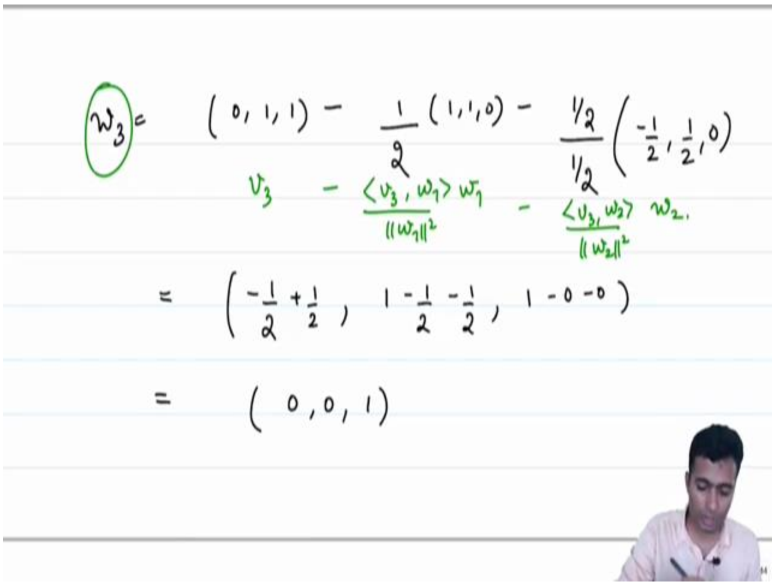

The calculations were done and we now have an orthogonal set of vectors as you can see this, this, this and this.

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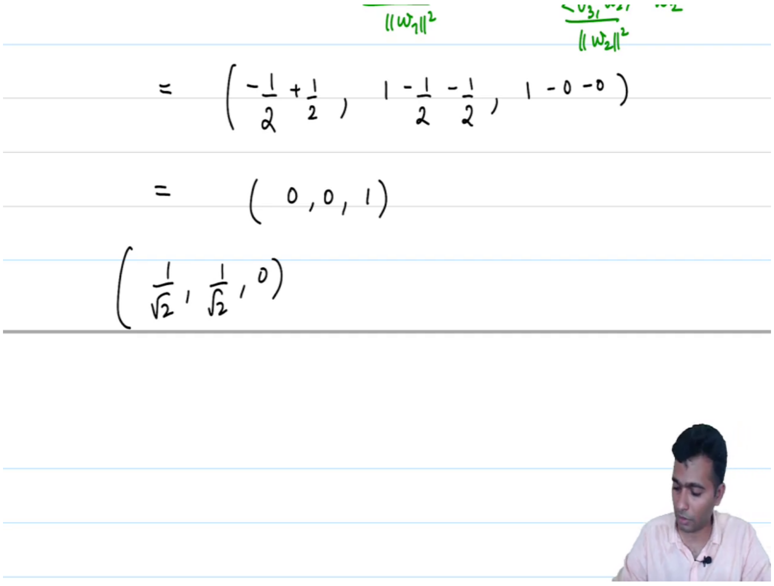

So, if we orthonormalize it, if we orthonormalize it, we will get w 1 prime. So, what will be our, I will say that 1 by root 2, 1 by root 2, 0, this is our first vector.

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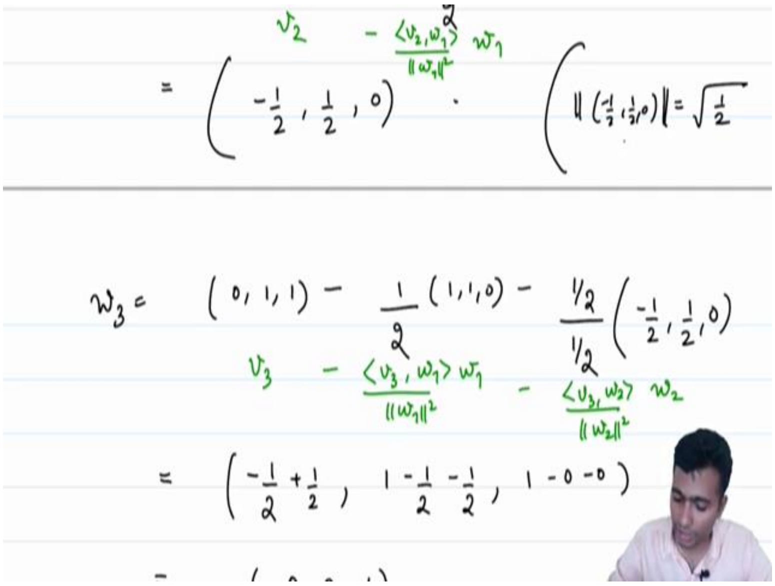

The second vector will just be 1 by 2, 1 by 2, 0 will have, what is the length of this if you carefully check what is the length of minus 1 by 2, 1 by 2, 0, this is equal to square root of 1 by 2. So, we divide it by 1 by 2, we finally get 1 by root 2. So, this divided by 1 by 2 is again 1 by root 2.

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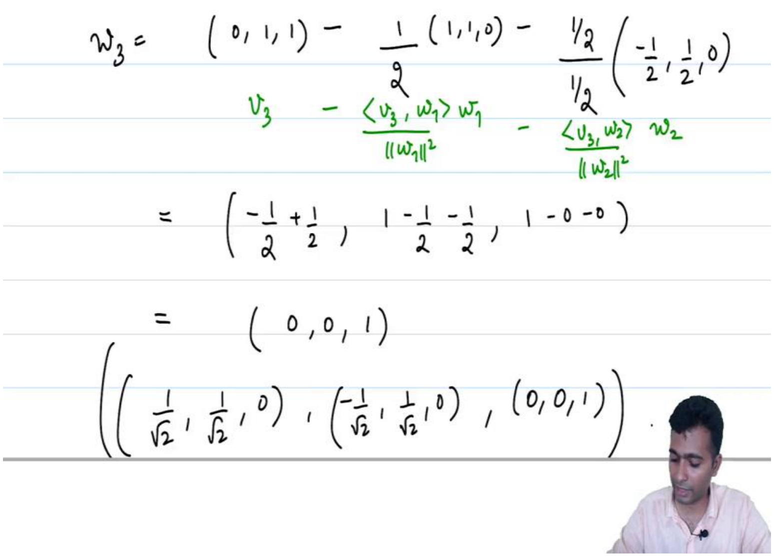

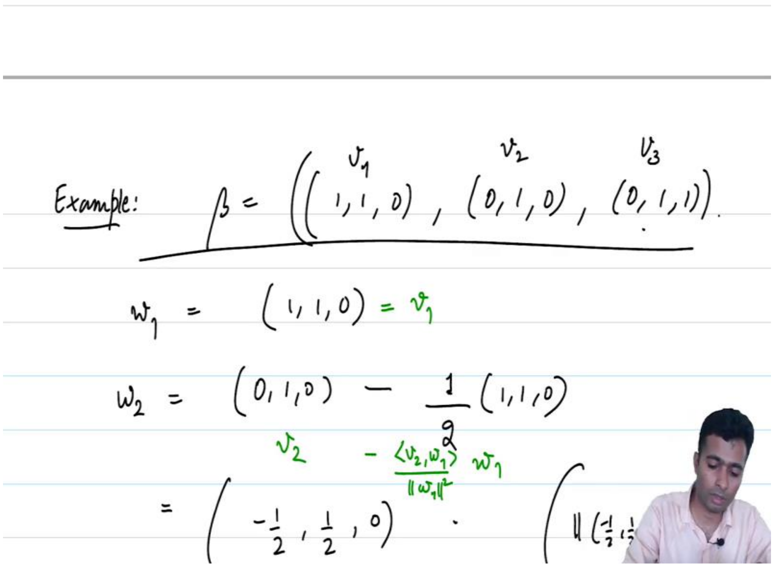

But now the sign changes, we get minus of 1 by root 2, 1 by root 2 and 0. The third one is already unit length. So, this is what we get as an orthonormal basis by applying Gram Schmidt orthonormalization to this basis. So, next, next video we will discuss orthogonal complements of a given vector subspace.