Linear Algebra

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Lecture 38
Orthonormal basis

So, in the last week we introduced the concept of an Inner Product Space. We defined the notion of length of a given vector in a inner product space and we also defined when it, what it means to say that two vectors are orthogonal to each other. We talked about a few properties of orthogonality. So, for example we prove that, if w is orthogonal to vectors v1, v2 up to say vk, then w is also orthogonal to a linear combination of this vectors and we proved the Pythagoras theorem in an inner product space.

We also generalized it to finitely many vectors. So, in this week we will begin by discussing the notion of an orthonormal basis. It is a basis, basically it is a basis, which consists of vectors which are orthogonal to each other and each of which has length 1. So, let us begin by discussing what it means to say that a collection of vectors.

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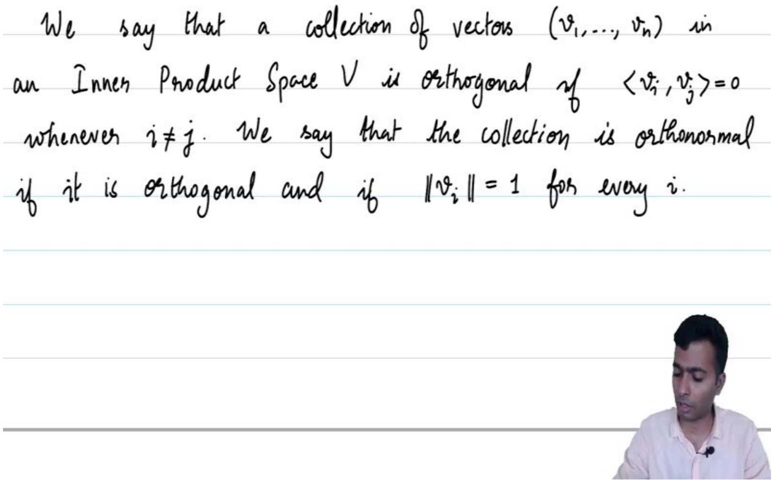

So, we say that a collection of vectors v1 to vn in an inner product space V is orthogonal, when it is orthogonal pair wise. If the inner product of v1 with vj is equal to 0, whenever i is not equal to j and we say that a collection is orthonormal, if it is not just orthogonal, but also if the length of each of the vectors is 1.

If it is of unit length, so we say that the collection is orthonormal, its normalized that is where the normal is coming from, it is orthonormal, if it is orthogonal and if the length of vi is equal to 1 for every i. So, it is not just enough for the vectors to be orthogonal pair wise orthogonal, but the length of the vector should also be 1. So, let us look at a few examples.

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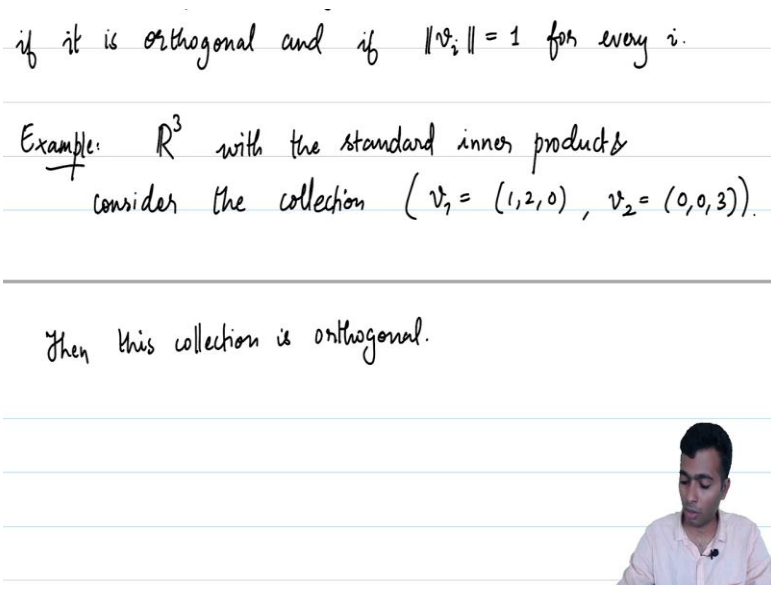

So, example, let us work with R2 or maybe R3 with the standard inner product, which is the dot product. So, consider R2 with the standard inner product and consider the collection of vectors. So, let us see v1, let it be equal to 1, 2, 0 and v2 be equal to 0, 0, 3. Then this is an orthogonal set of vectors. This collection, then this collection of vectors is orthogonal, is orthogonal. That is straightforward check, if you look at the inner product of v1 and v2 they will just turn out to be 0. However, if you observe carefully, this is not an orthonormal collection.

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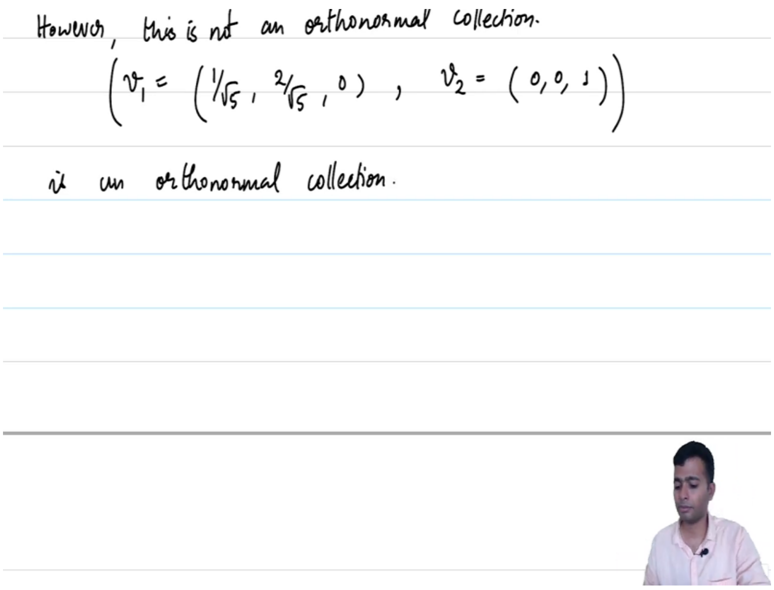

However, this is not an orthonormal collection. What would be an orthonormal collection, let us see may be, so v1 is equal to v by square root of 5, 2 by square root of 5, 0 and v2 equal to 0, 0, 1 by root 3 or maybe not just 1. This is an orthonormal collection this, so if you look at this collection is an orthonormal collection.

If you look at the length of v1, that is just going to be square root of 1 by 5 plus 4 by 5 plus 0, which is square root of 1, which is 1. So, this is an orthonormal collection of vectors. So, why are we considering such collections of course there are many, many good properties which come out, but let us see orthonormal properties, orthonormal collection, collection of orthonormal vectors has some really nice properties, one of which, is that it is linearly independent.

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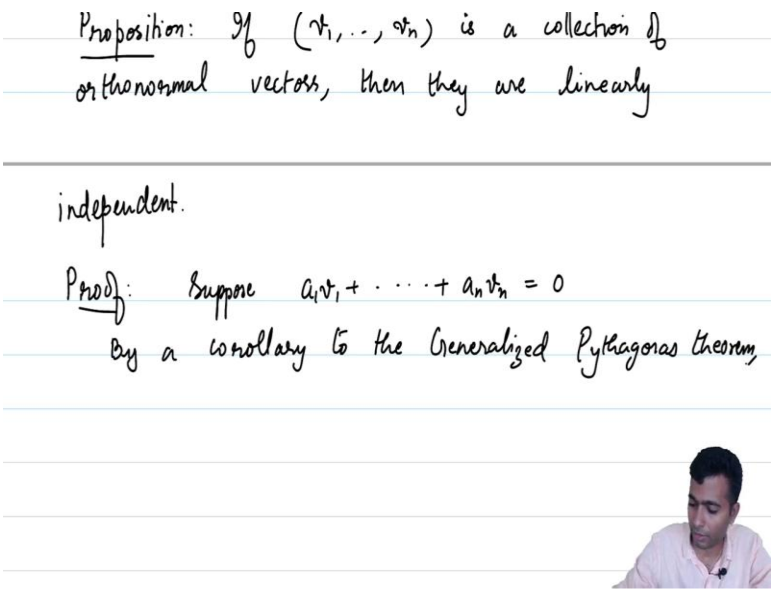

So, let us prove it, let us state the proposition if v1 to vn is a collection of orthonormal vectors, then they are linearly independent, they are linearly independent. Well, let us have a look at, a proof. Well, we could prove it in multiple ways, but let us make use of let us make use of what we already know, we have already proved. So, suppose we have a linear combination of v1 to vn which is equal to 0. Suppose, a1 v1 plus a2 v2 plus an vn is equal to the 0 vector.

So, as you can see I have slowly again stopped indicating, whether it is a complex vector space or real vector space or rather a complex inner product space or real inner product space. But unless and until it is mentioned explicitly all our results go through for both. So, let me not unnecessarily bring it up again and again. Our results whichever we are stating and proving are true for inner product spaces over both real numbers or inner product spaces over comple numbers.

So, suppose we have a linear combination which is equal to 0, then let us see what is the inner product of a1 v1 or maybe let us use the corollary to the generalized Pythagoras theorem by the corollary to the generalized Pythagoras theorem from the last week, what can we conclude.

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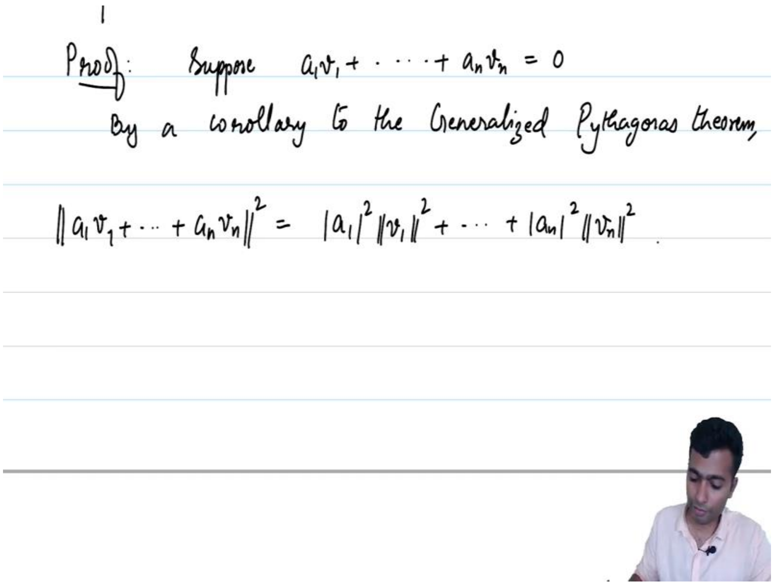

We can conclude that the length of this vector be length of a1 v1 plus to an vn square, this is going to be equal to mod a1 square length of v1 square plus up to mod an square length of vn square.

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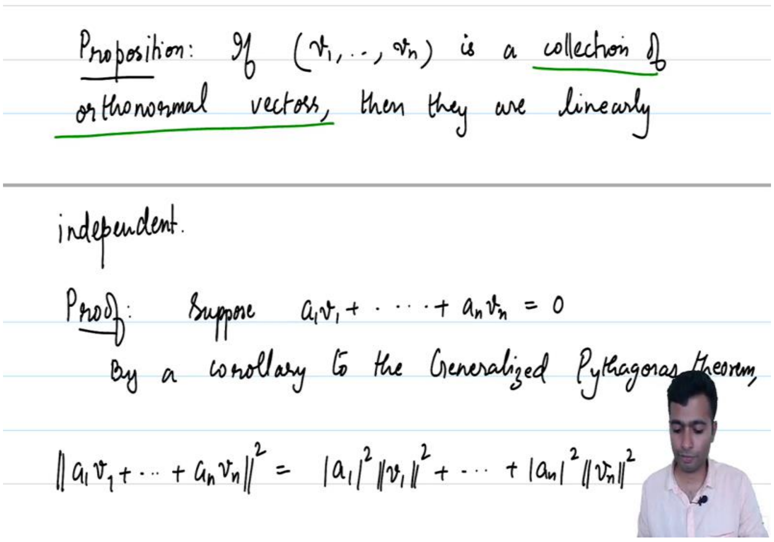

But what do we know about v1, v2 up to vn, it is after all as you can see, let me underline it, it is a collection of orthonormal vectors, so in particular each of these vis have length 1.

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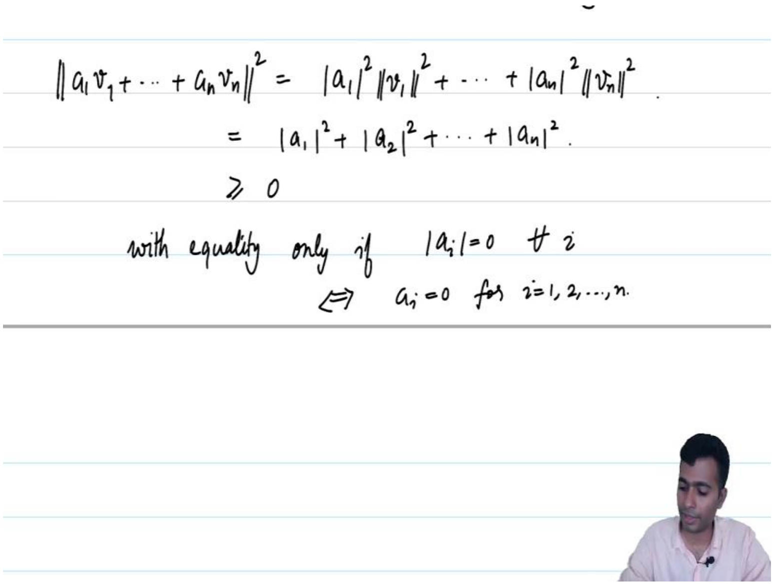

That means, this is just equal to, this is just equal to mod a1 square plus mod a2 square plus mod an square. But till now we did not have to worry about which field of scalars we were working on, but again irrespective of whether we are working over, working in an inner product space over complex numbers or whether it is an inner product space over real numbers. The absolute value the square of the absolute values are all positive numbers.

So, this is all a sum of, so this is greater than or equal to 0, when is the equality going to come up? With equality only if mod ai is equal to 0 for all i. So, this will happen, only if each of these positive numbers are 0, even if one of them is nonzero, it will be strictly greater than 0 and therefore it cannot be equal to 0. So, it is 0 means that each of them is forced to be 0. But when is the absolute value of a real number or a complex number equal to 0.

This happens only if the number itself is 0 only if the scalar itself is 0. So, this is if and only if ai is equal to 0 for i is equal to 1 to n and this forces our vectors v1, v2 up to vn to be linearly independent. Just to tell you or show you that the operation of inner products and the various consequences are powerful.

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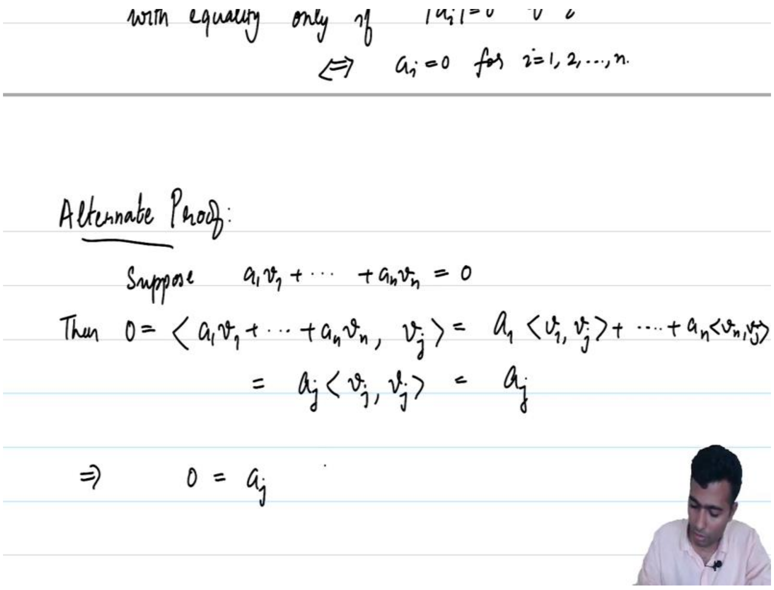

Let us look at an alternate proof, which is, it is not like this, it is not elegant. It is just that we can see that there are multiple approaches. So, suppose again a1 v1 plus up to an vn let this be equal to the 0 vector. Then we look if you look at the inner product of a1 v1 plus up to an vn and let us look at the inner product of this with vj, where vj is one of the vectors from the collection. By linearity, this is just going to be a1 times inner product of v1 vj plus a2 times inner product of v1 vj plus an times the inner product of vn vj.

Now what do you know about vis? They are orthonormal that means if i not equal to j inner product of vi and vj will be 0 and if i is equal to j, the inner product of vj vj will be the square of the length of vj which is equal to 1. So, this is just going to be equal to aj times the inner product of vj with vj, which is equal to aj, because length of vj is equal to 1 and therefore inner product of vj, with itself is equal to 1, all other terms vanish notice that. But what is a1 v1 plus a2 v2 up to an vn, it is just the 0 vector. So, this implies 0, so this is equal to 0. So, this implies 0 is equal to aj and if you do it for j is equal to 1, 2, 3 up to n.

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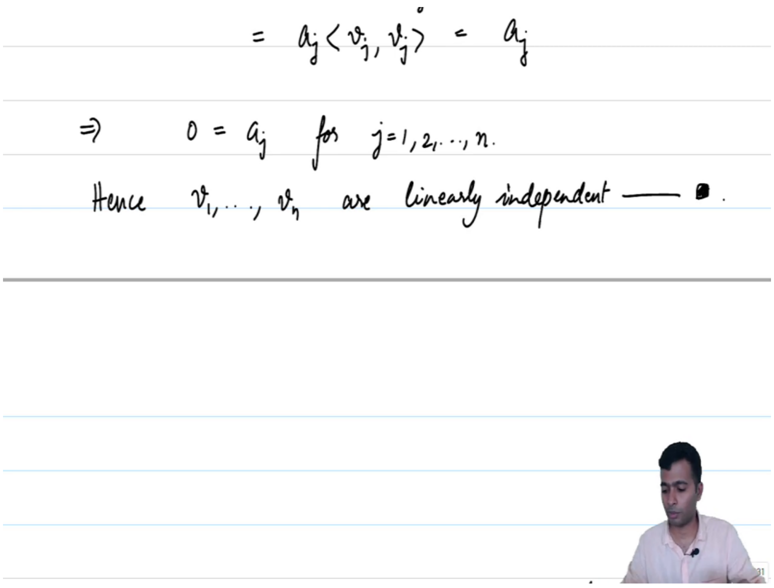

We conclude that, each of the ajs are forced to be 0, hence v1 to vn are linearly independent. So, we gave two proofs for this result, but nevertheless that is good result to keep in mind and this is just the beginning of many good things that we get out of orthonormal set of vectors. Let us next define what is meant by an orthonormal basis. As you can guess an orthonormal basis is a basis which is also an orthonormal collection.

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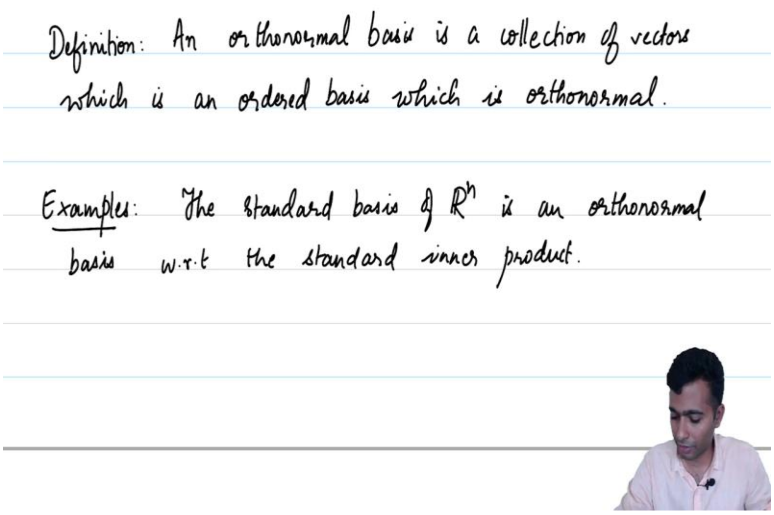

So, definition of an orthonormal basis, so let me just write definition here, a, an orthonormal basis is a collection of vectors, is of vectors which is both an ordered basis and an orthonormal set which is an ordered basis, which is an orthonormal set, which is an orthonormal. Let us look at a few examples, so the standard basis in Rn of Rn this is an orthonormal basis with respect to the standard inner product in Rn.

This is an orthonormal basis it is check for you to see that they are orthogonal to each other and that each of them has unit length. It is an orthonormal basis with respect to the standard inner product. Let us usually work with standard inner product. So, if I failed to mention which inner product we are working with in Rn it will be the standard inner product. Well let us be a little more adventurous.

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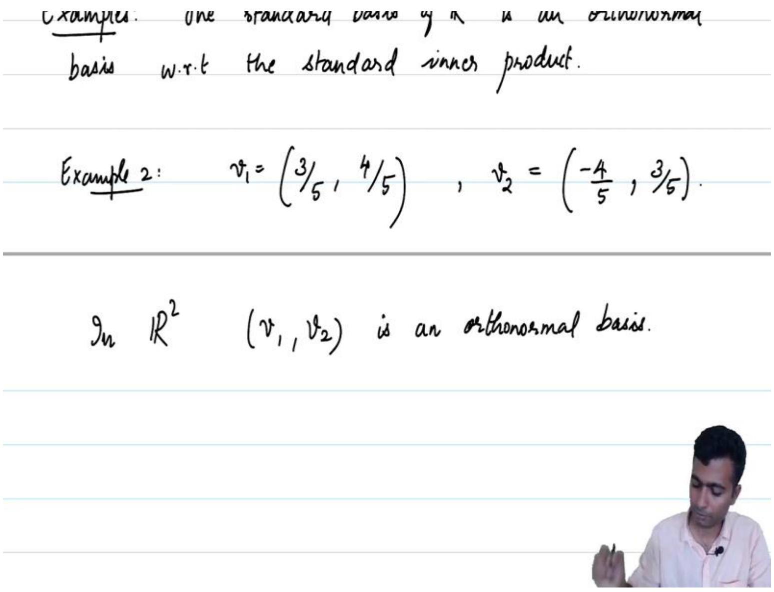

So, another example, so let us look at the following collections. So, let v1 be equal to 3 by 4, 3 by 5 and then 4 by 5. You can check that the length of v1 is 1 and what about v2 v2 be minus of 4 by 5 and 3 by 5. You can check that v2 has also length 1 and it is immediate to see that in R2 v1 and v2, they form an orthogonal collection if, they form an orthonormal set and therefore it is linearly independent.

Dimension of R2 is 2 and any linearly independent set of size 2 is a basis. So, this is an orthonormal basis. So, orthonormal basis are quite handy to work with. So, if for example, if you start with an arbitrary basis in say Rn and if you would like to. So, we know that every vector can be written as a unique linear combination of vectors in this ordered basis.

If we are to write down explicitly, what this linear combination is, it is a very tedious process. In fact, even in small, in low dimensional Rn. So, for example R4, if you take and if you want to write say 1, 5, 6, 7 in terms of a basis which is not the standard basis, it is quite complicated to find out what is the coefficient are and that is precisely where the orthonormal basis comes in a very handy manner.

We can write down explicitly what would be the coefficient featuring in the linear combination of the given vectors in a very simple manner we can do that. So, that will be our next goal. So, let me just note what I just said.

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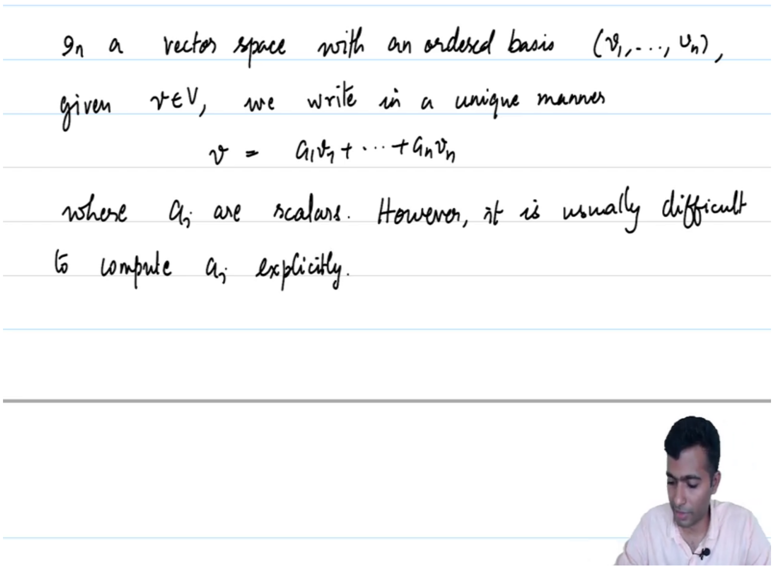

So, in a vector space with a basis, with an ordered basis given by say v1 to vn, we can write a vector, can write v. So, given v in capital V, we can write v has been equal to a1 v1 plus up to an vn, where ais are scalars, uniquely we can write that. We can write in a unique manner v as this particular linear combination. However, it is very difficult, it is usually difficult to compute ai explicitly, you know even in small vector spaces or in vector spaces like R3 R4 itself in R2 I would not say it is very difficult, but in R3, R4 itself it starts becoming quite tedious.

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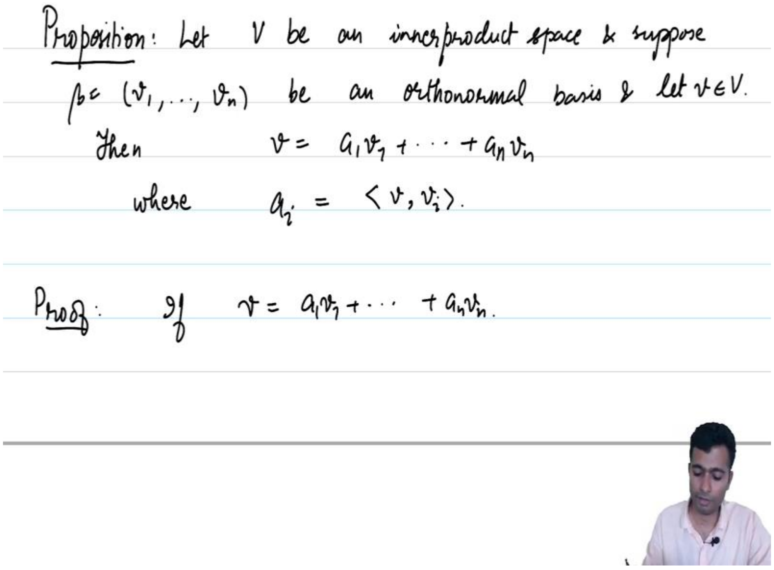

So, in order to capture what I said a few minutes back, let me state it down as a proposition. So, let V be an inner product space and suppose beta equal to v1 to vn be an orthonormal basis and let v be a vector in capital V, then v is equal to a1 v1 plus up to an vn, where ai is just the inner product of v with vi. So, we know explicitly what the coefficients of each of vis are in the linear combination of v1 to vn which is equal to v. So, let us give a proof of this proposition. So, suppose, so if v is equal to a1 v1 plus up to an vn, suppose this is the case and let us look at what is the inner product of v with say vi.

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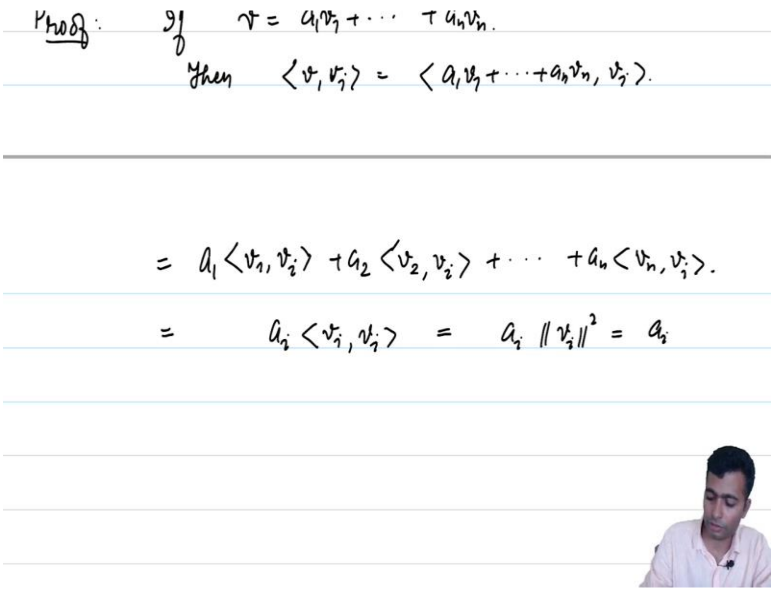

Then inner product of v with vi will just be the inner product of a1 v1 plus up to an vn inner product of that with vi, but what is this. This is equal to a1 times inner product of v1 with vi plus a2 times the inner product of v2 with vi plus up to an times inner product of vn with vi. But again, beta is an orthonormal, so inner product of vi with vij is 0, if i is not equal to j. So, this is just going to be equal to, we have already used this trick once before.

So, illustrate the power of it I had given the alternate proof earlier and this is just ai times the length of vi square which is 1 and therefore this is equal to ai and that is precisely what we had set out to prove. So, this makes a lot of things quite simple for us. So, for example if v is some vector.

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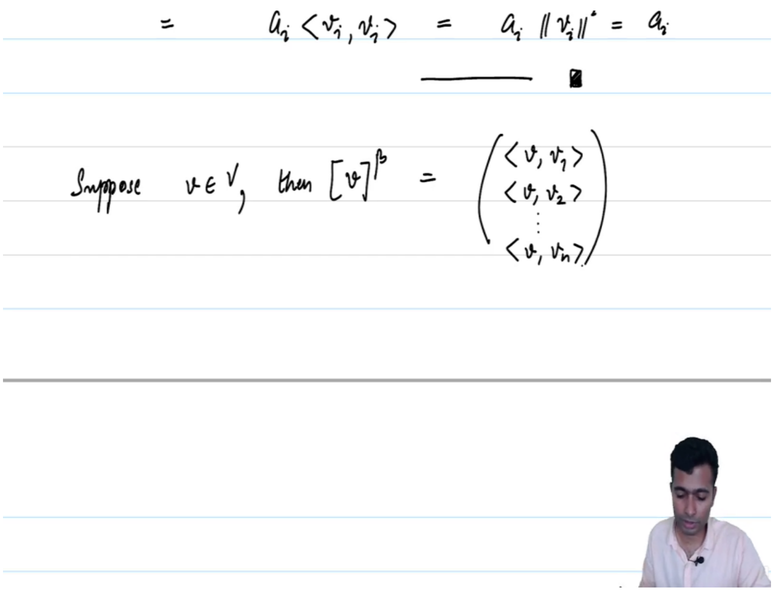

So, let me just say that we have proved it. So, suppose v is in capital V and then we are interested in what is the column vector of representation of v with respect to beta, then this is just equal to, this is just equal to the column vector which is given by the inner product of v with v1 v with v2 and so on v with vn. So, we now have an explicit column vector representation of v with respect to beta.

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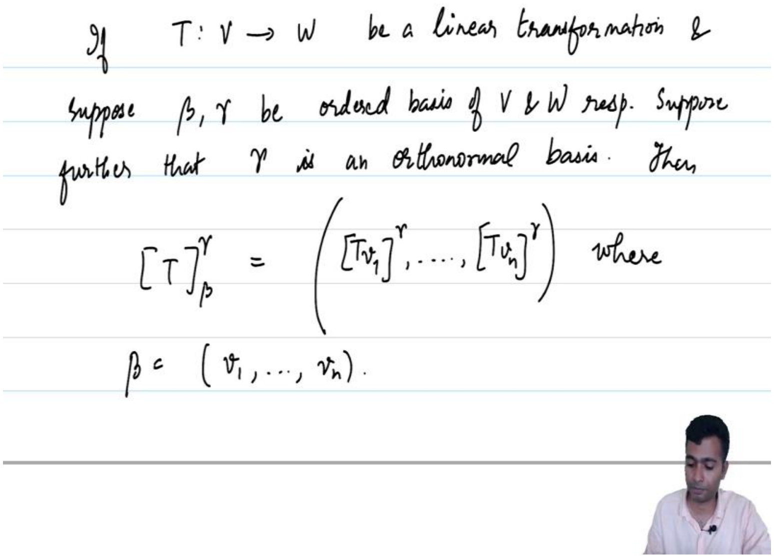

Even better, suppose so if T from V to W be a linear transformation. This also makes computing the matrix of T easier if W is an inner product space, where V and W are vector spaces over F transformation and suppose beta, gamma be ordered basis of V and W respectively.

Suppose further, suppose further that gamma is an orthonormal basis, then if you notice what T beta, gamma is. I will just write it as this, this is just going to be Tv1 gamma Tvn gamma, where beta is equal to v1 to vn is the ordered basis of v. So, we have written the column vector representation of each of Tv1, Tv2 up to Tvn and that will be the columns of our matrix. So, let us now look at a few examples of what we just did.

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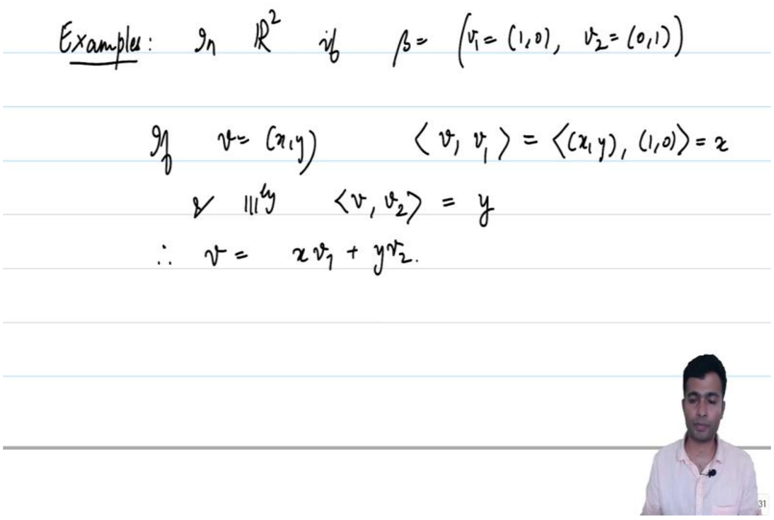

So, we have proved that given an inner product space and given an orthonormal basis. If we have any vector v in, so if we have a linear combination of v1 to vn giving vectors in v, then we know exactly what that linear combination is. So, in particular in R2, if beta is just say 1, 0 and 0, 1 here we do not really need to do this exercise, because it is quite straightforward already.

If you know if v is equal to say x, y then inner product of v with let us call this v1 and let us call this v1 as 1, 0 and v2 is 0, 1 v with v1 will just be the inner product of x, y and 1, 0 which is equal to x and similarly v with v, let me write v1 very carefully, v2 here this is just going to be similarly y and therefore we know that this is, therefore v is equal to x times v1 plus y times v2 which we know. So maybe I should not have written v1, v2 that is the standard basis and I think we had given E1, E2 as the notation for it, but that is okay.

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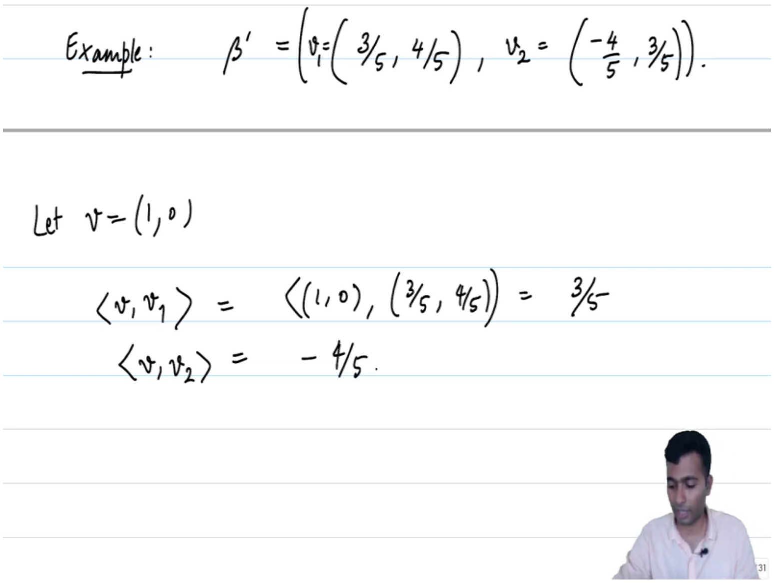

And let us look at something which is more complicated, this was quite straightforward. So let us look at another example where say beta prime is something like 3 by 5, 4 by 5 like we were considering earlier, this vector so let us call it some names, so this is v1 and v2 is minus of 4 by 5 and 3 by 5. So, let us take the simplest vector 1, 0 which we can consider. Let v be equal to 1, 0 and we would like to write v as a linear combination of v1 and v2.

This itself we have slightly increased the complication because you will now have to write down a linear set of linear equations and solve for them to get hold of what the coefficients would be, but now we have a tool. We know that the coefficient will just turn out to be v. So, let me write a to be equal to, what is v1? Let us calculate it explicitly, what is this? This is just 1, 0 inner product of 1, 0 with 3 by 5, 4 by 5 which is equal to 3 by 5 and how about the inner product of v with v2, it will just turn out to be equal, let me not write too much it is just minus of 4 by 5 and therefore we know.

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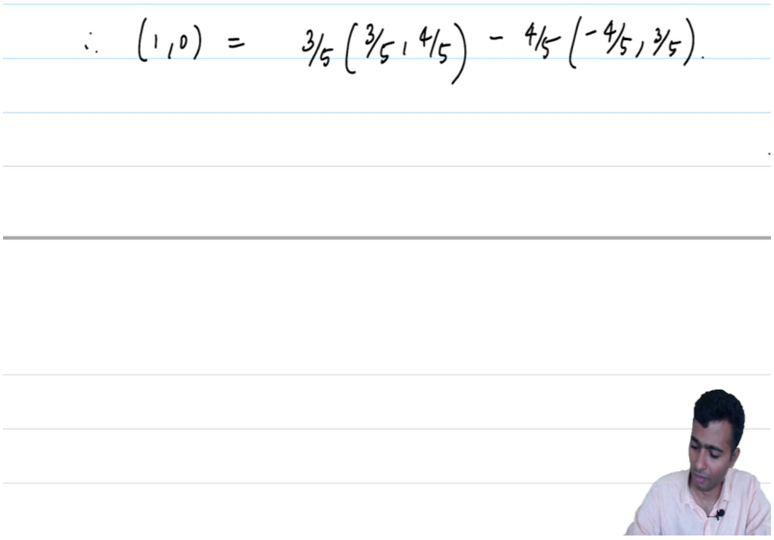

Therefore, by what we just did, this is equal to 3 by 5 times 3 by 5, 4 by 5 minus 4 by 5 times minus of 4 by 5, 3 by 5 and you can check that this is actually the case. So, without going down to solving a set of linear equations this is making our life much easier. It only gets better actually with higher the dimension, the complexity starts coming down accordingly. Let us now look at one more example which is of great importance.

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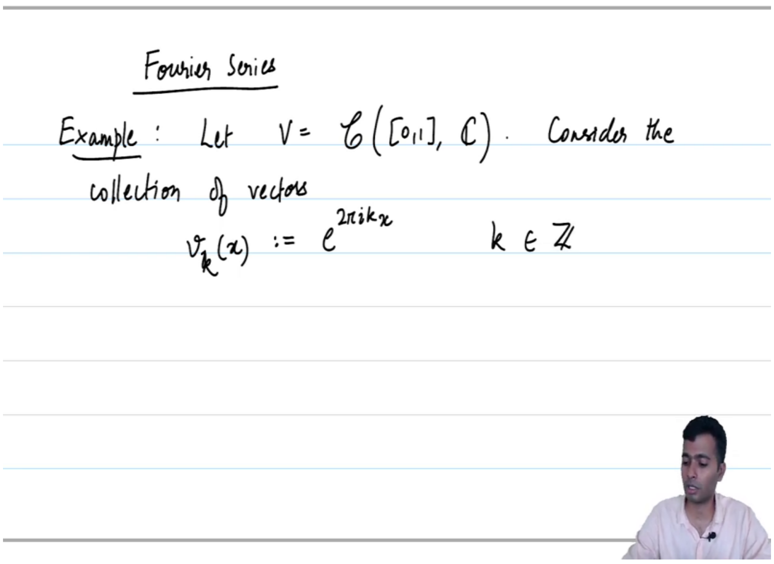

In general, to both the mathematical community and others, so even in engineering this example is of great interest. So, let us consider the vector space, let V be the vector space C, 0, 1, c and let us consider a collection of vectors. So, this example let me just put here the name as Fourier Series. This is a standard a Fourier, a course in Fourier analysis is there in graduate study.

So, because of the vastness of the subject, so this is just skimming over the tip of the iceberg, but nevertheless it is a good place to give an example. So, let us consider the vector space if you recall C 0, 1 to c is the vector space of all continuous functions from 0, 1 complex valued continuous functions and let us consider the following collection of vectors. Let consider the collection of vectors, collection of vectors v1 to be say vk let me put it as vk.

This is just e to the power 2 pi ikx. So, vk is a function, remember so this is vk of x is defined to be e to the power 2 pi ikx and this is being done for each integer. So, not just positive numbers not just natural numbers for k is equal to 0, for k is equal to minus 6, minus 10 for all such integers we are defining vk of x to be e to the power 2 pi ikx. So, let us see what is the, what are the properties of these collection of vectors vk of x, so if you and what is the inner product?

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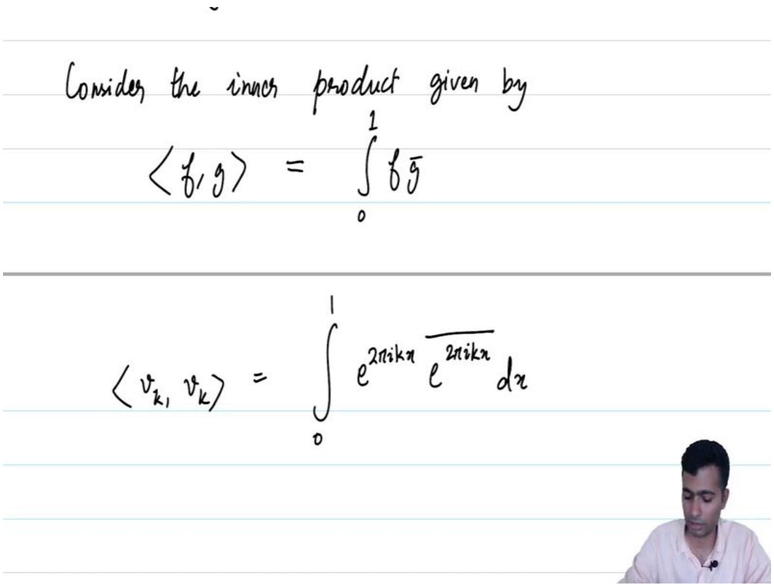

So, let us consider the inner product, so before I mentioned the inner product if I have given vk, does not really make sense, inner product, of course it makes sense, but why these functions that will not be justified. The inner product is given by f, g the inner product of this is already defined once in the previous week. This is just the integral of f, g bar. So, let us see what happens to our vks.

So, if you look at vk, vk for any k this is just going to be integral of vk of x, vk of x bar. So, this is e to the power 2 pi ix times e to the power 2 pi ikx from 0 to 1 and e to the power 2 pi ikx bar dx, this is exactly what our integral will be. But then what is the bar. If you go back and see, this is just cos 2 pi ikx plus i times 2 pi ikx, which will turn out to be.

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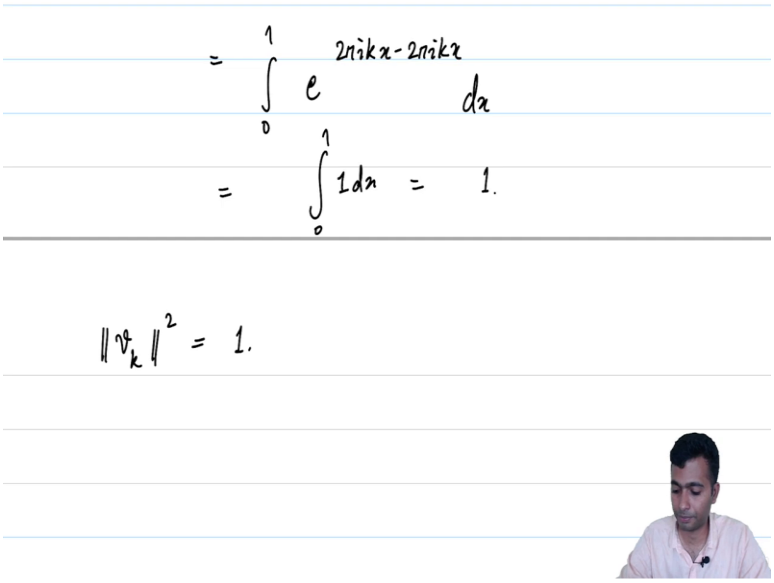

So, this is just, I will leave it to you, to check that this is integral 0 to 1 e to the power 2 pi ikx times e to the power minus 2 pi ikx dx and this will be e to the power 2 pi ikx minus 2 pi ikx dx, which is e to the power 0, which is integral of 1 dx, where the limits are from 0 to 1 which will just turn out to be 1. So, length of vk square is equal to 1. So, this is already a unit vector, it is normalized. This is a normalized vector. How about the inner product of vk with vj.

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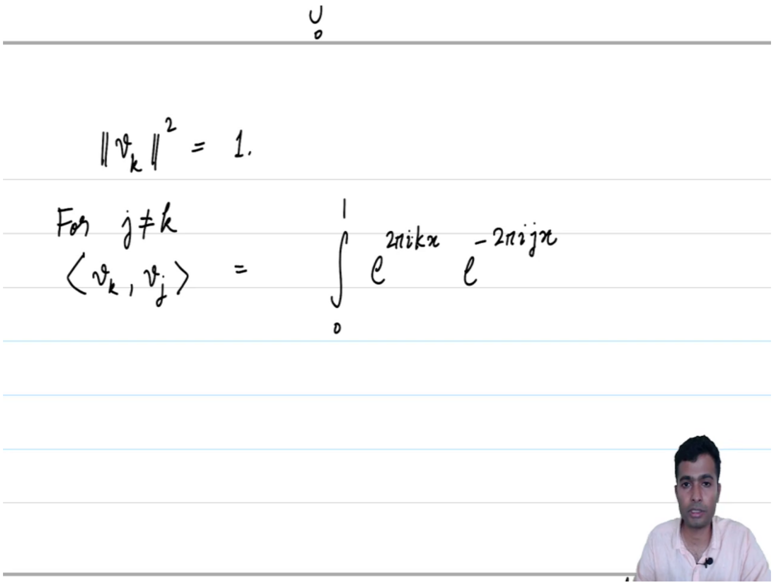

So, let us look at what is vk inner product with vj for j not equal to k. So, for j not equal to k. Let us look at what this is, this is just inner product of integral from 0 to 1 e to the power 2 pi ikx, let me jump a few steps and directly write as e to the power 2 pi i minus of e to the power 2 pi ijx. So, notice that the I featuring in the exponent here is the square root of minus 1 and it is not an index. However, k and j are indices.

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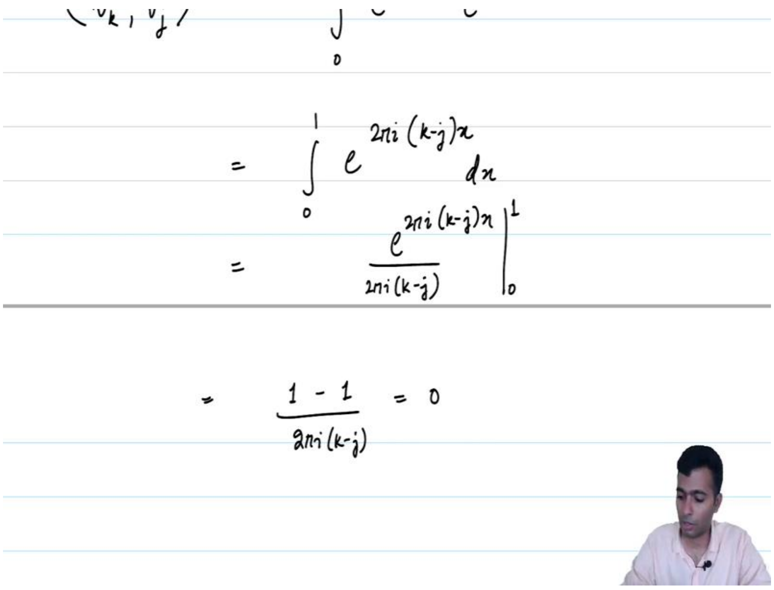

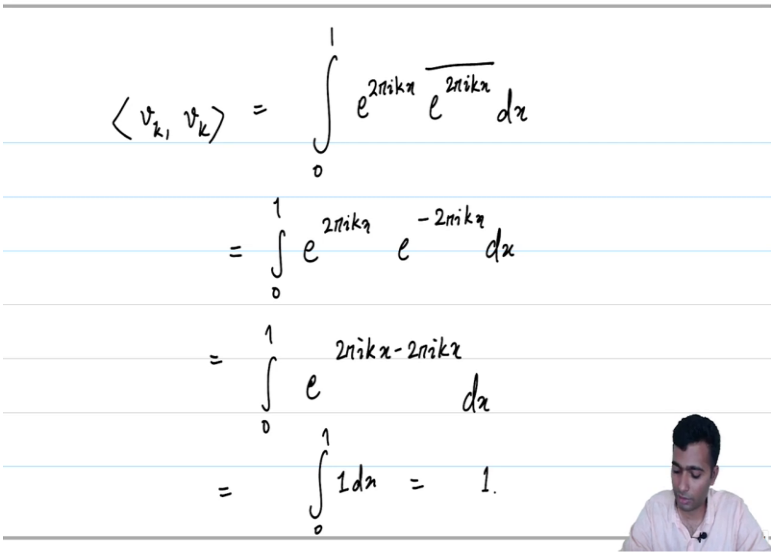

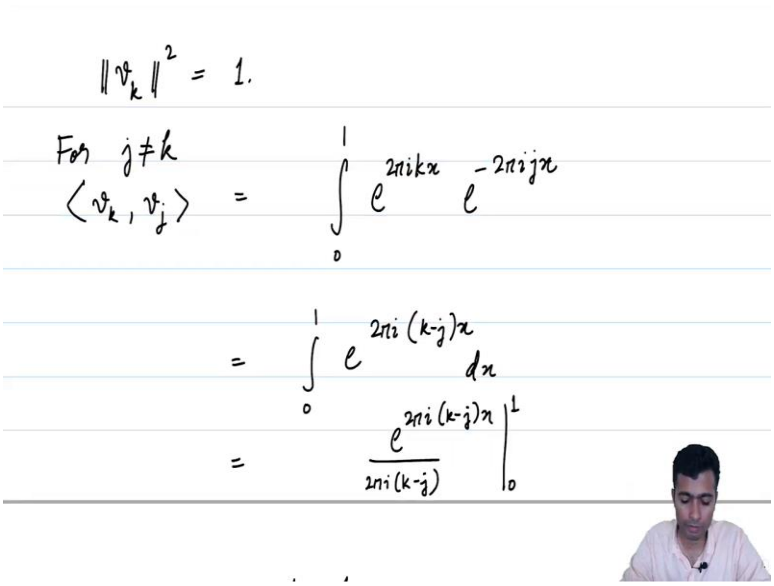

So, this, so this is equal to integral 0 to 1 e to the power 2 pi i times k minus j times x dx and this is just e to the power 2 pi ik minus jx by 2 pi i times k minus j from 0 to 1. This is exactly what the integral will turn out to be. You can check that this is just 1 minus 1 by 2 pi ik minus j which is equal to 0.

The one, the first one coming because e to the power 2 pi i times, any integer is 1 and e to the power 0 is also 1. So, the second one is from e to the power 0 and this is equal to 0. So, what we have established let me just go back and what we just proved. We showed that the inner product of vk which itself is 1 and the inner product of vk with vj is 0 for all k not equal to j.

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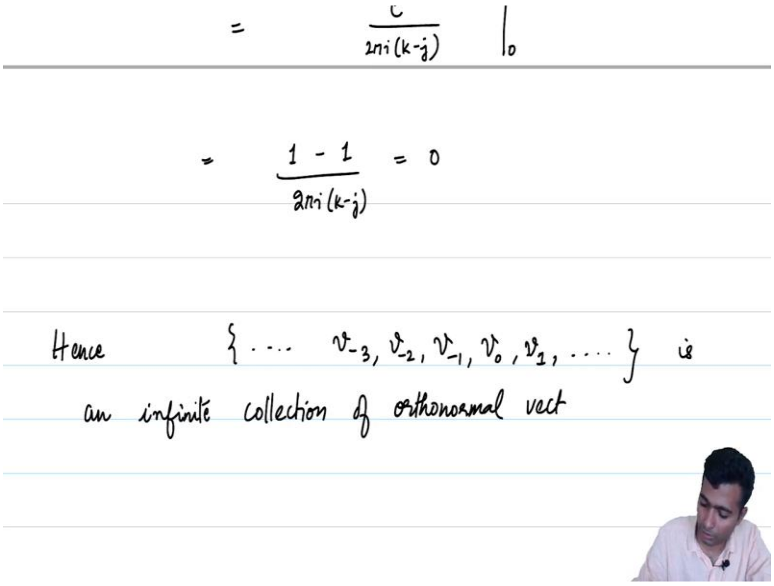

So, hence the collection of vk or let me put it this way v minus 3, v minus 2, v minus 1, v0, v1 dot, dot, dot. This is an infinite collection of orthonormal vectors. So, we will not talk about Fourier Series right now because that is beyond the scope of this course, we will have to develop convergence of an infinite series and such things, but rather what we will do is let us look at specific subspace.

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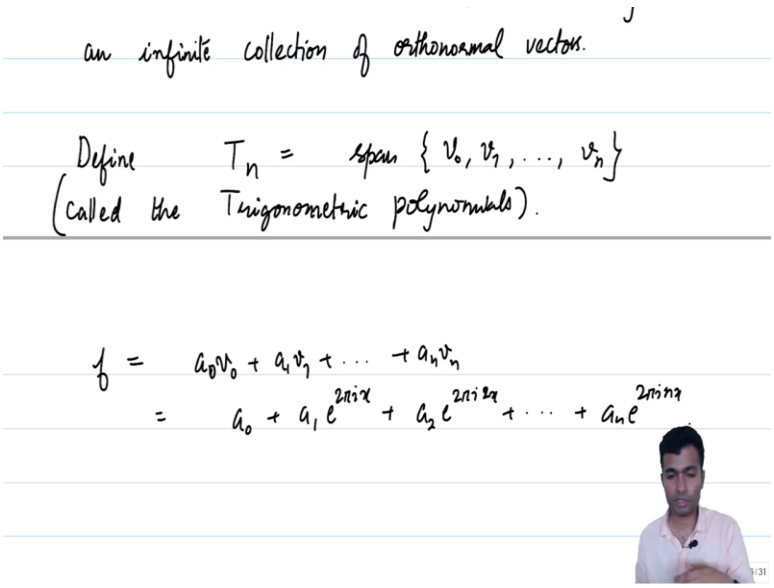

So, define Tn to be the span of v0, v1 up to vn. So, these are called the trigonometric polynomials called the trigonometric polynomials. So, what will be a typical element here, so any element here. So, a typical element f will be of the type a1 a0 v0 plus a1 v1 plus up to an vn and if we write it down, this is just a0 plus a1 times e to the power 2 pi i x plus a two times e to the power 2 pi i into 2x and up to an e to the power 2 pi inx. So, this is something like a polynomial.

So, a0 plus a1x a1 times say t plus a2 times t square and so on. So, that is why it is called trigonometric polynomials. So, let us now implement whatever we have just developed to an arbitrary element in Tn.

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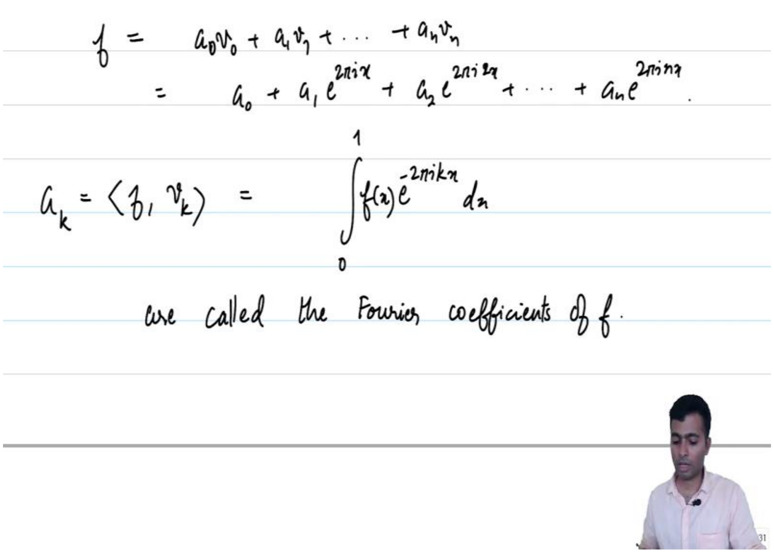

So, then for f inner product with vk, this is just going to give you the, by whatever we just developed this is our ak, but what is f inner product with vk. This is inner product of f with e to the power 2 pi ikx dx. So, I should be little more careful. This is 0 to 1 f of x e to the power 2 pi i minus e to the power minus 2 pi i kx dx. These are called the Fourier coefficients of f, are called the Fourier coefficients of f.

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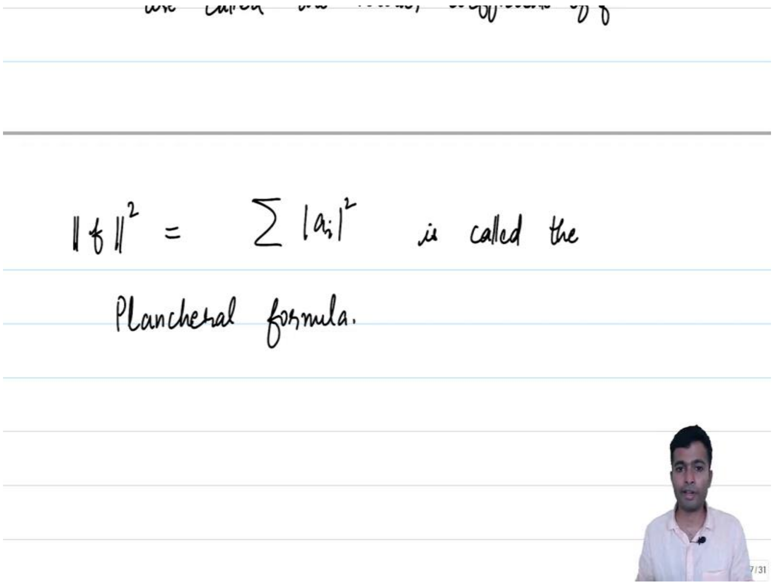

And we also have the famous formula which says that, if you look at the length of norm f square is equal to summation mod ai square. This if you write down with ai is now known explicitly. If you write it down like this, this is called the Plancherel formula. So, in the next video, we will discuss orthogonality in much greater detail by talking about when two vectors of spaces can be orthogonal to each other.