Linear Algebra
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Lecture 1.4
Linear Combinations and Span

So let us take some arbitrary vector in vector space capital V and ask this following question. Can we have at least can we obtain at least one subspace of capital V, which contains this particular vectors?

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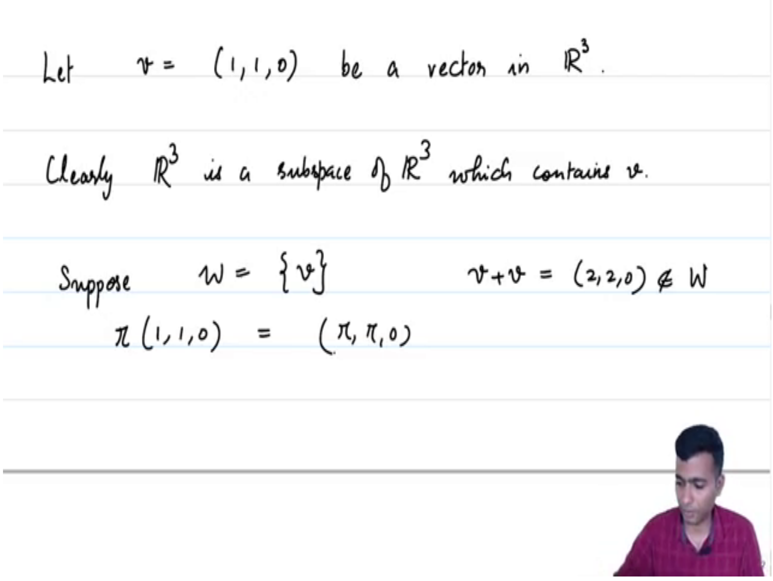

Let us, let us look at it with an example. So let v be equal to say 1, 1, 0 be an element in R 3, be a vector in R 3. So let us ask this question, can we obtain subspaces, at least one subspace of R 3 which contains v, so clearly R 3 itself is an example, right? R 3 is a subspace of R 3, which contains v, but that is (())(1:11) we need to, we would like to look at something which is a proper subspace, which is smaller than R 3, certainly the 0 vector space will not do, because it is too small that contains only the 0 element, it does not contain v, we would like to look for subspaces, which will contain v but which need not be the entire space. We might not be able to actually do that, but let us see if we can do that in this case.

It might happen that the only subspaces, which contain a few given vectors in the entire space that can happen, but let us see, let us focus on just this vector v 1, 0, 0, which is an R 3 and ask that question. So suppose our W is the set given by the single term v. Let us see, if this a subspace? Okay, good question. Is this a subspace? I would say that a few minutes of observation will tell you that it is not closed under vector addition or scalar multiplication. It is just certainly a subset of R 3, but it is certainly not closed under vector addition and scalar multiplication because what is two times v plus v is just equal to 2, 2, 0, right, v use 1, 1, 0 so v plus v which does not belong to our W, because W is just containing 1, 1, 0.

Similarly, you take you take any scalar multiple of v for that matter, let us look at Pi times 1, 1, 0, what is that I took Pi purposely because we were only considering two times, three times and all that, two times the vector, three times the vector and such examples, any real number should work as a scalar that is the reason why I have taken Pi. This is just going to be called 2 Pi comma Pi comma 0 and that certainly does not belong to our W. So the single term W is not a vector subspace of R 3. Okay.

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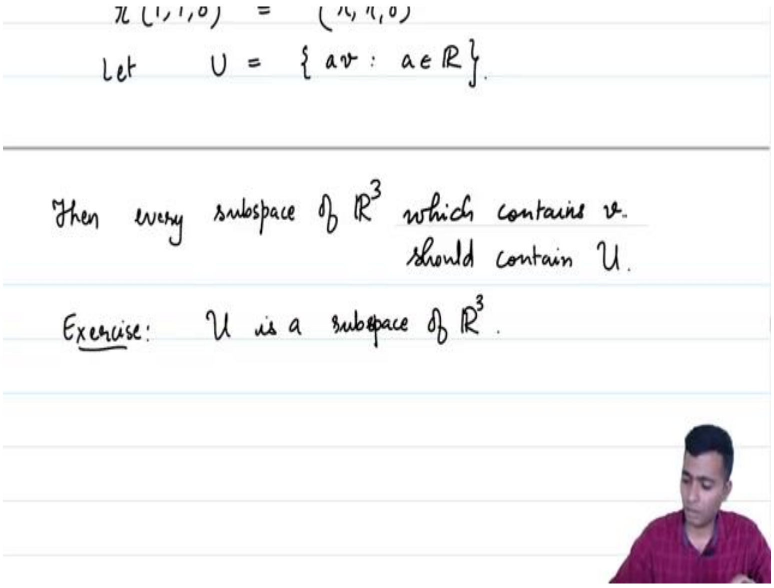

One thing which we will immediately notice is that if there is a vector subspace of R 3 which contains capital W, every scalar multiple should certainly be there, right? So let U be the setup all a times v where is in R all real numbers, or is in F, whichever you want to call it. Then notice, every subspace of R 3 which contains v certainly contains u because a subspace is closed under scalar multiplication in particular. Then my claim is that u itself is a subspace, so let me leave it as an exercise for you to check out that u is a subspace of R 3. So, not only have we obtained a subspace of R 3 which contains our vector v, we have also checked that any subspace of R 3 which contains the vector v should also contain this one. Sometimes it is the smallest subspace which contains our vector v.

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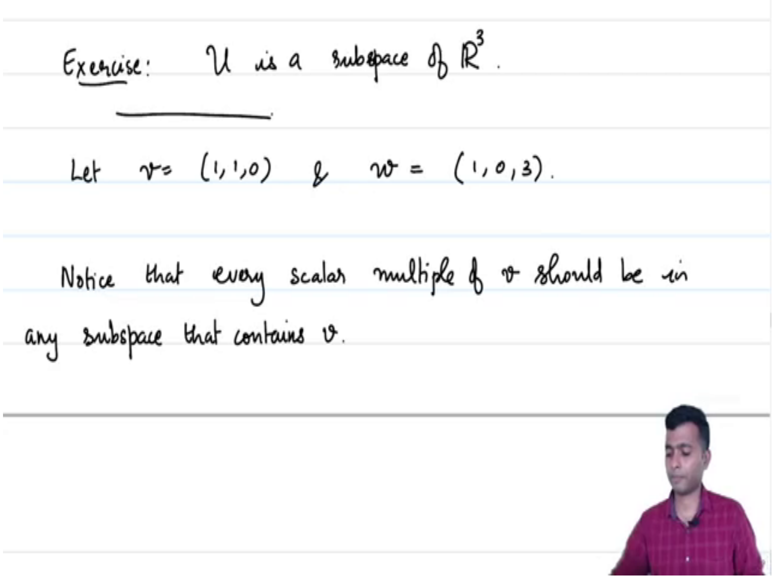

Okay, let us play the game with more vectors now. So, let v be equal to 1, 1, 0 and w be equal to small w b equal to 1, 0, 3 and let us ask the same question. What is the smallest subspace? Or let us not ask smallest or anything, let us ask me get hold of one subspace which contains v and w which might be you know, which could be a proper subspace and not the entire R 3 itself or which is a proper subspace of R 3, my term could will not fit there anyway. So, like in the previous case, the first thing to notice is that, if v is that any scalar multiple of v should also be there. So, notice that every scalar multiple of v should be in any subspace that contains v.

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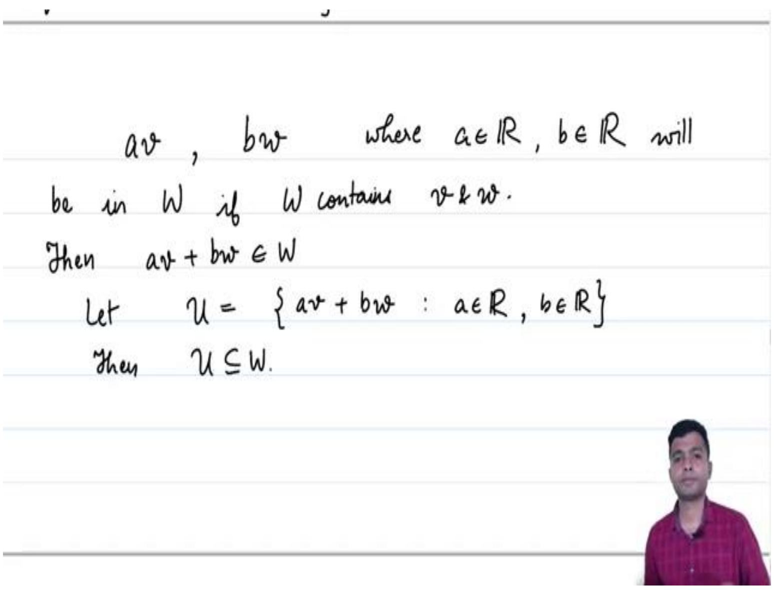

Similarly, any scalar multiple of W should also be there, should be in, just writing it again which is inefficient, but let me do it in any way, any subspace that contains w. So in particular, a times v and b times w, here a runs overall scalars and b runs over all scalars, we will be in w if w contains v and w, but are they all the elements they will contain we know more, then a v plus b w also belongs to W because W is a subspace, right. And subspace is by definition should be closed under vector addition as well. So, we know that for any scalar a, a v belongs to W and for any scalar b, b w also belongs to, b small w also belongs to capital W. But then these two are elements in capital W, their some, their vector addition of these two should also give you an element which is in capital W, and that is why a b plus b w also should be in capital W.

So, let us do one thing, let u be the set of all a b plus b w such that a is rising overall scalars, b is also rising over all scalars and we just noticed then u is contained in W.

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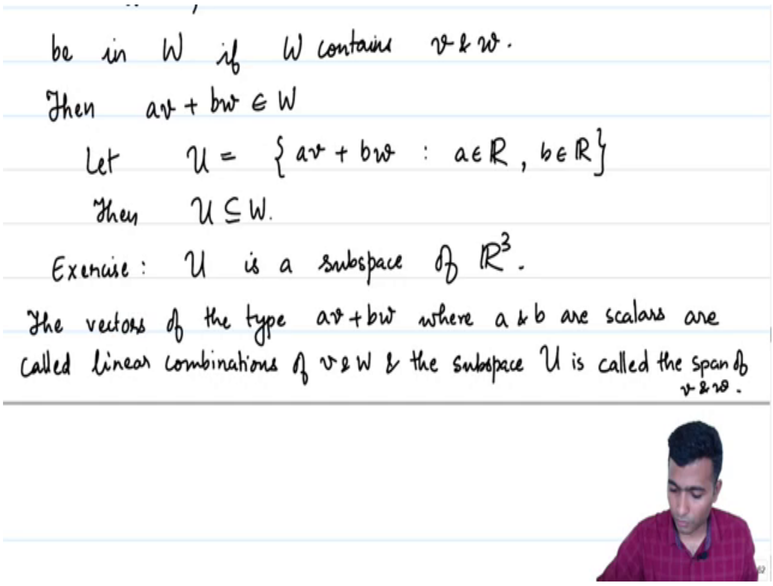

But then again I leave it as an exercise for you to check that u itself is a subspace. Therefore, it is, it so any it is in some sense the smallest subspace which contains v and w so, elements of this type a b plus b w is what is called as a linear combination of V and W and the collection of all such linear combination in this case, which is denoted by u is what is called as the span of V and W. So let us give a formal definition of linear combination and a span. So, let me just note it down here anyway. The element, the vectors of the type a v plus b w, where a and b are scalars, are called linear combinations of V and W, and the subspace u is called the span of V and W.

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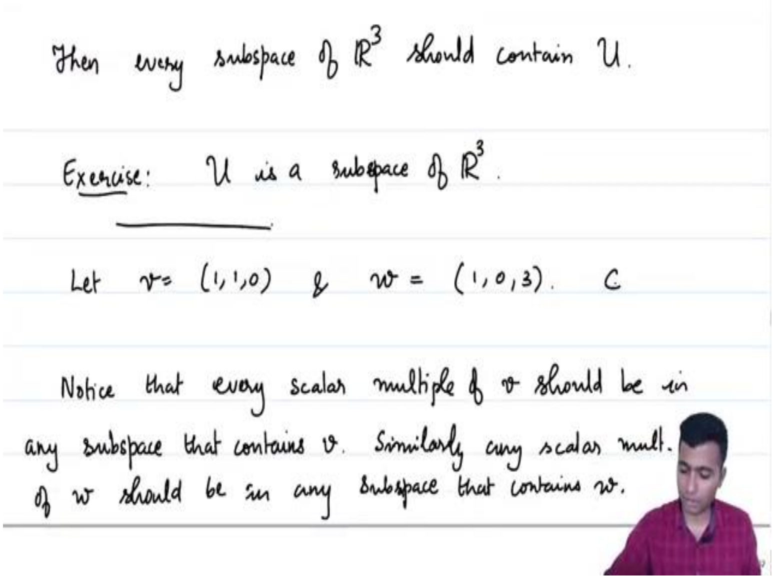

Of course, this was for a very special case, right? We took 2 vectors V and W in R 3, and then we were discussing all these things. So let us now give a formal definition of what is meant by a linear combination and what is meant by this path.

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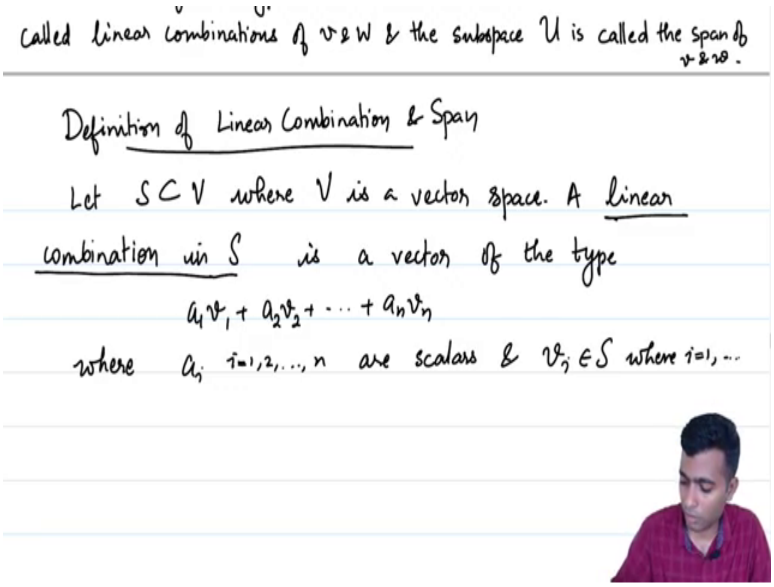

So let S be a subset of V, where V is a vector space, so I am giving you the definition of a linear combination and span. So let S be a subset of V where V is a vector space. Linear combination in S is an element of the type, so notice the language that is being used, a linear combination in S is an element, is a vector, let me not use the word, let me use the word vector to denote that it is an element of capital V, is a vector of the type a 1 v 1 plus a 2 v 2 plus dot dot dot a n v n where a i are fixed scalars, i equal to 1, 2, up to n are scalars and v i belongs to S, where again i is equal to 1 to n.

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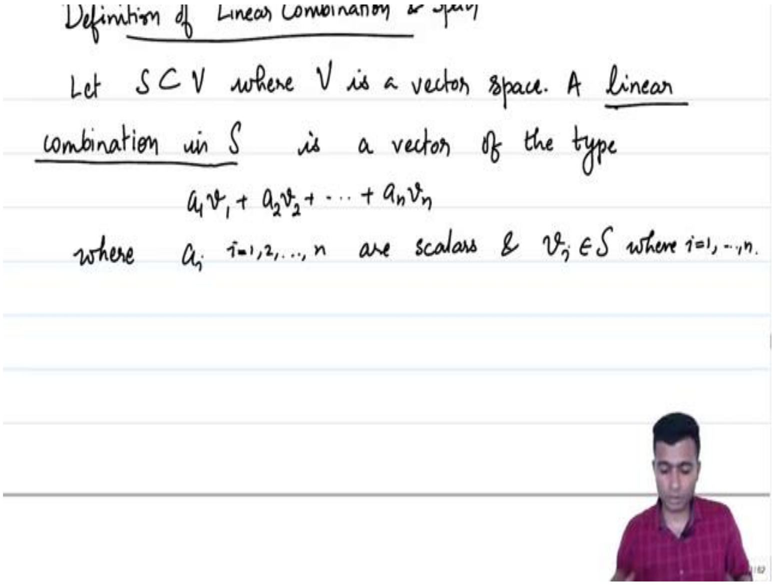

So let us revisit our definition, let S be any subset of our vector space V. And you pick v 1, v 2, up to v n arbitrary vectors in S and fixed scalars a 1, a 2, up to a n, look at a 1 v 1 plus a 2 v 2 up to a n v n, such an element such a vector is called a linear combination in S.

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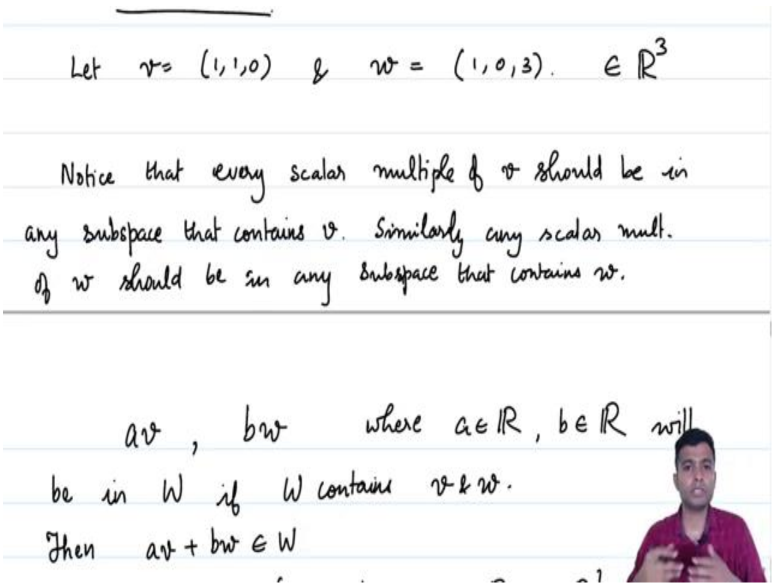

So if you come up to this example, two times w plus three times two times v plus three times W, where is the vector? Here it is, two times v plus three times w is an example of a linear combination of V and W, or Pi times V plus E times W is a linear combination of V and W, alright.

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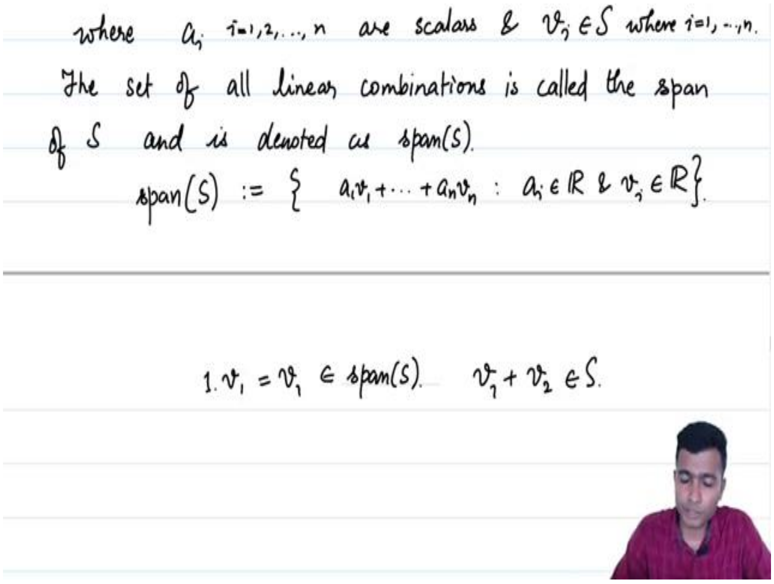

Now the set of ,the set of all linear combinations is called the span of S and is denoted oh it is called the span of S and that is what S is denoted as, is called the span of S and is denoted as span of S. So formally, span of S is the by definition is the set of all a 1 v 1 plus a n v n, where a i are scalars and v i belong to S. So, the first thing to note here is that our n is not fixed here. So, in particular one times v 1 equal to v 1 belongs to span of S. v 1 plus v 2 belongs to span of S, we have not put any restriction on the a I, the scalars a i could be anything. It could be anything in the sense any scalar, any real number, it could be 0, it could be a negative rational number, it could be a negative real number, which is not a rational and irrational number it could happen.

So we take any real number, look at combination with those any call a set of real numbers and look at a linear combination with those numbers that will be the span of S. Notice that we have not put any restriction on the set S.

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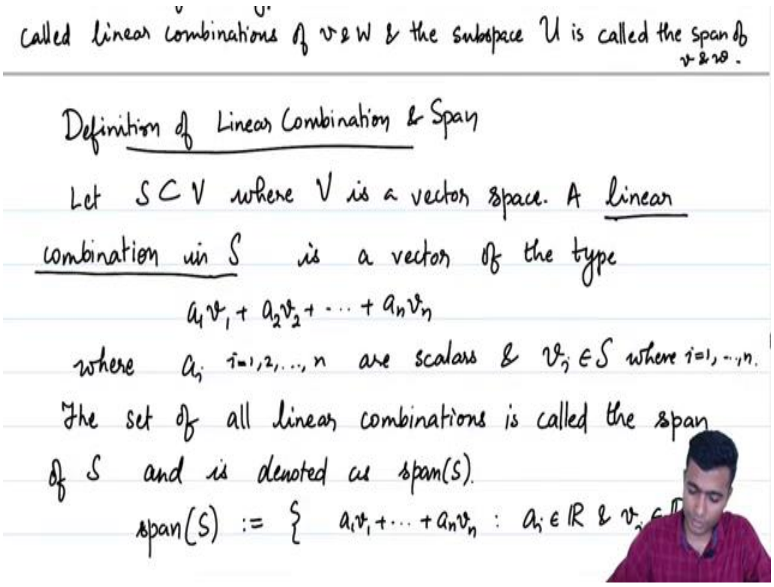

If you go back to our definition, we have just taken S to be some subset of v and we have defined the span and the linear combination in S right, it is the span of S and linear rather linear combination in a same span of S, so we have not demanded that S to be finite or anything, S could infinite. However, the linear combination will always be of finitely many elements in the set S.

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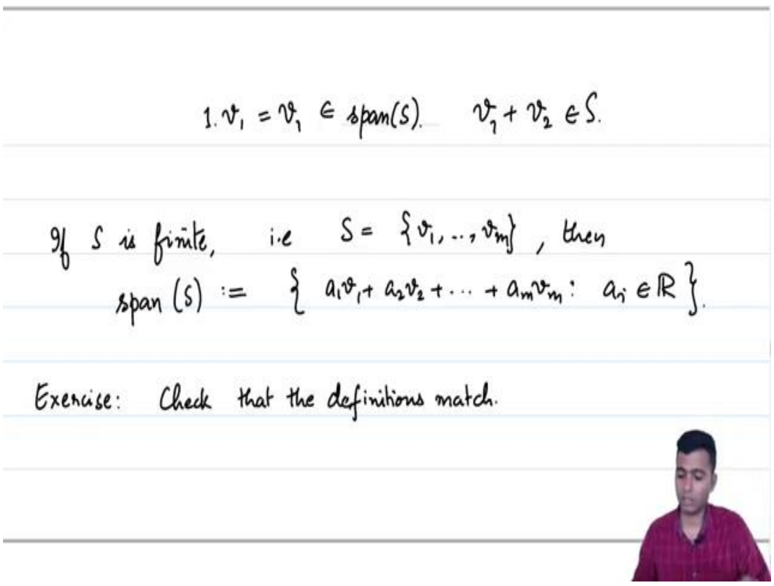

If S is finite, then we could always consider the linear, the set span of S, so let me just note it if S is finite, i.e. S is equal to say v 1 to v n, then I could say that span of S is the not this is already defined, this is equal to the set of all a 1 v 1 plus in fact, let me use a different and to not denote that it is fixed a 1 v 1 plus a 2 v 2 plus up to a m v m, where a i is any set of scalars. So, you if S is finite, then the span will consist of linear combinations of all the vectors in S, but that is not saying much. I will just leave it as an exercise for you to check that the definition has not changed. Check that the definition maybe I can put this thing here definitions match. So, what is their project at all is the question right.

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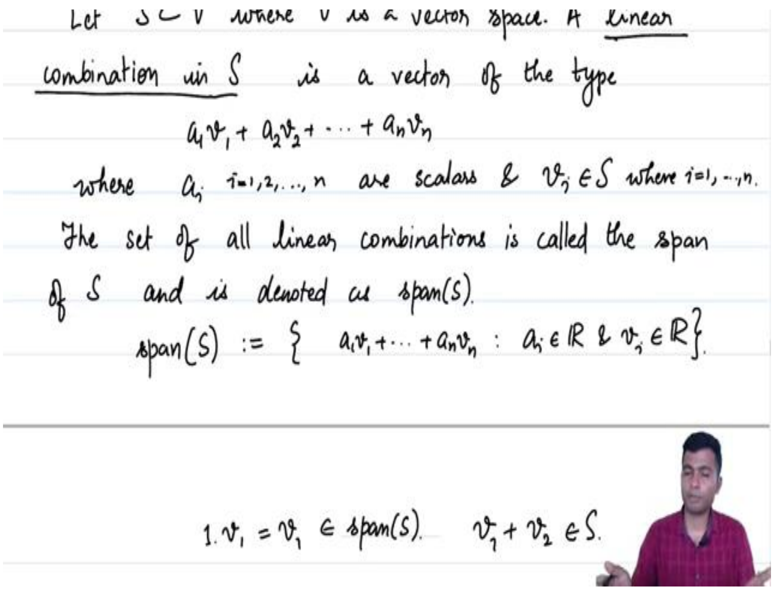

Here, all varying lengths where n could be 1 and could be 2 and so on are being considered when we define span of S in the formal definition, a couple of minutes back, we had no restriction on considering all elements of course we cannot. We are only talking about finite remaining sums.

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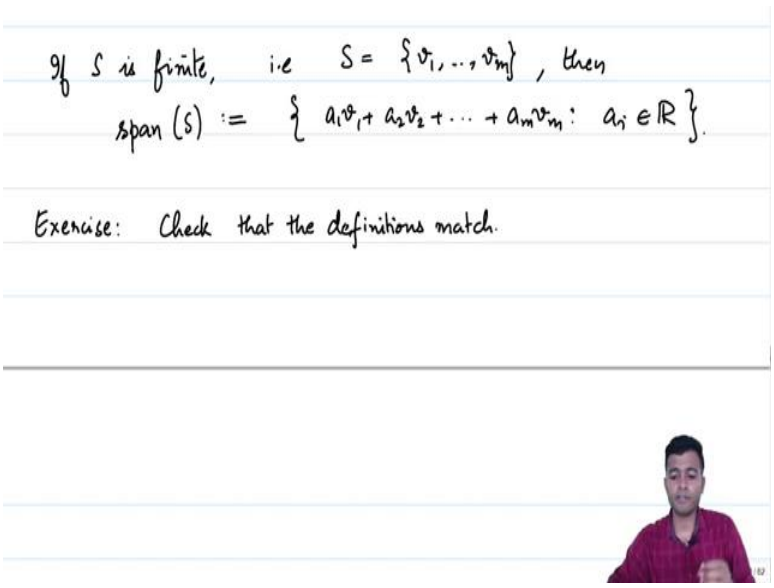

Here however, if S is finite, we are looking at the linear combination involving each and every element of S that is a difference. And it is an exercise to check that it makes no difference.

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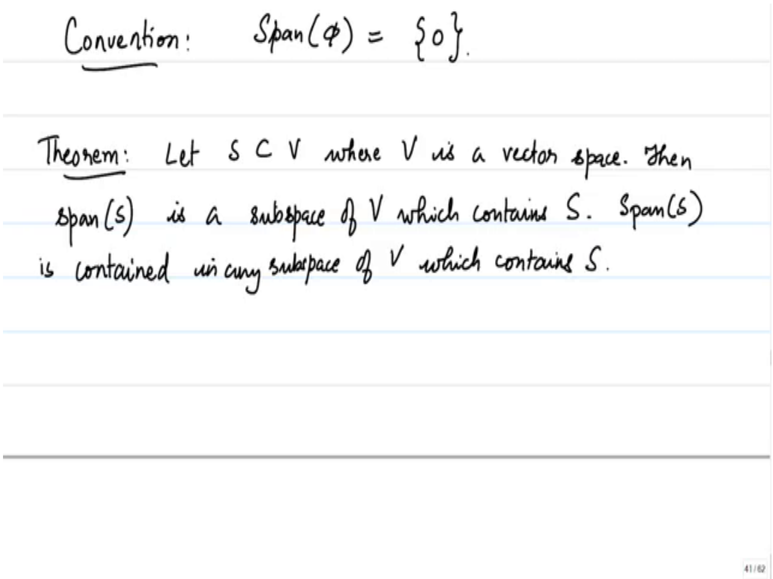

Alright, so let me introduce a convention here; span of the empty set. So in particular, empty set is a subset of V, so we could ask what is the span of the empty set? By convention, this is the 0 sub space, so this is taken as a convention or an axiom. And there is no clue some results about this span, we have already seen glimpses of what to expect, let me put it down as a theorem. So, let s contained in V, where V is a vector space then span of S is a subspace of V which contains S. Moreover, if W is any other subspace which contains S then span of S is contained in W, span of S is contained in any subspace of V which contains S Okay, let us have a look at the proof.

So, we will try to keep the proof as rigorous as possible because it is some the first few theorems we are proving okay. So what are we expected to prove here, S contained in V is a subset of V which is a vector space, we would like to first show that span of S is a subspace of V.

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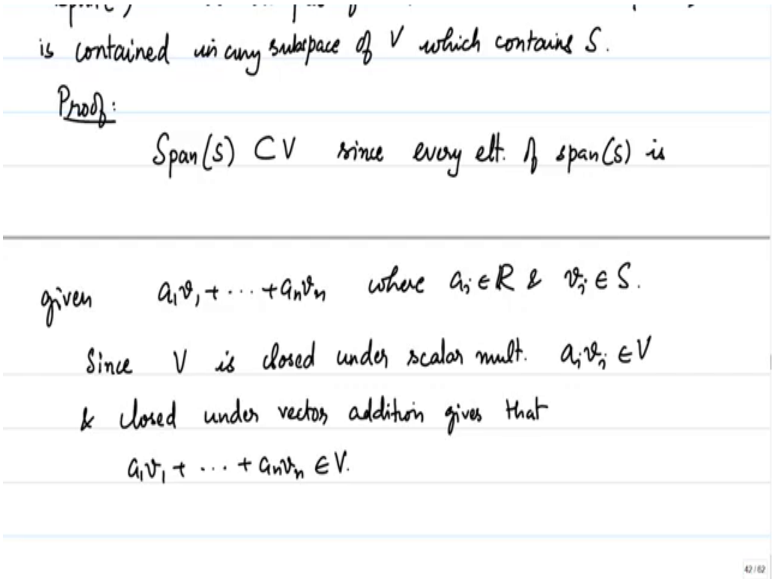

So, first note that span of S is contained in V since, every element of span of S is of the type is given by a 1 v 1 plus up to a n v n where a i are in capital S, I am sorry, a i are in capital R and v i are in capital S. Then but then since V is closed under both vector addition and scalar multiplication, under scalar multiplication, a i times v i belongs to V, and closed under vector addition gives that a n v n plus up to a n v n is in capital V. So, yes, span of S is certainly contained in capital V, so that was the first condition for something to be a subspace. W is a subspace of V if W is contained in V, and two more conditions are to be satisfied. We said W is closed under vector addition and scalar multiplication. So, in this case we would like to show that span of S is closed under vector addition and scalar multiplication.

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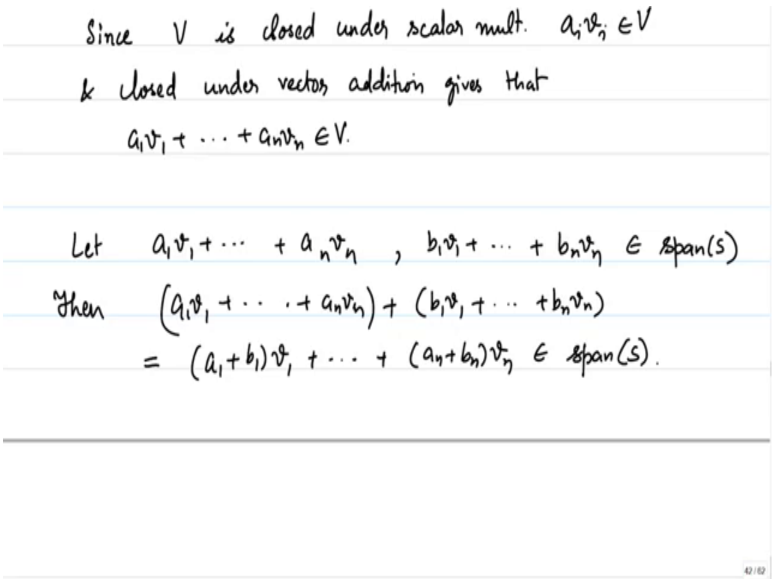

Let us first check closed under vector addition. So, let a 1 v 1 plus a m v m comma b 1 v 1 plus okay so let me write down a statement and let me then justify that; a n v n plus up to b n v n in span of S. Then, a 1 v 1 plus up to a n v n, this plus b 1 v 1 plus up to b n v n is equal to after all the properties 1 to 8, many of the properties will be used extensively, you should sit and see which are the properties which are being used, we will get this is equal to a 1 plus b 1 times v 1 plus dot dot dot a n plus b n times v n which belongs to span of S. So, if you take two vectors which are the span of the same vectors v 1 v 2 up to v n then the vector addition of these two put you back in the span of S, but if…

My point is that we are done with the checking of closeness under a vector addition because suppose, you take two linear combinations of linear combinations in S, then suppose v 1 to v n and w 1 to w n are the two different linear combinations, you look at the union and then extend the existing linear combination by adding zeros as coefficients of the ones which are not, then you can write any linear combination as the linear combination of this is the starting set with a few more elements with 0 as the coefficients. And therefore, the two different vectors can be written down as a linear combination of the same vectors. And by what we have just said, they were going to be the, the sum of them is going to be in this span of S. Check that this is closed under a vector addition. Just write down whatever I just said, for the sake of completion completeness.

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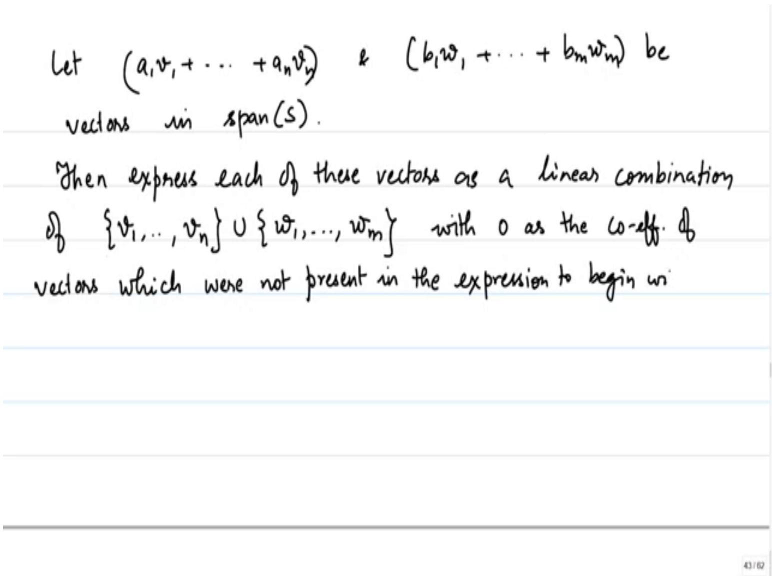

So, let a 1 v 1 plus up to a n v n and b n w n plus up to b m w m be vectors in span of S, then express each of these vectors as a linear combination of v 1 to v n or m, v 1 to v n and w 1 to w m, v 1 and w 1 to w m by… Oh, maybe that is a bad idea to write it like this. Let me put it this way, union w 1 to w m by with 0 as the coefficient, with 0 as the coefficient of vectors, which were not present to begin with.

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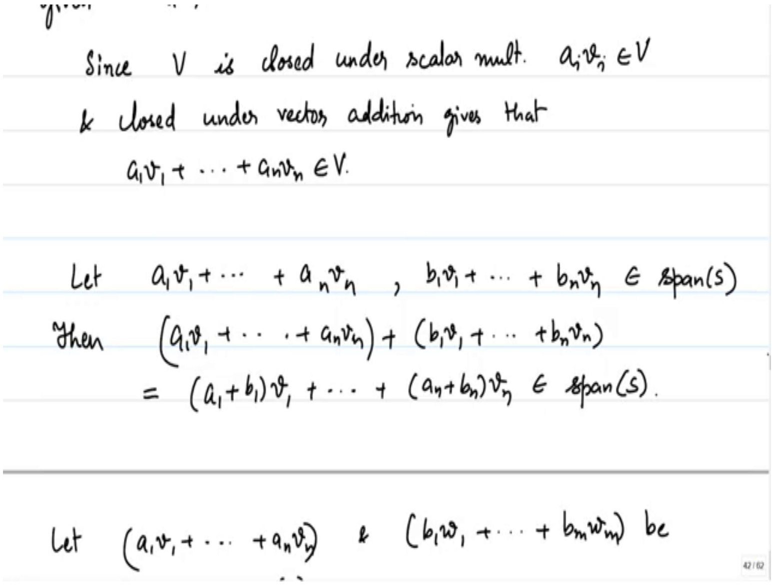

Then let us now invoke what we have just noted, as you know the sum of two such vectors can again be written as a linear combination of v 1, v 2 up to v n union w 1, w 2, up to w n, which is an element of span of S.

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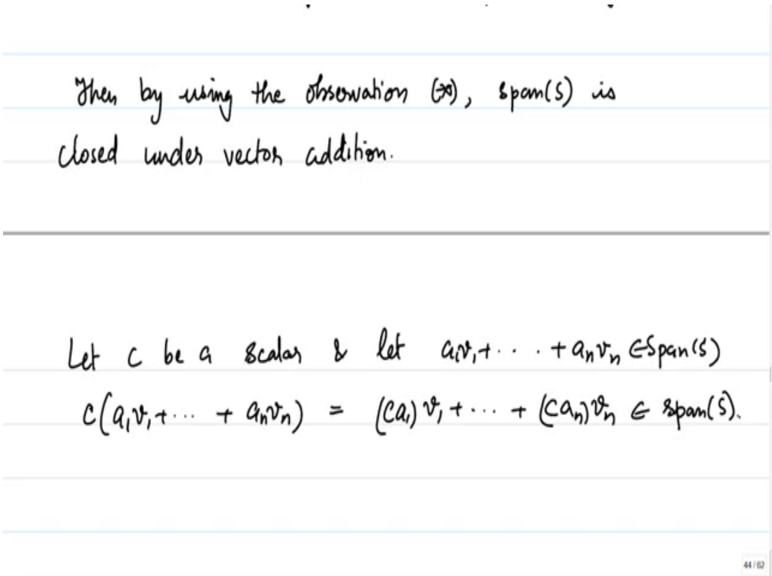

Then by using the above observation let me put this as star so this, let me call it as star. Observation star span of S is closed under a vector addition, the span of S being closed under scalar multiplication is far more straightforward So, let us C be some scalar, let it be a real number and let a 1 v 1 plus up to say a n v n be an element in the span of S. Then C times a 1 v 1 plus up to a n v n, what is this? This is a scalar multiplied to a vector and by the various properties of the operations in a vector space which were listed in the definition, we have this is equal to c 1 v 1 plus up to c a and v a. But notice that c three times a 1 is just another scalar similarly, c times a 2 is just another scalar, c times a n is also just another scalar. So, this again is an element in the span of S after span of S contains all the linear combinations, this is also in particular another linear combination right.

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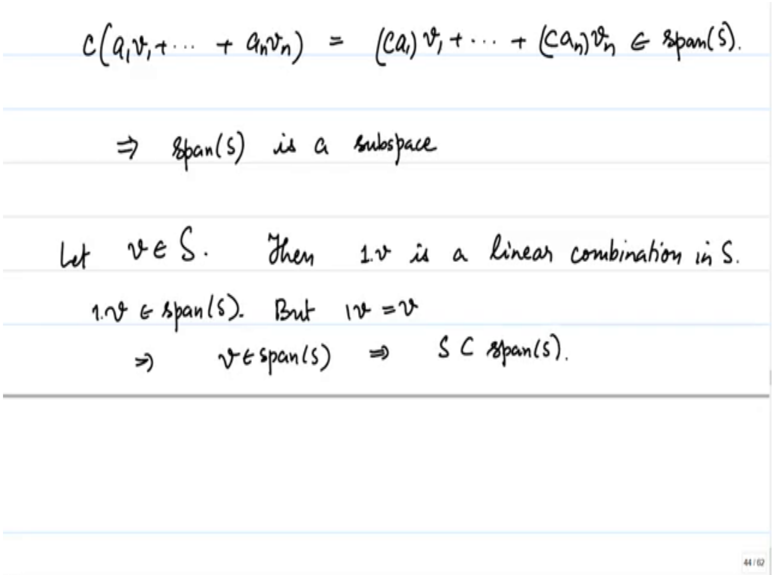

So, this gives us the span of S is a subspace, we are only partially done, we would like to now show that S is contained in this span of S but we have already observed right, S is clearly contained in the span of S. So let us prove it and then write it, so let v is some element in capital S then 1 times v is a linear combination in S by the very definition, but one times v just v, it is the multiplicative identity. So, this gives 1 times v is in span of S. But 1 times v is equal to v which implies v belongs to span of S, every element in S is in span of S, this implies S is contained in the span of S. So we have shown one statement in the theorem.

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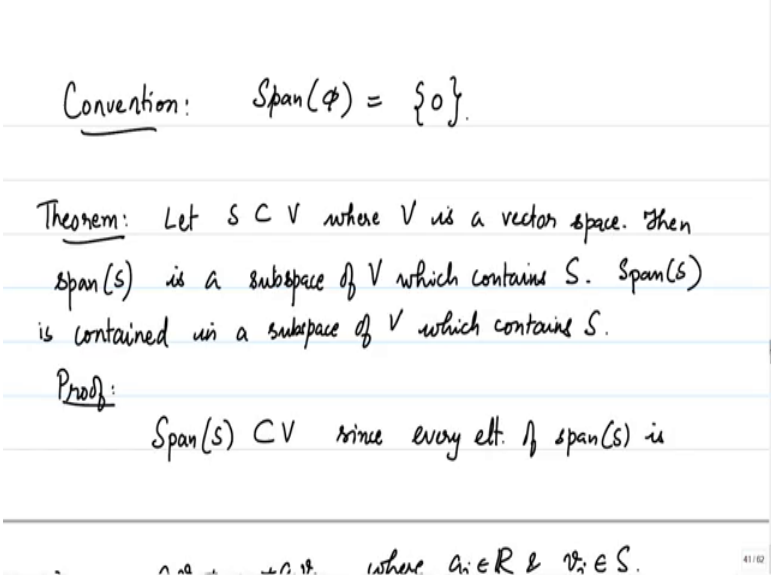

Let me recall what the theorem is, let S be a subset of V, where V is a vector space span of S is subspace of V which contains S. All that is left to show is that if W is a subspace of V which contains S then span of S is contained in V.

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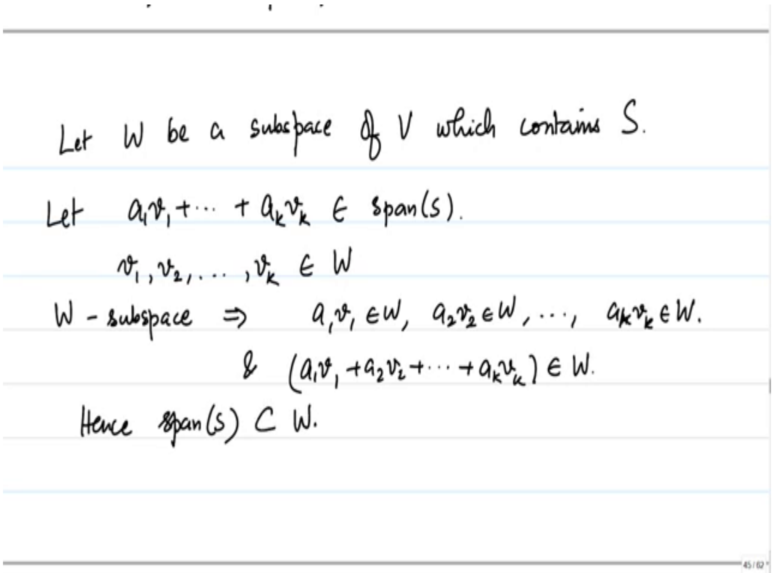

So, let W be subspace of V which contains S, we would like to show that span of S is contained in V. So, let us take some arbitrary element in the span of S. So, let a 1 v 1 plus let me use a different index so that you know, and it gives a feeling that it is always n so, let this be in the span of S. We would like to show that this element is in W as well. But notice that v 1, v 2 up to v k which are all elements in S belong to capital W, and W is a subspace, W is a subspace and therefore, all scalar multiples of this also belongs to capital W, it implies that a let me write it as a 1 v 1 belongs to capital W, a 2 v 2 belongs to capital W and so on, a n or a k v k also belongs to capital W because of scalar multiples belong.

But again W is a subspace implies that it is closed under vector addition as well and a 1 v 1 plus a 2 v 2 plus a k v k is belonging to capital W. Therefore, span of S is contained in S because we took some arbitrary vector in span of S and we showed that that has to be necessarily in W, so this means that this is contained in W. And that is precisely what we are supposed to proof. So, next let us look at the notion of a spanning set.

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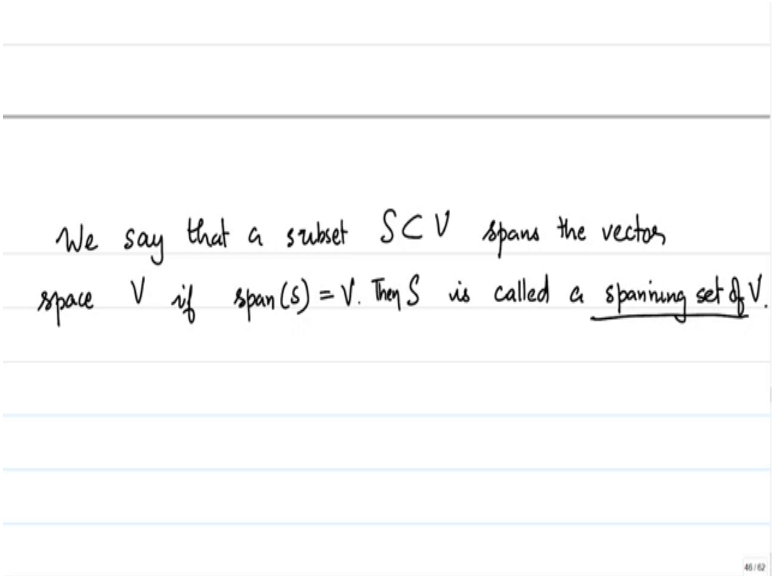

So, we say that as set spans the vector space V if the span of S is equal to v. So, we say that, a set a subset S contained in V spans the vector space V if span of S is equal to V. Notice that span of S is certainly a subspace of V as we have just proved, we say that it is a it spans the vector space V if it spans the entire vector space. S is called a spanning set of S then S called a spanning set of S spanning set of V, this is the definition of a spanning set. So, what is a spanning set, you take some subset S of V, check whether the span of S is the entire vector space, if it is the entire vector space, then we say that S is a spanning set of V, or that S spans V. Let us look at some examples...

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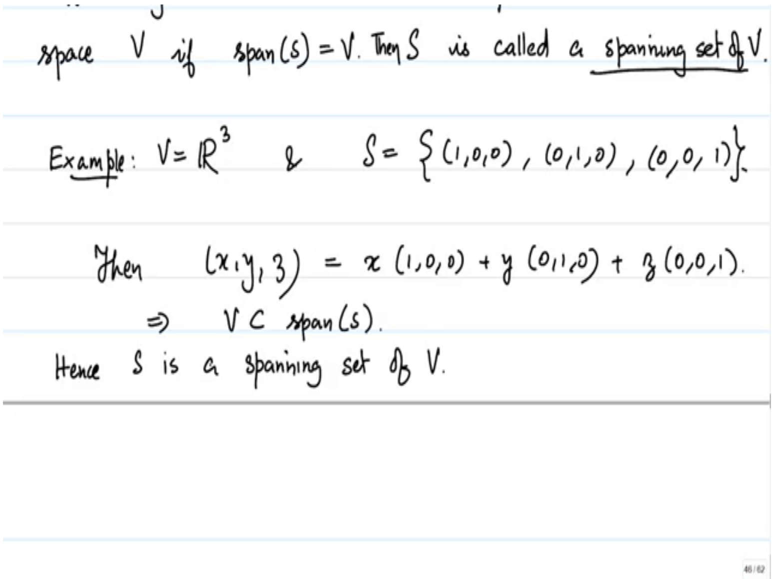

So, the first example would be to consider say R 3, let V be equal to R 3, and S be equal to the set 1, 0, 0; 0, 1, 0 and 0, 0, 1. Then you consider any x comma y comma z, this can be written, I have just write down explicitly what the linear combination of the above elements or above vectors are, which will give us x, y, z. This is going to be x times 1, 0, 0, what is that x is a scalar plus y times 0, 1, 0 plus z times 0, 0, 1. And therefore, any x, y, z can be written as an element in the span of S, which means that V is contained in the span of S. Hence, S is a spanning set of capital V, spanning set are quite remarkable, right. R 3 contains infinitely many elements, but we just showed that any element of R 3 could be written down as a linear combination of these three elements. Right.

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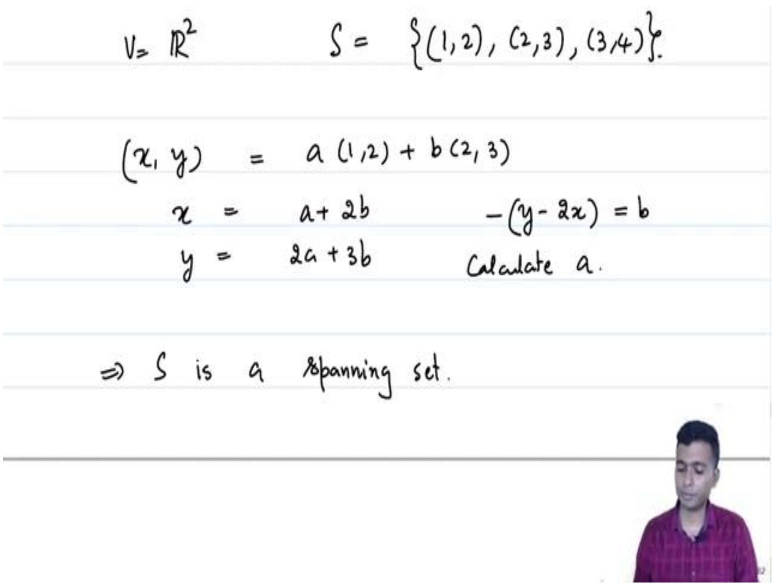

Next example. Let us look at V to be equal to say R 2. And consider S to be equal to 1, 2; 2, 3; 3, 4. I would like to show that 1, 2 and 2, 3, if you look at any element x comma y, suppose this is equal to a times 1, 2 plus b times 2, 3. Just let us focus on 1, 2 and 2, 3 and let us see what we can do. Can we get hold of some a and b such that this happens? Well, this just tells us that x is equal to a plus 2 b, and y is equal to 2 a plus 3 b and this tells us that y minus 2 times x is equal to or minus of this is equal to b. And similarly, we can get hold of a, calculate a, I leave it to you. And we have just shown that given any x y, we will be able to write x, y as a linear combination of 1, 2 and 2, 3.

In particular, if you look at S which has 1,2; 2,3 and 3, 4, you can write any element as a linear combination of S or any element as an element in the linear combination in the span of S. So, this implies that S is a spanning set again, but if you just go look at this example, you will notice that the 3, 4 was not needed right, 3, 4 was lying there as an unnecessary wastage. We could have written any element in R 2 as a linear combination of just 1, 2 and 2, 3. So the junk vector 3, 4 is in some sense dependent on 1, 2 and 2, 3, and that will be the content of the next week that will that is how we will begin the next week by discussing what is called as linear dependence and independence, so let me stop here.