Linear Algebra
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Lecture 38
Problem Session

So, we begin this week by a brief problem session on the material which was covered in week six and week seven of this course. So, we start by looking at the first problem, wherein we try to write a given matrix as a product of elementary matrices. So, let us begin.

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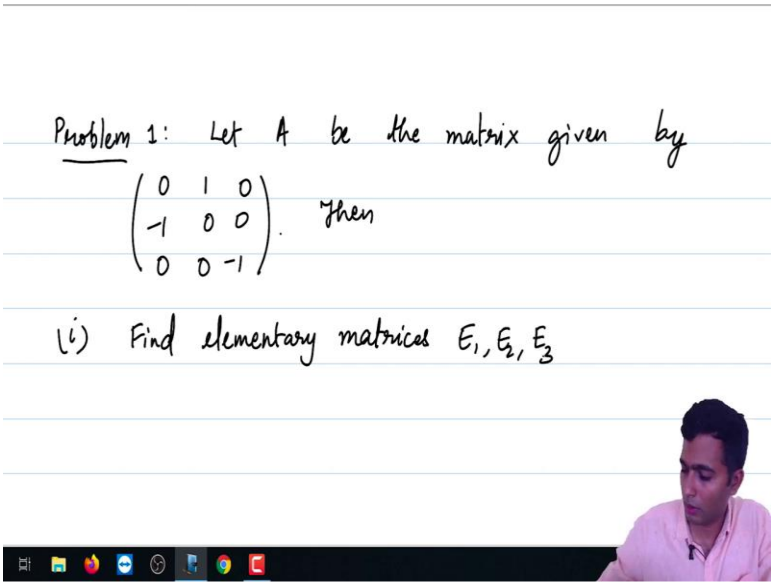

Problem 1: Let A be the matrix given by

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad \text{Then}$$

(i) Find elementary matrices E_1, E_2, E_3

So, problem one. So, let A be the matrix given, let A be the matrix given by 0, 1, 0, minus 1, 0, 0, 0, 0, minus 1. Then the first one says find elementary matrices such that we can write as a product of these elementary matrices. Such that A is equal to E1 E2 E3.

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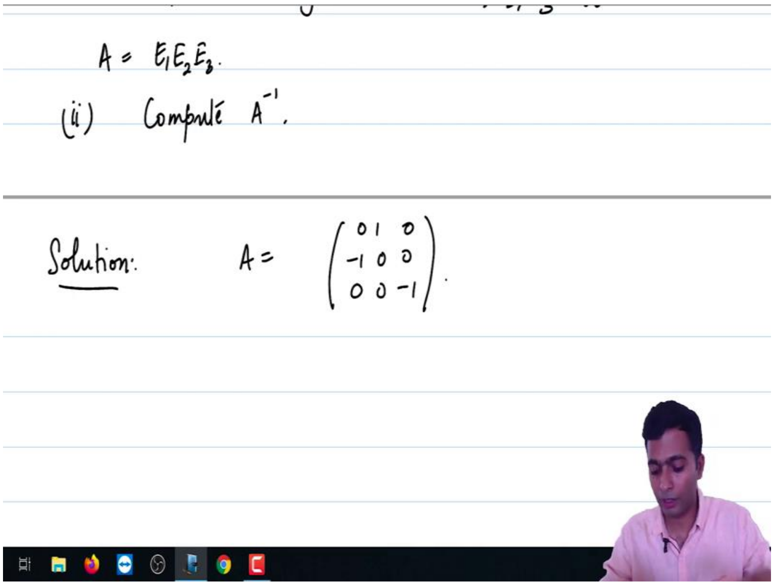
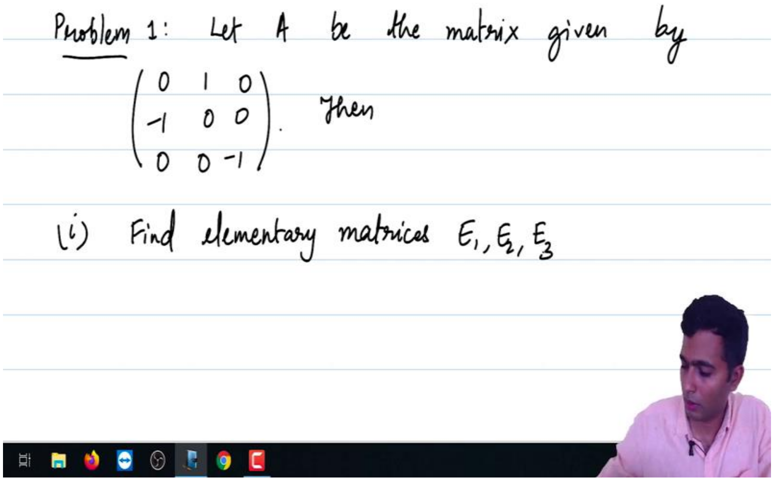


$A = E_1E_2E_3$.

(ii) Compute A^{-1}.

Solution: $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.



Problem 1: Let A be the matrix given by $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then

(i) Find elementary matrices E_1, E_2, E_3

That is the first part of the problem and the second part of the problem is to compute A inverse. So, needless to say, the second part is actually simple corollary to the first part. So, let us focus on solving this problem. Solution, so let us recall what our A is, you recall A is given by, let us just go up and check out what A was 0, 1, 0, minus 1, 0, 0, 0, 0, minus 1 the rows of A 0, 1, 0, minus 1, 0, 0, 0, 0, minus 1. So, let us see, what are the elementary matrix operations that we will be applying to A in order to get hold of A as a product of these elementary matrices. So, the strategy we would follow is the following.

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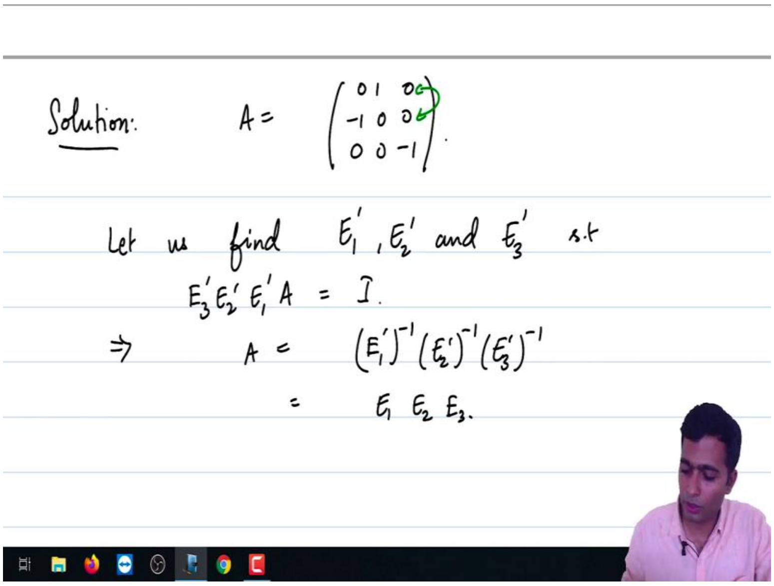

Solution:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let us find E_1', E_2' and E_3' s.t

$$E_3' E_2' E_1' A = I.$$

$$\Rightarrow \quad A = (E_1')^{-1} (E_2')^{-1} (E_3')^{-1}$$

$$= E_1 E_2 E_3.$$

Let us find E1 prime, E2 prime and E3 prime such that, after doing a row reduction of A we will get the identity such that E3 prime, E2 prime, E1 prime times A is the identity matrix, then this would imply that A is the matrix which is given by E1 prime inverse, E2 prime inverse, E3 prime inverse. But we know that the inverse of an elementary matrix is again an elementary matrix.

So, this will just turn out to be this is exact y what we want, this is the E1, E2, E3 that we want and we will come to the inverse of A in the second part of the problem. So, let us focus on this. So, our immediate goal is to find E1 prime, E2 prime and E3 prime such that after the relevant row operations we get back the identity. So, the first one I would say is to exchange these two rows the first row and the second row and what was the elementary matrix that was needed to exchange the first and the second row.

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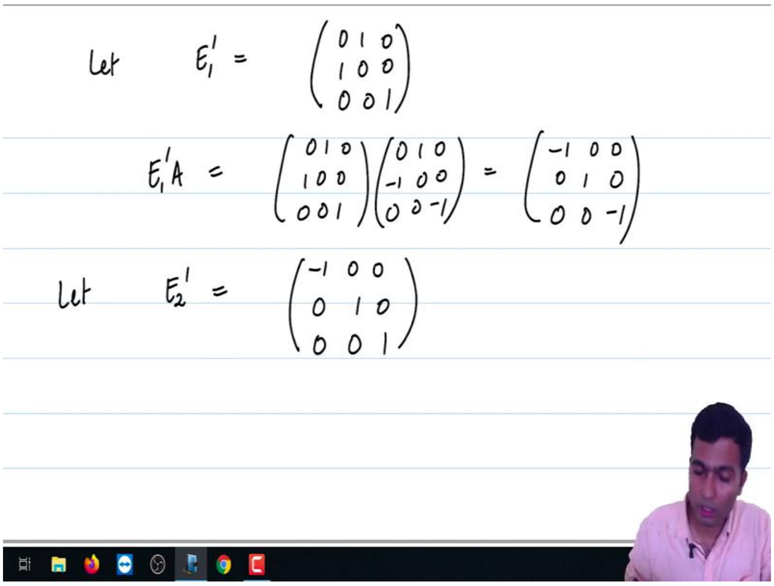

Let $E_1' = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$

$$E_1'A = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

Let $E_2' = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

So, let E1 prime be equal to, so the first row and the second row. So, this will be 0, 1, 0, 1, 0, 0, 0, 0, 1 this is precisely the matrix you need to multiply to exchange the first and the second row. So, so let us check what is E1 prime times A this will just be 0, 1, 0, 1, 0, 0, 0, 0, 1 times our matrix A which I think is 0, 1, 0, minus 1, 0, 0, 0, 1, 0, minus 1, 0, 0, 0, 0, minus 1 and after the multiplication, this will just turn out to be equal to minus 1, 0, 0, 0, 1, 0, 0, 0, minus 1. So, we have exchanged the first row and the second row. The next is to multiply the first row by minus 1. So, let E2 prime be equal to minus 1, 0, 0, 0, 1, 0, 0, 0, 1.

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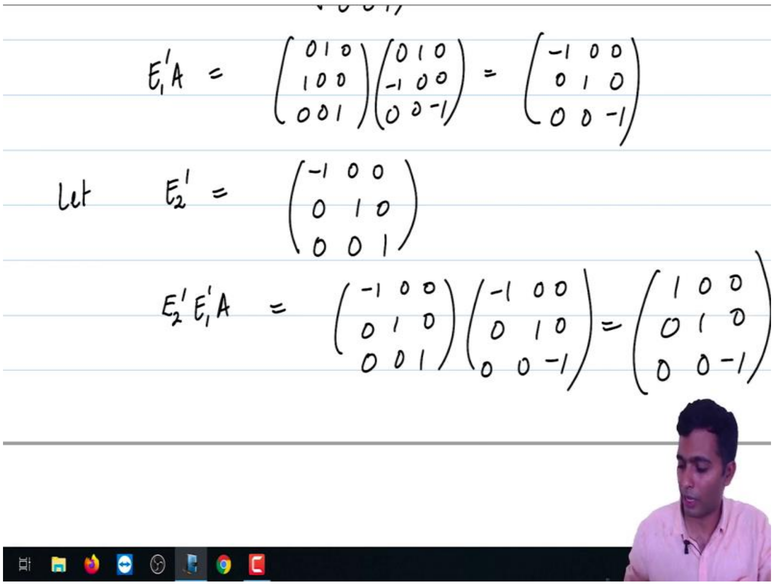

$$E_1'A = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

Let $E_2' = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

$$E_2'E_1'A = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

And now let us check what is E2 prime times E1 prime A which is minus 1, 0, 0, 0, 1, 0, 0, 0, 1 times minus 1, 0, 0, 0, 1, 0, 0, 0 minus 1, which will just be 1, 0, 0, 0, 1, 0, 0, 0, minus 1. So, we are quite close now.

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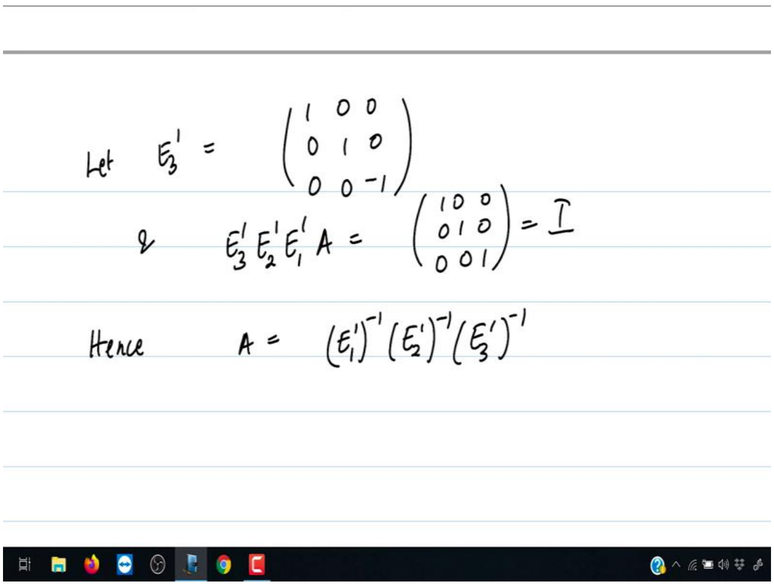

Let $E_3' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

& $E_3' E_2' E_1' A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{I}$

Hence $A = (E_1')^{-1} (E_2')^{-1} (E_3')^{-1}$

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And then let E3 prime be equal to, you need to multiply the third row with minus 1. So, this is going to be 1, 0, 0, 0, 1, 0, 0, 0, minus 1 and can check that E3 prime E2 prime, E1 prime A is equal to 1, 0, 0, 0, 1, 0, 0, 0, 1 which is the identity matrix. So, we have essentially obtained what E3 prime E2 prime and E1 prime is. So hence A is nothing, but the matrix E1 prime inverse, E2 prime inverse and E3 prime inverse which is quite straight forward to compute because what was E1 let us look at what E1 was E1 prime was.

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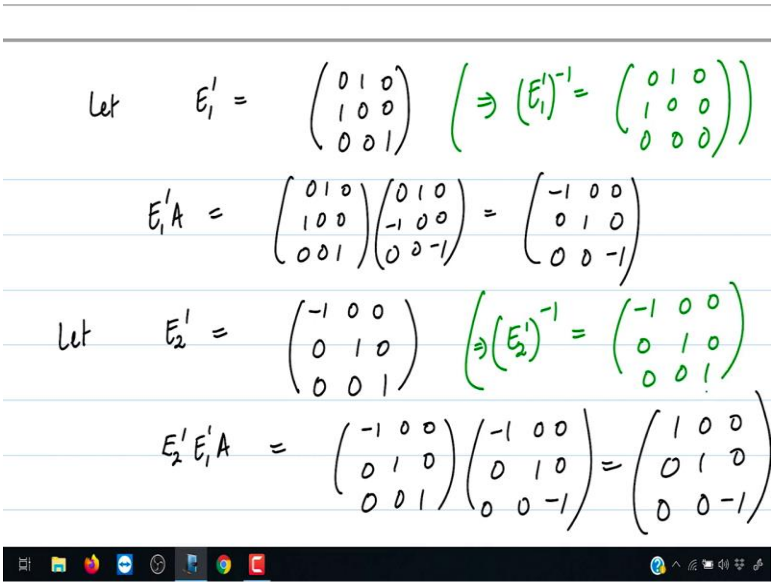

Let $E_1' = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ $\left(\Rightarrow (E_1')^{-1} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\right)$

$E_1'A = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

Let $E_2' = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $\left(\Rightarrow (E_2')^{-1} = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$

$E_2'E_1'A = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

Well I will put it in green here this would imply E1 prime inverse is the same matrix. This you should check we have in fact seen that if you exchange the first and the second row twice you will get back the same identity matrix. Similarly, E2 prime inverse will also be the same matrix, because if you multiply, if c was the diagonal entry 1 by c would have come up which is the same here as minus 1. So, this is going to be our E2 prime inverse.

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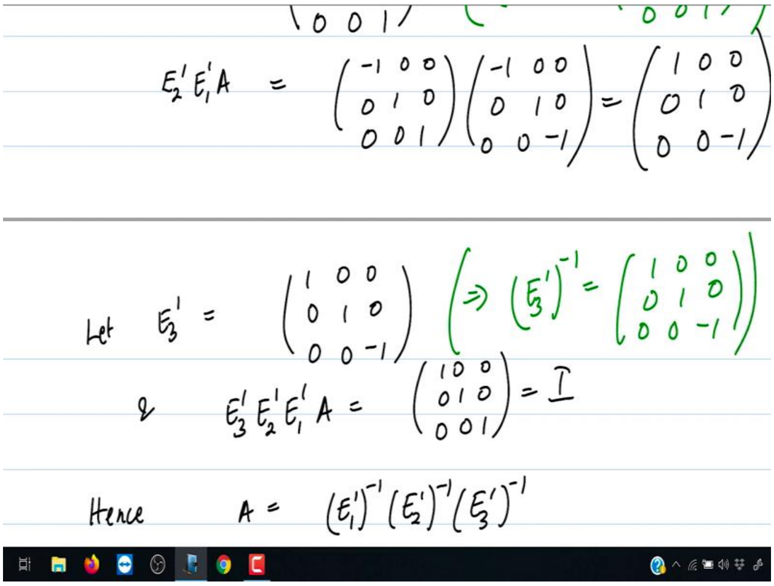

$E_2'E_1'A = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

Let $E_3' = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ $\left(\Rightarrow (E_3')^{-1} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right)$

& $E_3'E_2'E_1'A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \underline{I}$

Hence $A = (E_1')^{-1}(E_2')^{-1}(E_3')^{-1}$

And how about E3 prime inverse, E3 prime inverse is again the same matrix as E3.

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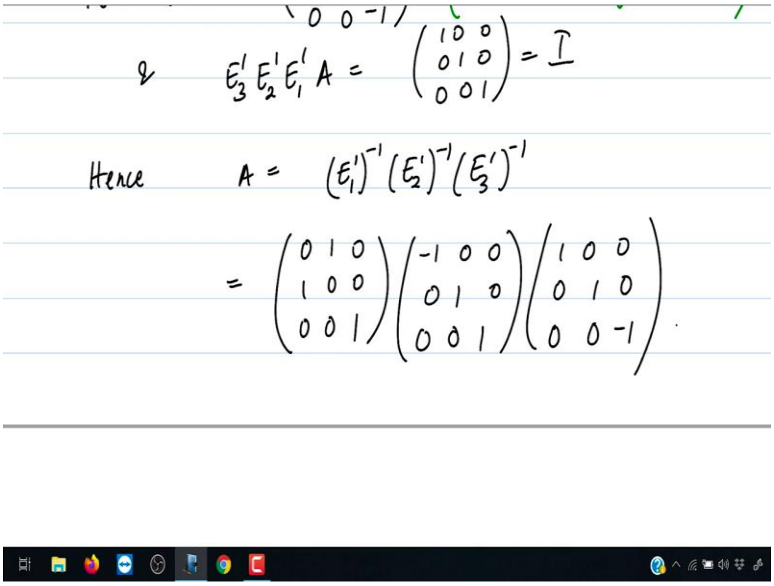

$0 \quad 0 \quad -1$

& $E_3' E_2' E_1' A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \underline{I}$

Hence $A = (E_1')^{-1}(E_2')^{-1}(E_3')^{-1}$

$$= \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

Therefore, we know exactly what this object is. So this is going to be equal to first one was 0, 1, 0, 1, 0, 0, 0, 0, 1 times E2 was minus 1, 0, 0, 0, 1, 0, 0, 0, 0 and third one was 0, 1, 0, 0 this should not be 0, I am sorry this is 1, this is 0, 1, 0, 0, 0, minus 1. So, these are precisely the matrices we were looking for.

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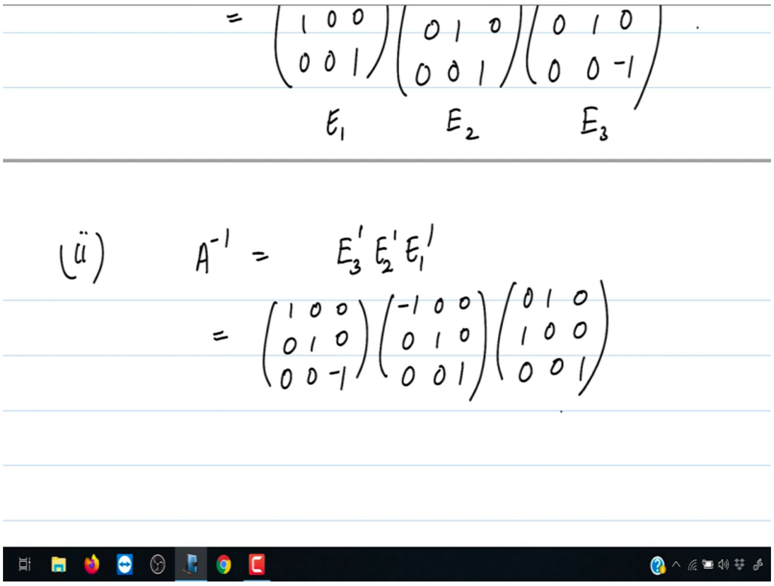
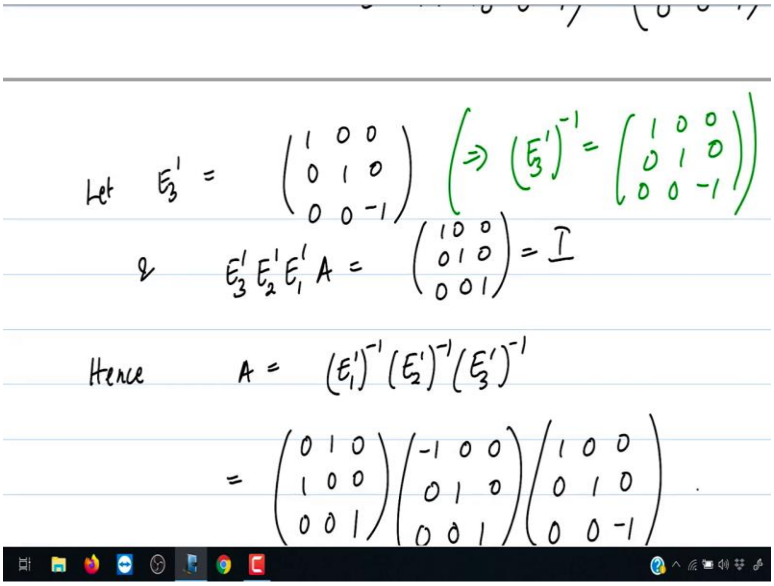


$$= \begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&-1\end{pmatrix}.$$

$E_1 \qquad E_2 \qquad E_3$

(ii) $A^{-1} = E_3' E_2' E_1'$

$$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$



Let $E_3' = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ $\left(\Rightarrow (E_3')^{-1} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right)$

& $E_3' E_2' E_1' A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I$

Hence $A = (E_1')^{-1}(E_2')^{-1}(E_3')^{-1}$

$$= \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

This is our E1, this is our E2 and this is our E3, that is good. Now, let us look at what is A inverse, well A inverse was. So, this is the second part, which leads compute to A inverse. A inverse if you notice is from here just the product of E3 prime, E2 prime, E1 prime. So, A inverse is the product of E3 prime, E2 prime and E1 prime which is let us just write it down E3 prime was 1, 0, 0, 0, 1, 0, 0, 0, minus 1.

E2 prime was minus 1, 0, 0, 0, 1, 0, 0, 0, 1 and how about E1 prime, E1 prime was the ones which exchange the first and the second rows. So, this is going to be exactly this. So, when you multiply something from the right, some elementary matrix from the right it is just going to be a column operation.

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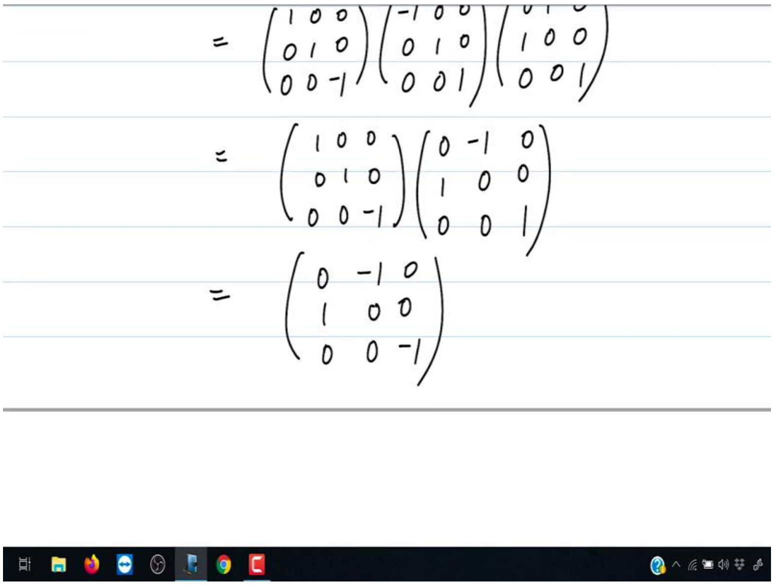

$$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}$$

So, this is just going to be 1, 0, 0, 0, 1, 0, 0, 0, minus 1 times the first and the second column get interchanged, so this is going to be minus 1, 0. So the first column now will be this second column will be this, the third column remains unchanged and how about this we can think of this as a operation from the left, so this is going to be 0, minus 1, 0, 1, 0, 0, 0, 0, minus 1. So, this is going to be the inverse of our given matrix. You can check that this indeed is the inverse it should just get hold of the A which was 0, 1, 0, minus 1, 0, 0.

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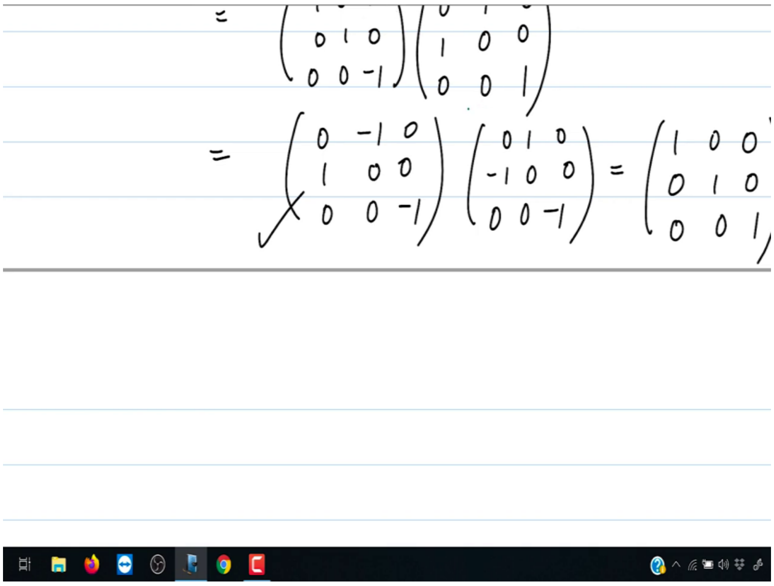
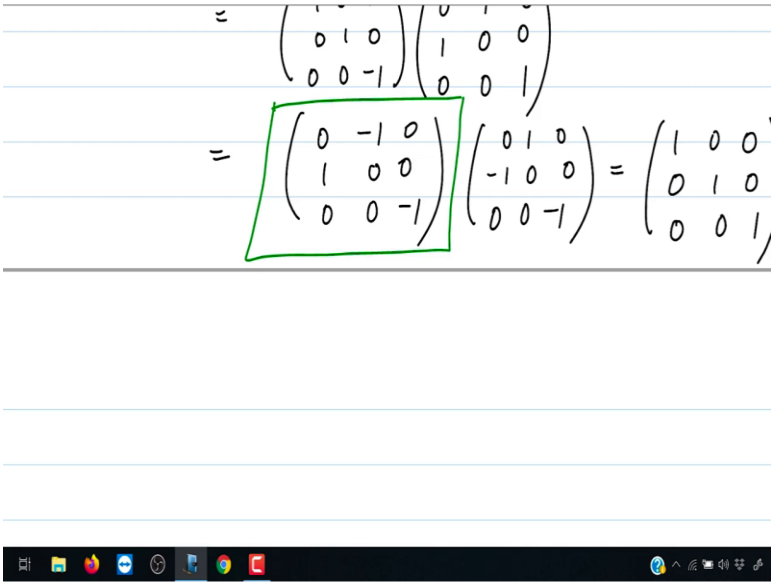


$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$= \boxed{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let us say 0, 1, 0, minus 1, 0, 0, 0, 0, minus 1. If you notice this is the first one is identity 0, 0, 0 the second one is 1 third entry 0, 0, 0, 1 yes this is indeed the inverse. So, we have indeed obtained the correct inverse. So, let me put this in green for observing that this is our A inverse. So, even in the case of very simple matrices, the computations tend to be become a bit tedious.

But these are concrete methods with which we will be able to compute a simplified echelon form from the given matrix in this case which turns out to be the identity matrix. The next problem deals with the relation between a matrix B and a matrix obtained by realizing B as a block matrix entry. So, it will be more clear when I write down the problem for you.

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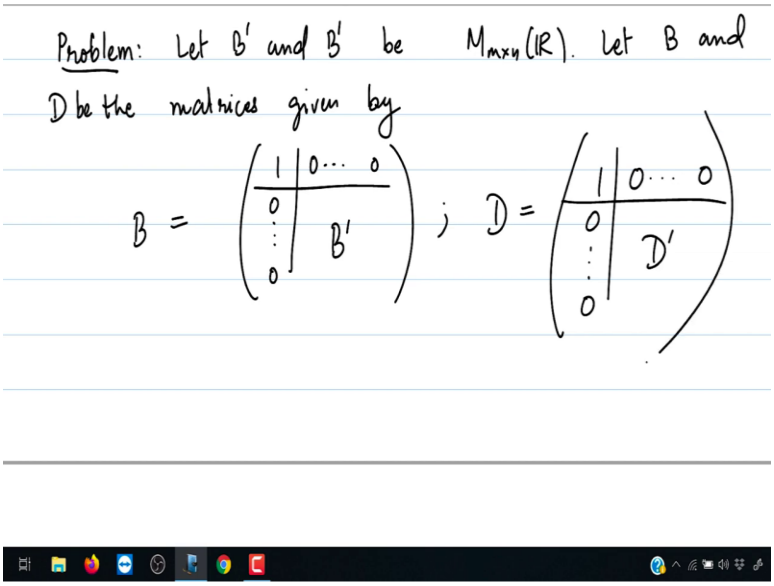

Problem: Let B' and B' be $M_{m \times n}(\mathbb{R})$. Let B and D be the matrices given by

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right) ; \quad D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right)$$

So, let B prime and D prime be m cross n matrices with entries in R. So, let me just write it down in this manner m cross n of R. Let us obtain two new matrices let B and D be obtained as below or let me not say obtained as well, be the matrices let B and D be the matrices given by B is equal to 0, 1 then the remaining entries are 0 in both the column and the row and there is the B prime here and how do we get D, D is obtained in the exact similar manner 0 here and then there is a D prime here. So, the problems demands us to prove that if you can obtain D prime from B prime by an elementary row operation, then you can obtain D from B by an elementary row operation.

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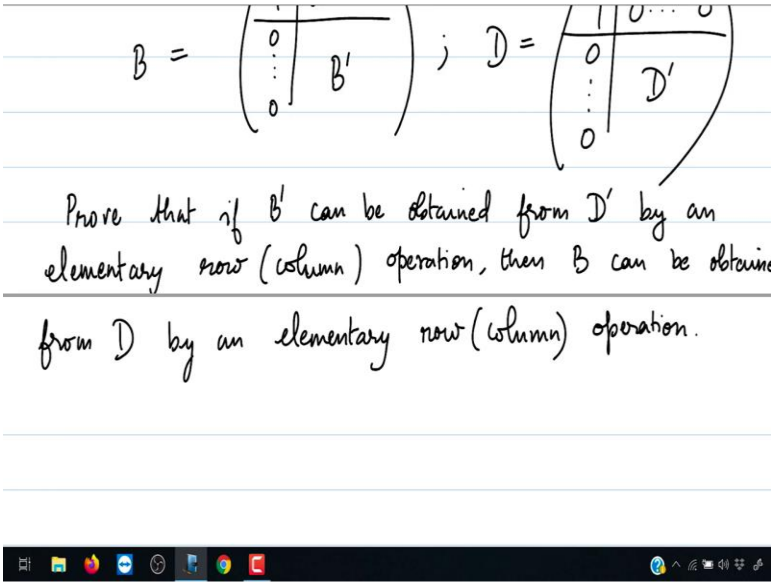

$$B = \begin{pmatrix} 0 & \\ \vdots & B' \\ 0 & \end{pmatrix} ; \quad D = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & D' \\ 0 & \end{pmatrix}$$

Prove that if B' can be obtained from D' by an elementary row (column) operation, then B can be obtaine

from D by an elementary row (column) operation.

So, prove that if B prime can be obtained from D prime by an elementary row or rather for that matter even a column operation, then B can be obtained from D by an elementary row operation. Further conclude that if rank of B prime is, if rank of B is say r then rank of B prime is r minus 1.

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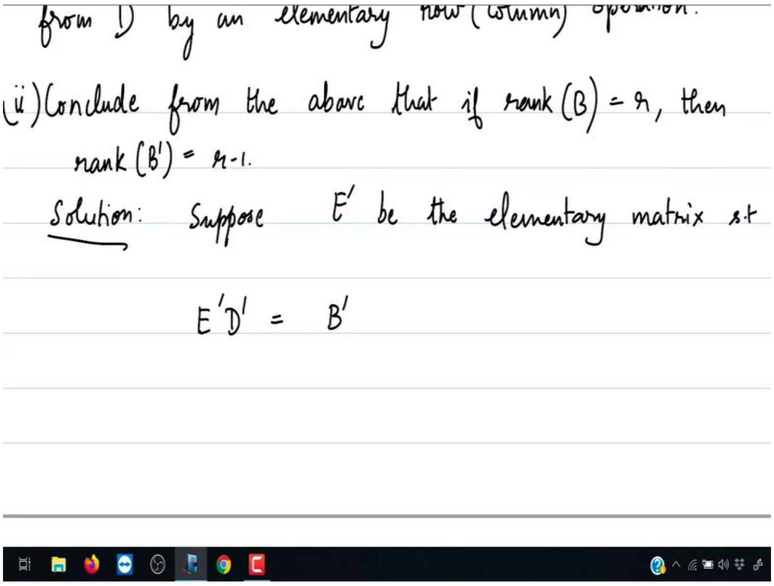
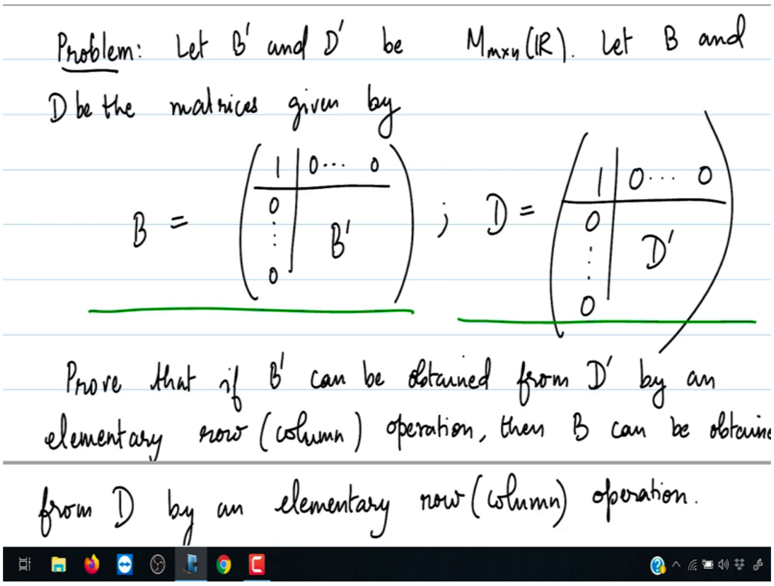


from D by an elementary row (column) operation.

(ii) Conclude from the above that if $\text{rank}(B) = r$, then $\text{rank}(B') = r-1$.

Solution: Suppose E' be the elementary matrix s.t

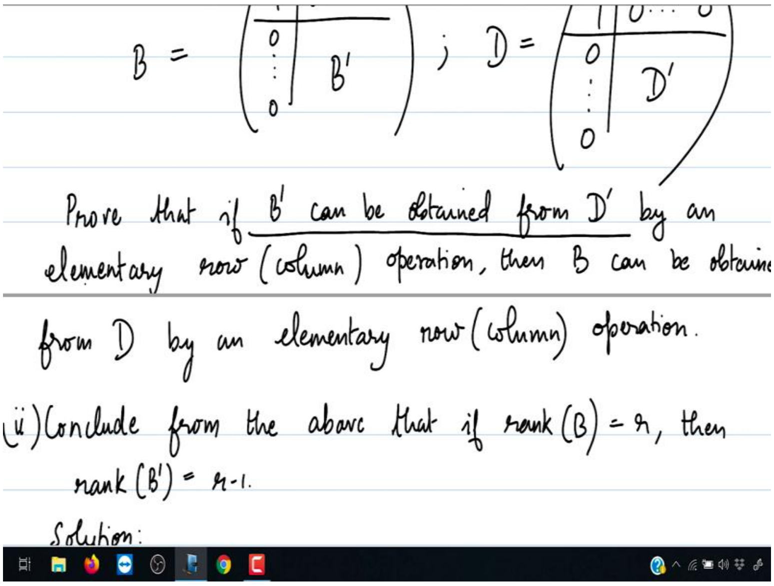
$$E'D' = B'$$



Problem: Let B' and D' be $M_{m\times n}(\mathbb{R})$. Let B and D be the matrices given by

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right) \; ; \; D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right)$$

Prove that if B' can be obtained from D' by an elementary row (column) operation, then B can be obtaine from D by an elementary row (column) operation.



$$B = \left(\begin{array}{c|ccc} & & & \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right) \; ; \; D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right)$$

Prove that if B' can be obtained from D' by an elementary row (column) operation, then B can be obtaine from D by an elementary row (column) operation.

(ii) Conclude from the above that if $\text{rank}(B) = r$, then $\text{rank}(B') = r-1$.

Solution:

So, conclude from the above that if, so this is let me call it 2. Rank of B prime is equal to, rank of B is equal to r then rank of B prime is equal to r minus 1. So, let us look at the problem once more and try to see what it says. The problem says that, you are say given two matrices.

So, let B prime and D prime was a problem. So, suppose B prime and D prime are two m cross n matrices such that you get B prime by applying a row operation to D prime. This problem says that suppose our B is obtained from B prime in this manner and say D is obtained from D prime in this manner, then we have a, some other obviously it will not be the same, the row operations applied to B prime is by an m cross m matrix.

Now in the case of B, there will be an m plus 1 cross m plus 1 matrix which will give you row operation from well B to D or rather D to B, B is obtained from B by a elementary row operation. Let us prove this and understand exactly how the solution works.

So, a solution, so what is the information that we have the information that we have is that B prime can be obtained from D prime by an elementary. So, we will do the solution for a row operation, the column operation proof will be exactly the same. So, suppose E be the elementary matrix, so notice that this is going to be this is going to be a m cross n elementary matrix, such that E, so let me call it E prime, E prime D prime is equal to B prime.

So, that is precisely what B statements says. B prime can be obtained from D prime by an elementary row operation. We would like to show that D can be obtained or rather B can be obtained from D by an elementary row operation. Obviously, this elementary operation E prime will play an important role is deciding or finding out what the elementary operation E corresponding to which you know you get B from D. So, we will see how it works. So, recall that B is obtained by having 1 in the first 1, 1 entry 0 elsewhere in the first row and the first column and then in the other block there is a B prime.

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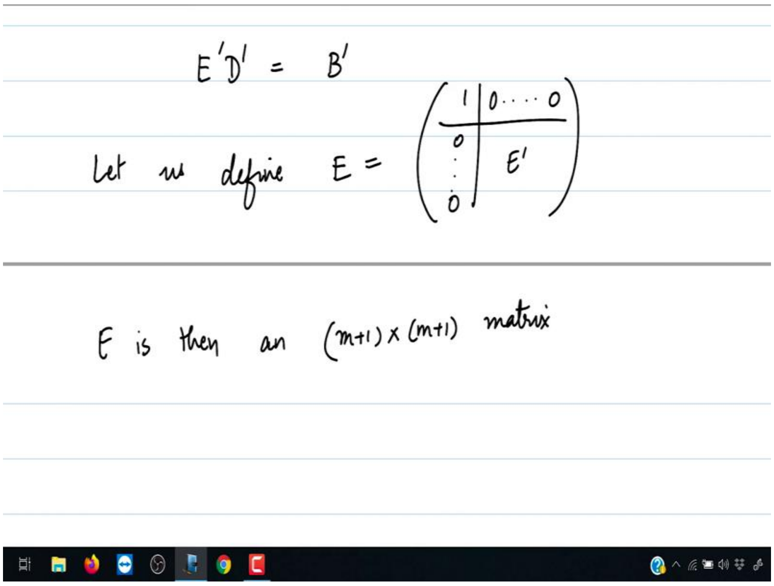

$$E'D' = B'$$

Let us define $E = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & E' \\ 0 & \end{array}\right)$

E is then an $(m+1) \times (m+1)$ matrix

So, let us define let us define E to be defined as very similar manner, let us define it to be once here and then this is going to be E prime here. So, notice that E is then an m cross an m plus 1 cross m plus 1 matrix and let us try to see what E times D is.

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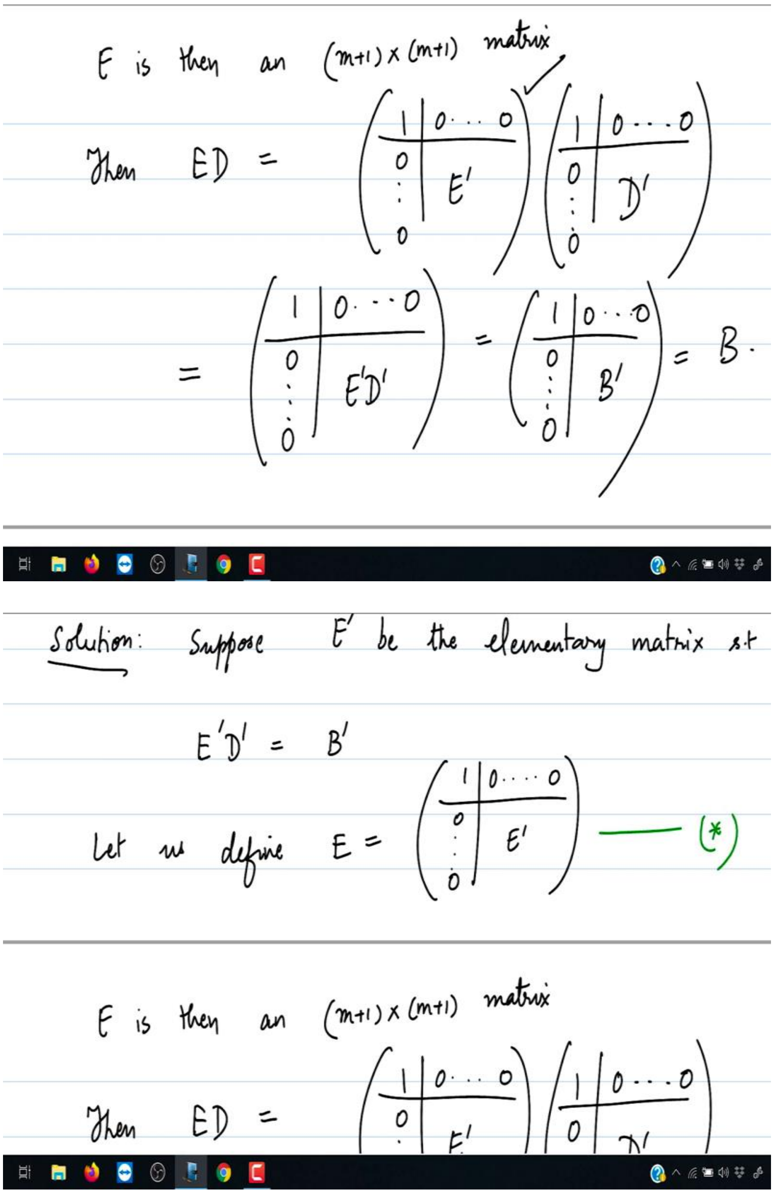


E is then an $(m+1) \times (m+1)$ matrix

Then $ED = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & E' \\ 0 & \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & D' \\ 0 & \end{array}\right)$

$= \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & E'D' \\ 0 & \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B' \\ 0 & \end{array}\right) = B.$

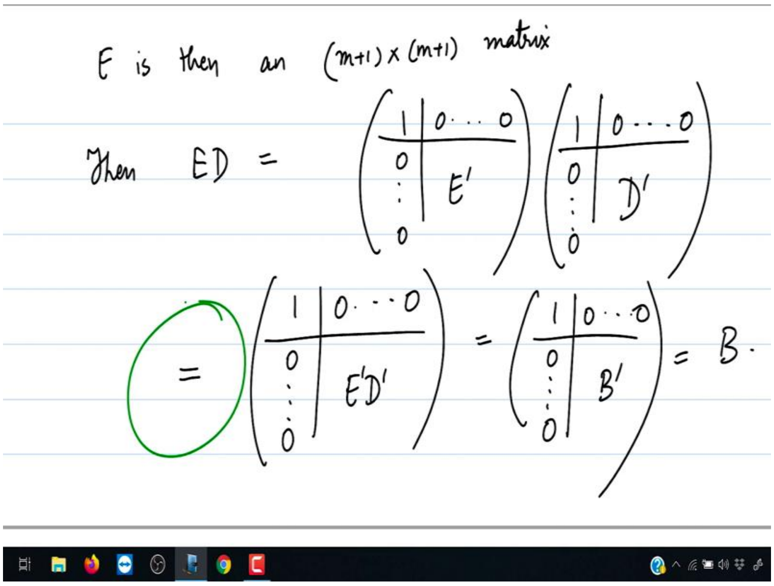
Solution: Suppose E' be the elementary matrix s.t

$E'D' = B'$

Let us define $E = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & E' \\ 0 & \end{array}\right)$ —— $(*)$

E is then an $(m+1) \times (m+1)$ matrix

Then $ED =$



E is then an $(m+1) \times (m+1)$ matrix

$$\text{Then } ED = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & E' \\ 0 & \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & D' \\ 0 & \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & E'D' \\ 0 & \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B' \\ 0 & \end{array}\right) = B.$$

Then E times D is what. So, let me just write down in figures what this is, this is 1, 0 here in the first row and first column and then there is an E prime here and then there is 1 and there are 0s here and there will be a D prime.

So, let us see, so I would actually ideally like to leave it as an exercise for you to check that this is going to get a matrix for this type. This is going to be E prime D prime, which is nothing but E prime times D prime is B prime which is exactly equal to B. So, basically from our elementary matrix E prime, we have defined, this particular matrix, as it is done here, green star. So, there is one step which I am now circling in green wherein I have asked you to check the relevant details. I will just give an indicator of what the check that is needed to look like.

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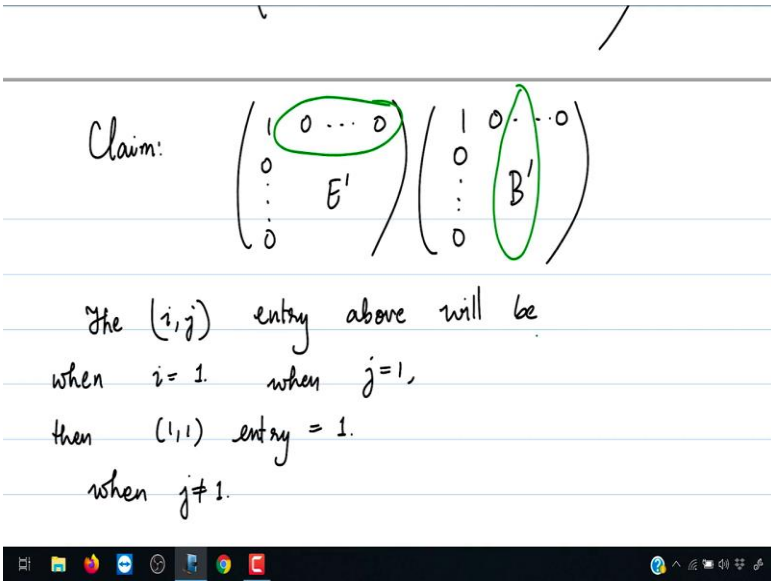

Claim: $$\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & E' \\ 0 & \end{pmatrix}\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B' \\ 0 & \end{pmatrix}$$

The (i,j) entry above will be

when $i = 1$. when $j = 1$,

then $(1,1)$ entry $= 1$.

when $j \neq 1$.

This matrix 1, 0s, 0s, here and then E prime times 1, 0s, 0s is here and then the B prime, this let us see what the ijth entry will look like. So, the ijth entry, the i j entry above will be as, will be, let us see what happens when i is equal to 1 when i is equal to 1. So, that means the first row is being considered in the product matrix first product, let us see what happens. So, when j is equal to 1, then the 1, 1 entry is equal to 1 and when j is not equal to 1, what will happen.

What does that mean we are considering I am putting some j is the column in this part and when you multiply it to the first row, the first entry is 0 in this column, which gets multiplied with 1 and gives you 0 and the remaining entries here are 0s, which will get multiplied with the remaining entries and which will give you back 0. So, the remaining entries will just turn out to be 0.

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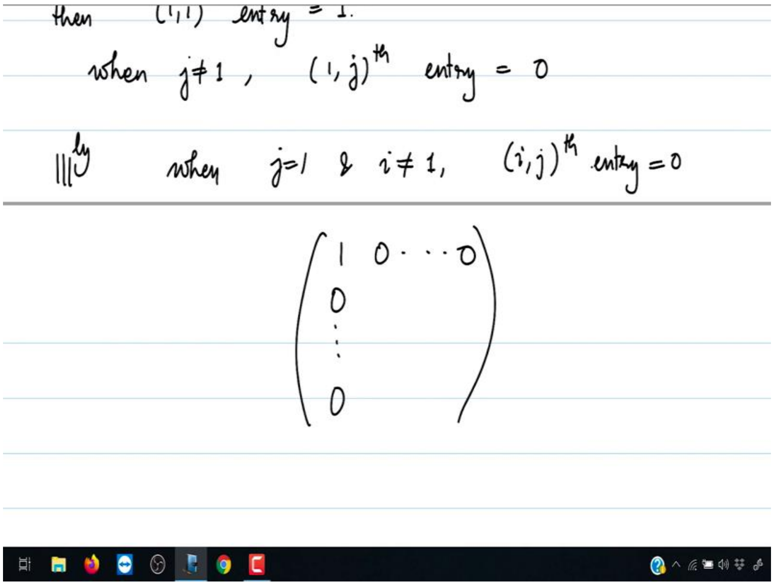

then (1,1) entry = 1.

when $j \neq 1$, $(1,j)^{th}$ entry $= 0$

$|||^{ly}$ when $j=1$ & $i \neq 1$, $(i,j)^{th}$ entry $=0$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

So, hence when j is not equal to 1, 1, jth entry is equal to 0. Similarly, when j is equal to 1 and i is not equal to 1 i, jth entry is equal to 0. So, this establishes that in the product matrix our first row will look like this the (())(21:07) will look like this. The only thing that is to be checked is in the remaining part of the entries. Now the remaining one, again the same argument will tell us that j is also not equal to 1, i is also not equal to 1. So, the first entry will not contribute anything.

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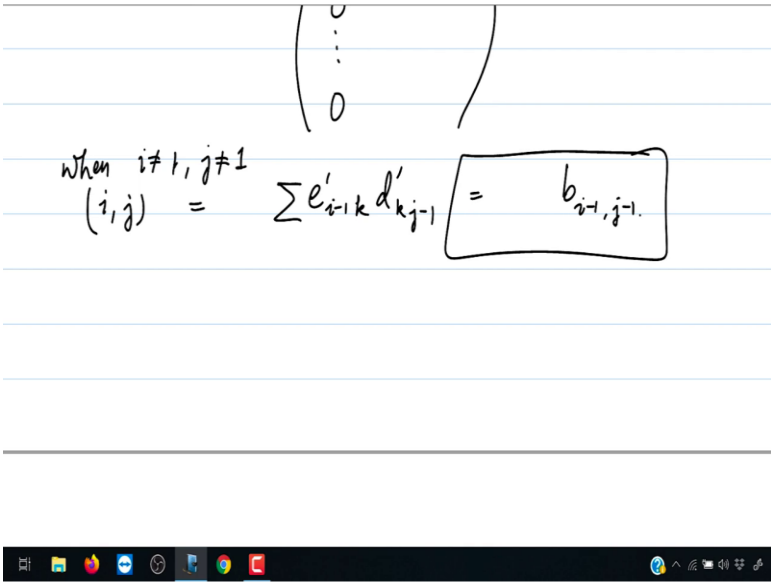
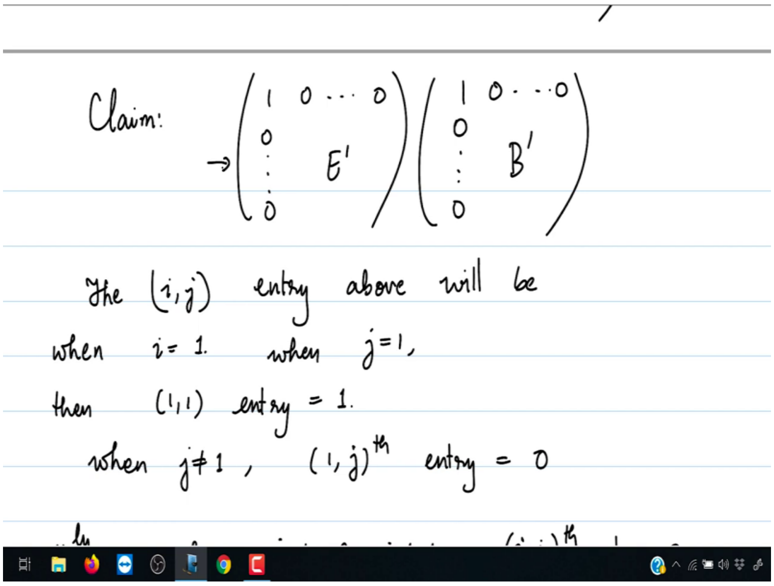


$$\begin{pmatrix} 0 & \\ \vdots & \\ 0 & \end{pmatrix}$$

when $i \neq 1, j \neq 1$

$$(i,j) = \sum e'_{i-1\,k}\, d'_{k\,j-1} \boxed{= \quad b_{i-1,j-1}.}$$



Claim: $\rightarrow \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & E' \\ 0 & \end{pmatrix} \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B' \\ 0 & \end{pmatrix}$

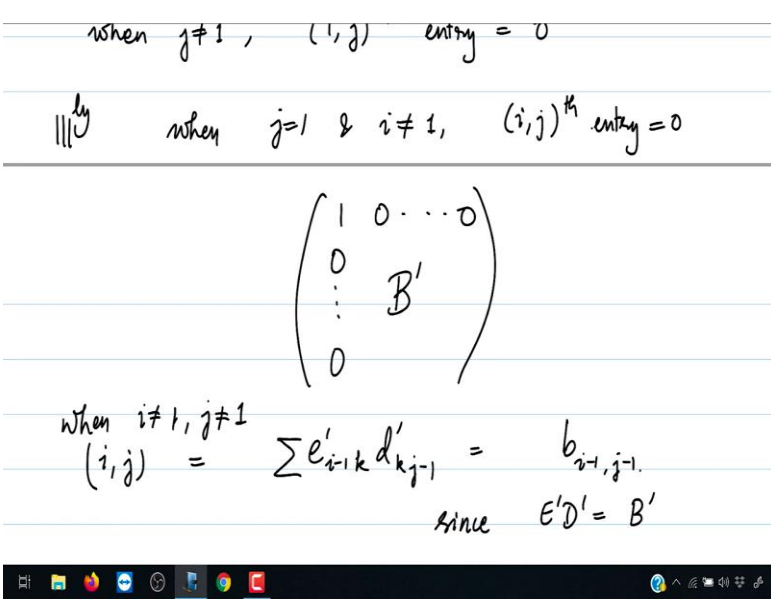
The (i,j) entry above will be

when $i = 1$. when $j=1$,

then $(1,1)$ entry $= 1$.

when $j \neq 1$, $(1,j)^{th}$ entry $= 0$

when $j \neq 1$, $(1,j)^{th}$ entry $= 0$



$|||^{ly}$ when $j=1$ & $i \neq 1$, $(i,j)^{th}$ entry $= 0$

$$\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B' \\ 0 & \end{pmatrix}$$

when $i \neq 1, j \neq 1$

$(i,j) = \sum e'_{i-1\,k} d'_{k\,j-1} = b_{i-1,j-1}.$

since $E'D' = B'$

The ij entry when i is not equal to 1, j is not equal to 1. The ijth entry will just be the ith row of let us see the ith row here will be the i minus 1th row of E prime and the jth column of B prime will be the j minus 1 column of B prime.

So, this will just give you e i minus 1 k, so let me put a prime here, times bkj minus 1, b prime k j minus 1, which is just the entry not b, d which is just the entry b i minus 1 j minus 1 and that is precisely, how do we get this entry is equal bi minus 1 j minus 1? It is because since, E prime times D prime is equal to B prime, this will give us this particular entry and hence the matrix here will just turn to be equal to the matrix B prime.

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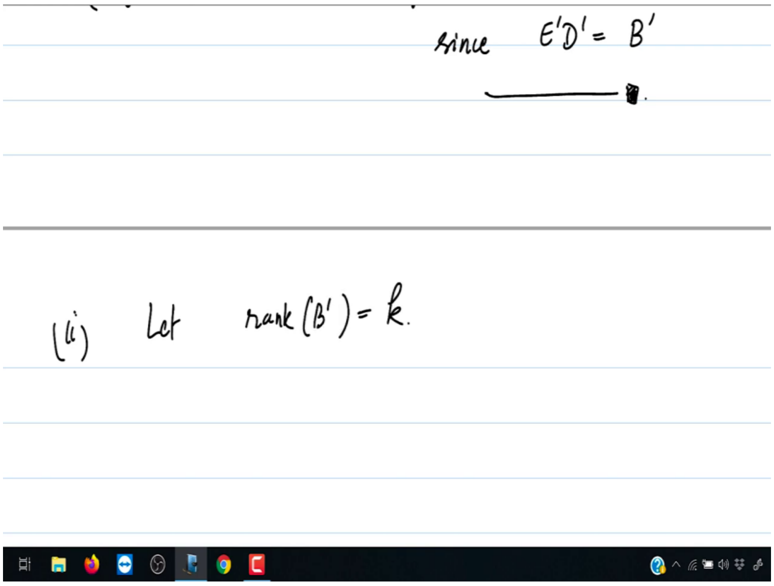
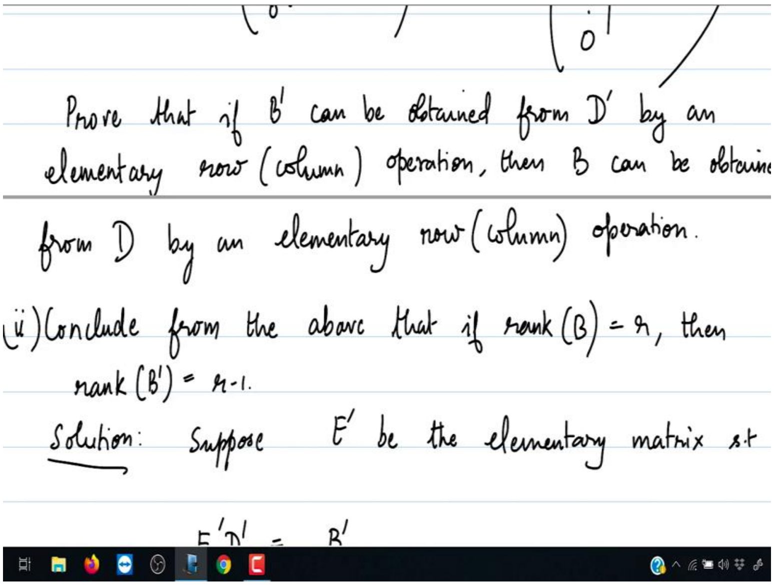


since $E'D' = B'$

(ii) Let $\text{rank}(B') = k$.



Prove that if B' can be obtained from D' by an elementary row (column) operation, then B can be obtaine

from D by an elementary row (column) operation.

(ii) Conclude from the above that if $\text{rank}(B) = r$, then $\text{rank}(B') = r-1$.

Solution: Suppose E' be the elementary matrix s.t

$E'D' = B'$

So, we have established the first part of the proof, the first part of the problem. What was the second part of the problem let us go back. It is saying that using this first part conclude that if rank of B is equal to r then rank of B prime is equal to r minus 1. So, let us see how to go about with this. Let a rank of B prime be equal to some k, that is B equal to some k. But what does that mean? this means that you recall that if a given matrix m cross n has rank r, then you can get hold of elementary matrices E1, E2 up to say El and F1, F2 up to Fs such that.

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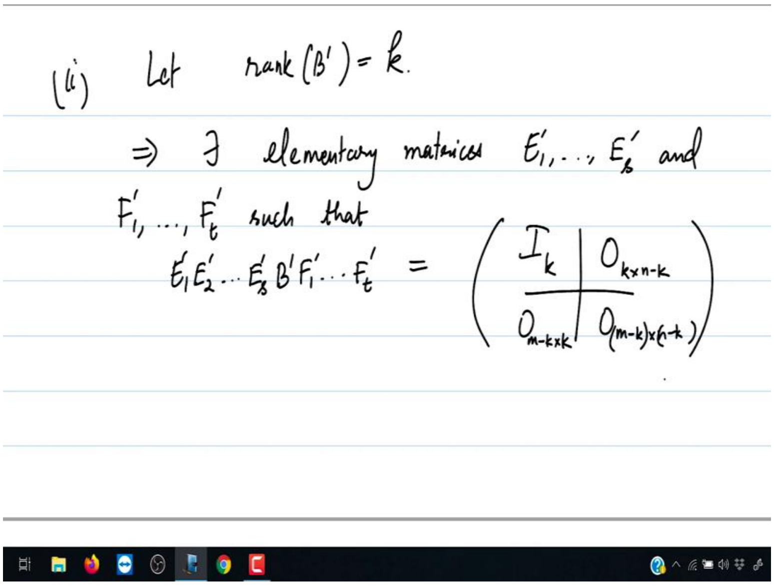

(ii) Let rank$(B') = k$.

$\Rightarrow \exists$ elementary matrices $E'_1, \ldots, E'_s$ and $F'_1, \ldots, F'_t$ such that

$$E'_1 E'_2 \cdots E'_s B' F'_1 \cdots F'_t = \left(\begin{array}{c|c} I_k & 0_{k\times n-k} \\ \hline 0_{m-k\times k} & 0_{(m-k)\times(n-k)} \end{array}\right)$$

So, let me write it down, this implies that there exist elementary matrices E1 to Er let me not use r, r is given in the problem. So, Es and F1 to Ft such that E1, E2, Es times B prime F1 up to Ft. So, let me put primes here. This is equal to an Ik here and the remaining block matrices. So, this is going to be a k cross m minus k, n minus k, m cross n it is, this will be a 0 m minus k cross k and this is going to be m minus k cross n minus k.

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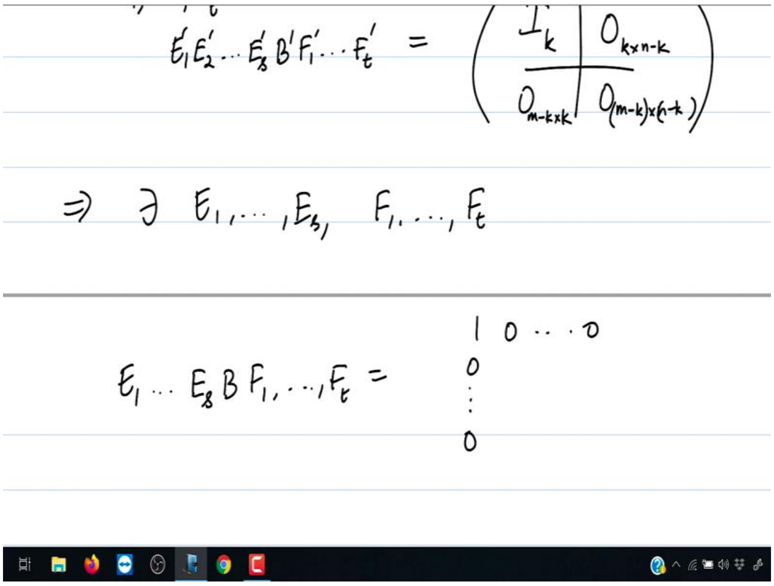

$$E'_1 E'_2 \cdots E'_s B' F'_1 \cdots F'_t = \left(\begin{array}{c|c} I_k & 0_{k\times n-k} \\ \hline 0_{m-k\times k} & 0_{(m-k)\times(n-k)} \end{array}\right)$$

$\Rightarrow \exists\ E_1, \ldots, E_s,\ F_1, \ldots, F_t$

$$E_1 \cdots E_s B F_1, \ldots, F_t = \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{matrix}$$

Now by what we just proved, this implies that there exist E1, E2 up to Es, F1, F2 up to Fs at Ft at every stage. Such that you multiply this E1 to Es B prime. So, B F1 to Ft by putting one,

one in the first 1 1 entry 0s in the first row and the first column elsewhere, this will just give you a 1 and then there will be 0s, there will be 0s here

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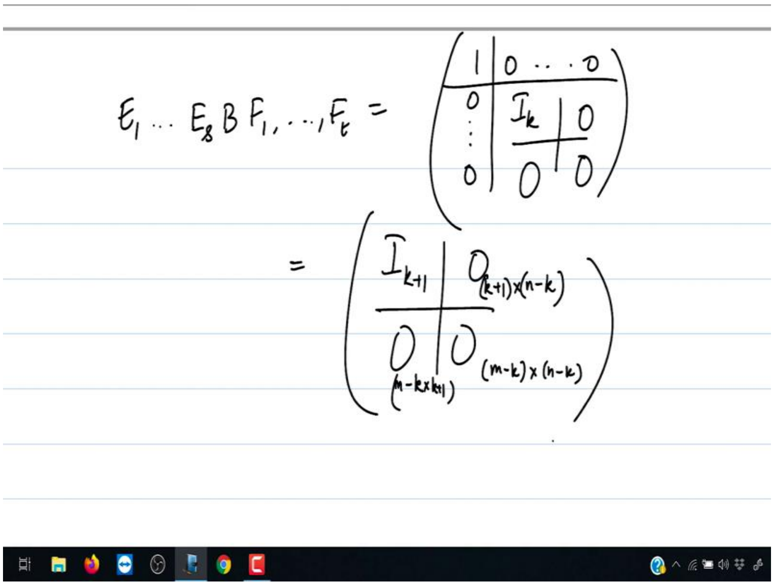

$$E_1 \cdots E_s B F_1, \ldots, F_t = \left(\begin{array}{c|cc} 1 & 0 \cdots & 0 \\ \hline 0 & I_k & 0 \\ \vdots & & \\ 0 & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} I_{k+1} & 0_{(k+1)\times(n-k)} \\ \hline 0_{(m-k \times k+1)} & 0_{(m-k)\times(n-k)} \end{array}\right)$$

And then there will be an Ik, there is a 0 here, there is a 0 here, there is a 0 here this matrix. But what is this matrix this is nothing but, Ik plus 1, 0, 0, 0 just to be careful about what we are writing this is going to be m plus 1, k plus 1 cross n minus k minus 1, n plus 1 minus k minus 1, which will be n minus k.

Similarly, here this is going to be m minus k minus 1 again, m plus 1 minus k minus 1 which will be m minus k cross k plus 1 and this will be m minus k cross n minus k, but what does that mean?

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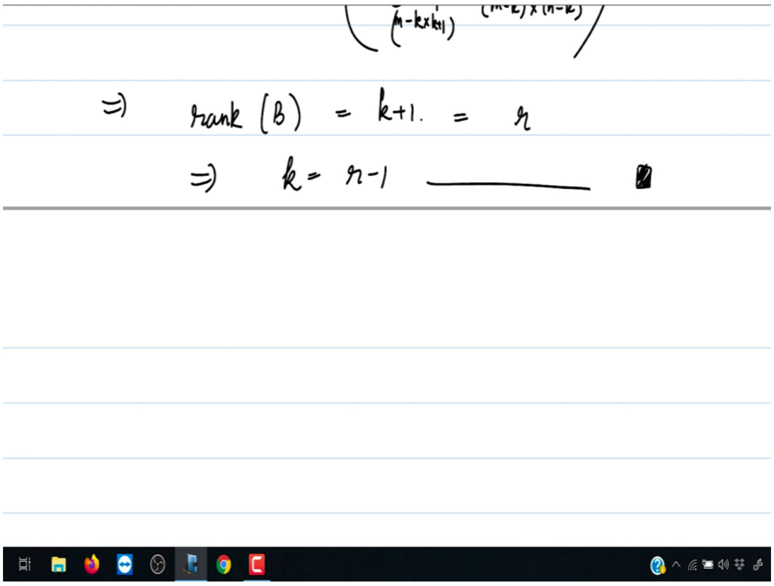

$$\Rightarrow \quad \text{rank}(B) = k+1. = r$$

$$\Rightarrow \quad k = r-1 \quad \_\_\_\_\_\_ \quad \blacksquare$$

This means that rank of B is equal to k plus 1 by the very definition of rank and the theorems that we proved in the relevant week in the sixth week. But rank of B is already given to us which is equal to r, this would imply that k is equal to r minus 1 which we had set out to prove. So, that completes the second problem. So, the next problem is a very important problem which talks about the determinant of a matrix which is given in one such block form, so let us see what the problem is.

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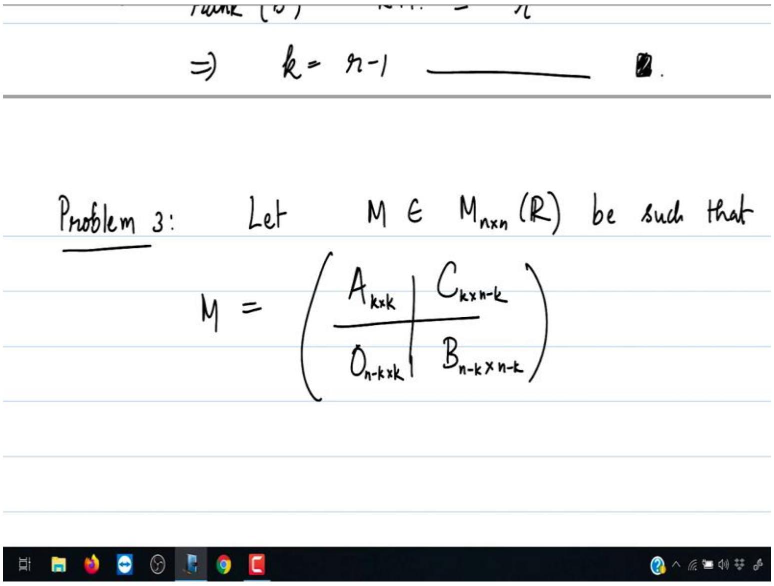

$$\Rightarrow \quad k = r-1 \quad \_\_\_\_\_\_ \quad \blacksquare.$$

Problem 3: Let $M \in M_{n\times n}(\mathbb{R})$ be such that

$$M = \left(\begin{array}{c|c} A_{k\times k} & C_{k\times n-k} \\ \hline O_{n-k\times k} & B_{n-k\times n-k} \end{array}\right)$$

Problem 3, so let M be an n cross n matrix, such that M is equal to A, say k cross k block matrix it is. So, this is 0 matrix, below A which is of n minus k cross k and there is a C which is k cross n minus k and there is a B which is n minus k cross n minus k. So, suppose M can be written in this form. So, M be Mn of R be such that, M is in this form.

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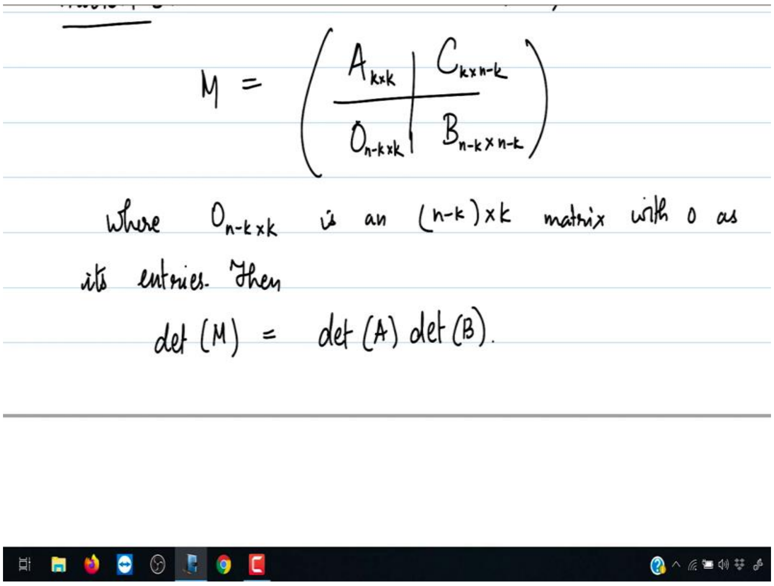

$$M = \left(\begin{array}{c|c} A_{k\times k} & C_{k\times n-k} \\ \hline 0_{n-k\times k} & B_{n-k\times n-k} \end{array}\right)$$

where $0_{n-k\times k}$ is an $(n-k)\times k$ matrix with 0 as its entries. Then

$$\det(M) = \det(A)\det(B).$$

So, notice that where 0 n minus k cross k is an n minus k cross k matrix with 0 entries, with 0 as its entries. So, then what is the problem? The problem is to establish that, then determinant of M is equal to determinant of A times the determinant of B. So, we have used this problem crucially in some of the theory that we have developed. So, let us give a proof of this, so let us see how this particular statement can be proved. So, notice once more that our M is in a block matrix form.

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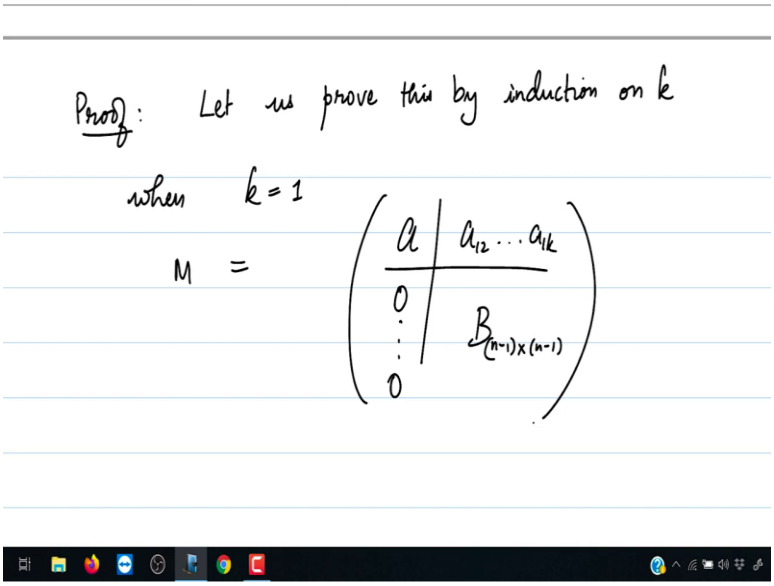
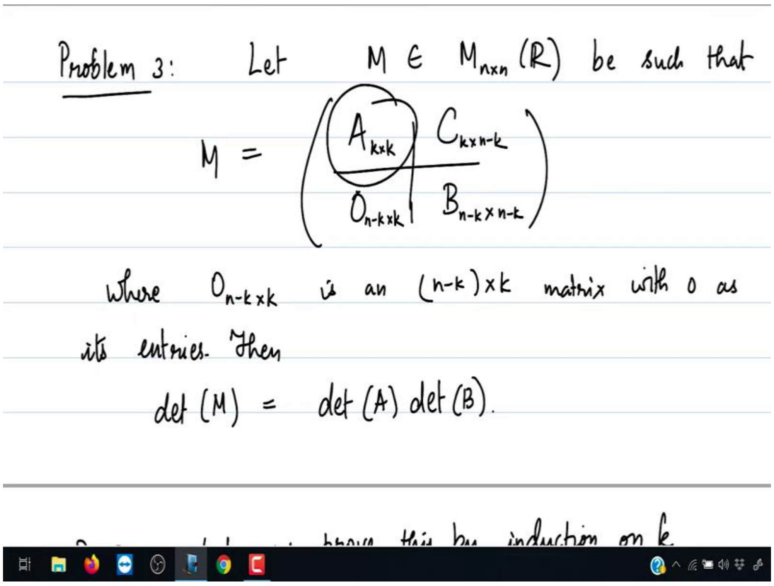


Proof: Let us prove this by induction on k

when $k=1$

$$M = \left(\begin{array}{c|c} a & a_{12}\cdots a_{1k} \\ \hline 0 & \\ \vdots & B_{(n-1)\times(n-1)} \\ 0 & \end{array}\right)$$



Problem 3: Let $M \in M_{n\times n}(\mathbb{R})$ be such that

$$M = \left(\begin{array}{c|c} A_{k\times k} & C_{k\times n-k} \\ \hline O_{n-k\times k} & B_{n-k\times n-k} \end{array}\right)$$

where $O_{n-k\times k}$ is an $(n-k)\times k$ matrix with 0 as its entries. Then

$$\det(M) = \det(A)\det(B).$$

this by induction on k

So, the proof is by induction. So, let us prove this by induction on k, what was k, k was the size of the matrix k here. So, we will prove it by induction on this k. So, when k is equal to 1, let us see what happens, M will just turn out to be equal to, the first one will be a 1 here and then there will be a C here which is, this is not 1 this is going to be a11, let me put it this way.

There is an a which is some real entry and there are a12 to a1k, a this will be 0s and there will be B here, which is n minus k, which is n minus 1 cross n minus 1. So, what will be determinant of M?

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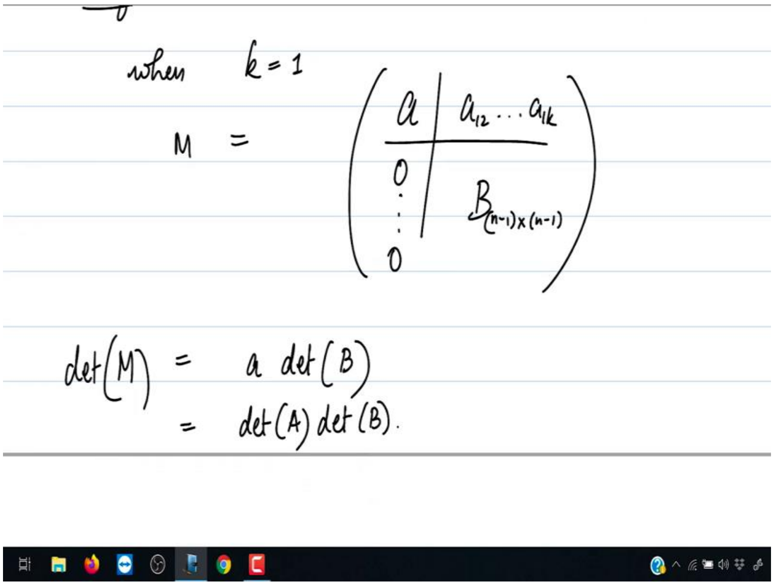

when $k = 1$

$$M = \left(\begin{array}{c|c} a & a_{12} \cdots a_{1k} \\ \hline 0 & \\ \vdots & B_{(n-1)\times(n-1)} \\ 0 & \end{array}\right)$$

$$\det(M) = a \det(B)$$
$$= \det(A)\det(B).$$

Determinant of M is equal to, let us look at the cofactor expansion along the first column. This will be a times determinant of B, which is the minor and the remaining terms will be just

0 times something and that turns out to be 0. So, this is just a times the determinant of B, but what is a? a is the only entry in capital A and this is hence the determinant of capital A as well, which is determinant of A times B now.

So, yes for the case k equal to 1 it is really satisfying the condition, that determinant of A times determinant of B is equal to the determinant of M. So, let us go further with the induction hypothesis.

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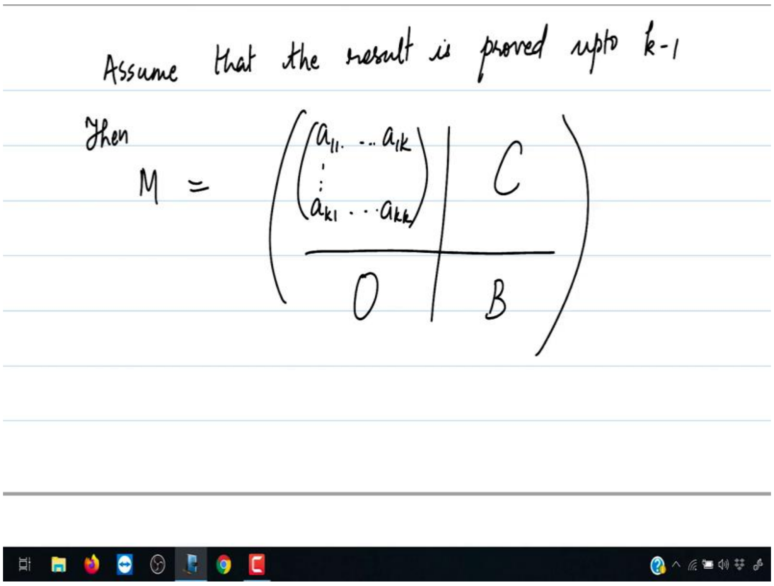

Assume that the result is proved upto k-1

Then

$$M = \left(\begin{array}{c|c} \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} & C \\ \hline 0 & B \end{array}\right)$$

Assume that the result is proved up to k minus 1. Let us now prove it for the case, when our a is of size k cross k. So, for the case A M, then M be equal to let me write down a explicitly for you, will be a1k, ak1 to akk, this will be the first block, the capital A and below will be a 0, which is of the relevant size there will be a C and there is a B.

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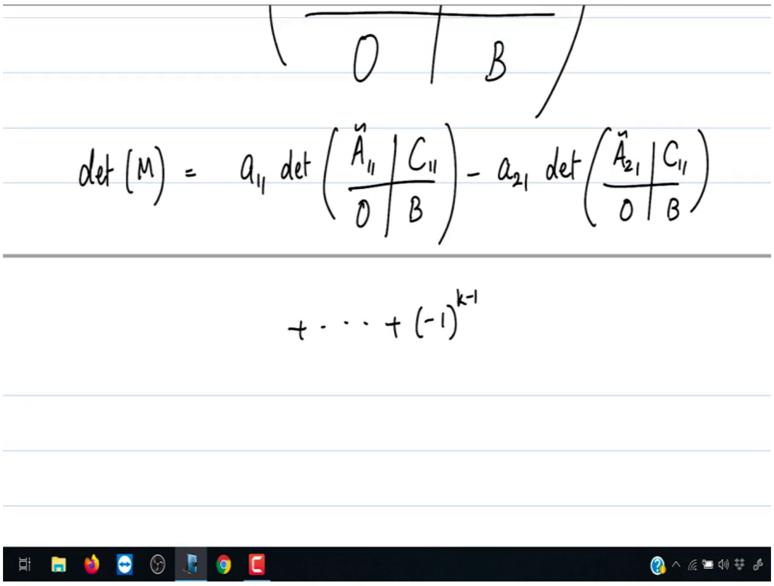

$$\left(\begin{array}{c|c} 0 & B \end{array}\right)$$

$$\det(M) = a_{11} \det\left(\begin{array}{c|c} \tilde{A}_{11} & C_{11} \\ \hline 0 & B \end{array}\right) - a_{21} \det\left(\begin{array}{c|c} \tilde{A}_{21} & C_{11} \\ \hline 0 & B \end{array}\right)$$

$$+ \cdots + (-1)^{k-1}$$

So, again let us look at the determinant of M here, by looking at the cofactor expansion along the first column. So, this will be a11 times the determinant of the matrix, first row and the first column is taken out here for the cofactor expansion. So, this will be a22 to a2k, ak2 to akk, there will be a C prime here. So, let me do one thing rather than writing it like this, I will use the expression we have already used earlier.

So, this is going to be A11 tilde, there will be some C11 here, which will not be C. The first row will be deleted from this, there will be a 0 here and there is a B here. How about the second entrant, it will be minus a21 times the determinant of A21 tilde C11 0 B so on and what will be the final term this will be minus 1 to the power k plus 1 times.

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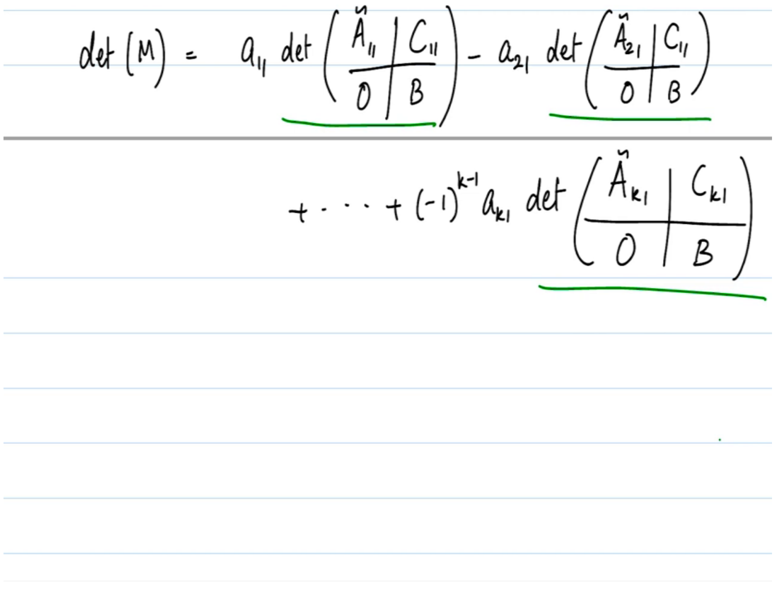
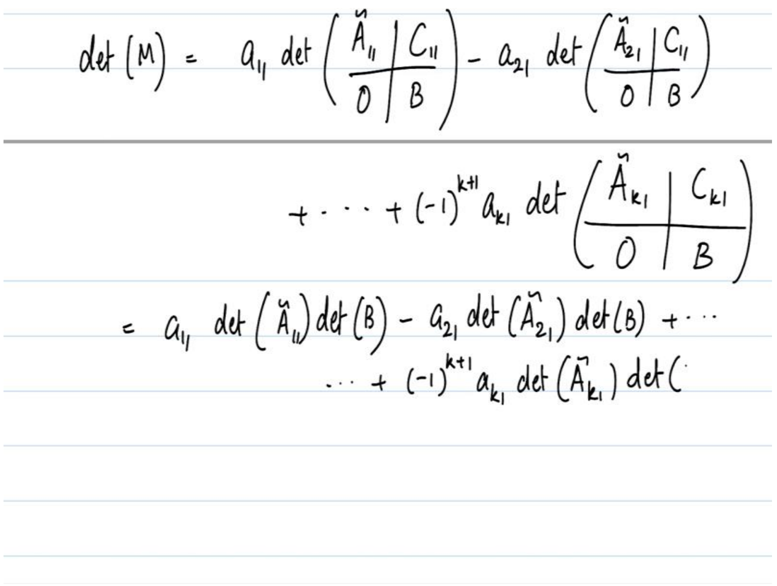


$$\det(M) = a_{11}\det\underline{\left(\begin{array}{c|c}\tilde{A}_{11} & C_{11}\\ \hline 0 & B\end{array}\right)} - a_{21}\det\underline{\left(\begin{array}{c|c}\tilde{A}_{21} & C_{11}\\ \hline 0 & B\end{array}\right)}$$

$$+\cdots+(-1)^{k-1}a_{k1}\det\underline{\left(\begin{array}{c|c}\tilde{A}_{k1} & C_{k1}\\ \hline 0 & B\end{array}\right)}$$



$$\det(M) = a_{11}\det\left(\begin{array}{c|c}\tilde{A}_{11} & C_{11}\\ \hline 0 & B\end{array}\right) - a_{21}\det\left(\begin{array}{c|c}\tilde{A}_{21} & C_{11}\\ \hline 0 & B\end{array}\right)$$

$$+\cdots+(-1)^{k+1}a_{k1}\det\left(\begin{array}{c|c}\tilde{A}_{k1} & C_{k1}\\ \hline 0 & B\end{array}\right)$$

$$= a_{11}\det(\tilde{A}_{11})\det(B) - a_{21}\det(\tilde{A}_{21})\det(B) + \cdots$$

$$\cdots + (-1)^{k+1}a_{k1}\det(\tilde{A}_{k1})\det($$

This C keep it there, ak1 times the determinant of Ak1 tilde Ck1, as a 0 entry here and then there is B, which is remaining untouched. So, let us just go up and check what are the expressions which I am now underlying, there are three of these one which I am underlying and if you notice carefully. So, these three elements which I have underlined in green, they are going to be covered by the induction hypothesis.

I can write this as a11 times the determinant of A11 tilde times the determinant of B minus a21 times the determinant of A21 tilde times the determinant of B and so on plus, plus, plus dot minus 1 to the power k minus 1 or k plus 1 whichever is suitable for you ak1 times the determinant of Ak1 tilde times the determinant of B. Now if you notice carefully, there is a determinant of B here which is common.

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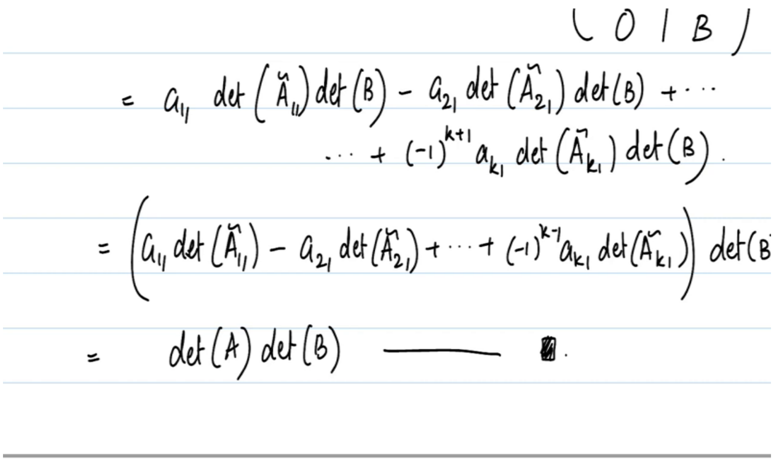

$$\left(0 \mid B \right)$$

$$= a_{11} \det(\tilde{A}_{11}) \det(B) - a_{21} \det(\tilde{A}_{21}) \det(B) + \cdots$$

$$\cdots + (-1)^{k+1} a_{k1} \det(\tilde{A}_{k1}) \det(B) .$$

$$= \left(a_{11} \det(\tilde{A}_{11}) - a_{21} \det(\tilde{A}_{21}) + \cdots + (-1)^{k-1} a_{k1} \det(\tilde{A}_{k1}) \right) \det(B$$

$$= \det(A) \det(B) \quad \text{———} \quad \blacksquare .$$

Which is hence equal to a11 times the determinant of A11 tilde minus a21 times the determinant of A21 tilde plus up to a minus 1 to the power k plus 1 ak1 determinant of Ak1 tilde, the whole times the determinant of B. Now that is interesting, because this first term is the determinant of A and then the second term is the determinant of B and that is precisely what we had set out to prove.

So, hence we have established the result that determinant of M is determinant of A times B. So, the next problem really deals with computing the eigenvalues and the eigenvectors of a given linear operators. So, let us look at it.

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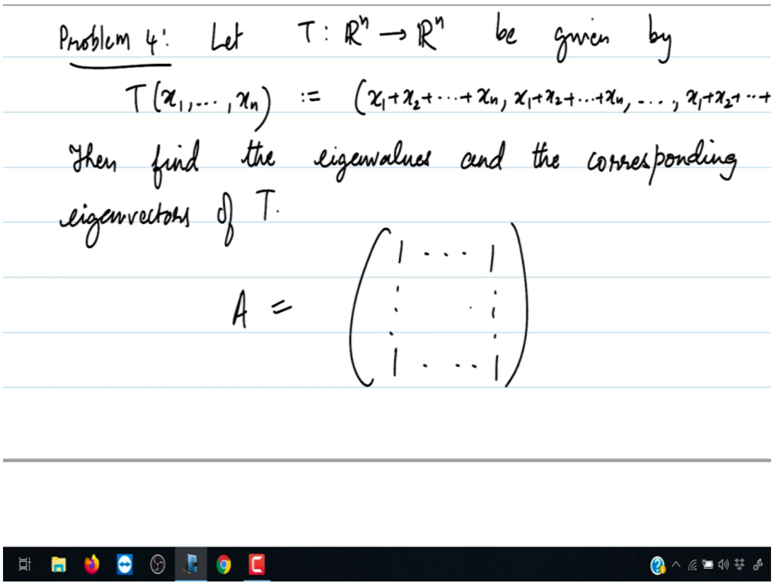
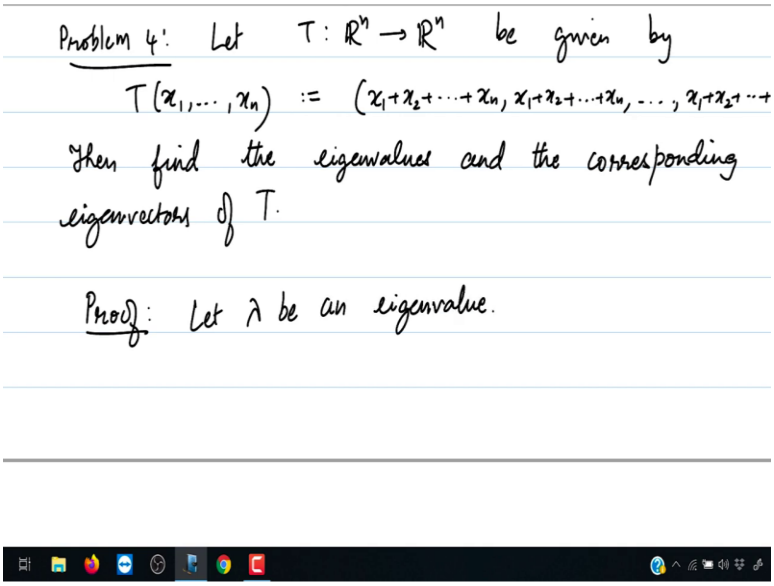


Problem 4: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$T(x_1, \dots, x_n) := (x_1 + x_2 + \cdots + x_n, x_1 + x_2 + \cdots + x_n, \dots, x_1 + x_2 + \cdots +$$

Then find the eigenvalues and the corresponding eigenvectors of T.

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$



Problem 4: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$T(x_1, \dots, x_n) := (x_1+x_2+\dots+x_n, x_1+x_2+\dots+x_n, \dots, x_1+x_2+\dots+$

Then find the eigenvalues and the corresponding eigenvectors of T.

Proof: Let λ be an eigenvalue.

Let us look at the problem specifically. The problem is it 4, yeah it is 4. So, let T from Rn to Rn be given by T of x1, x2 up to xn is equal to by a definition equal to x1 plus x2 plus up to xn, x1 plus x2 plus up to xn, x1 plus x2 plus up to xn. So, we have a T which maps to every coordinate is the same, which is basically the sum of (())(36:47). So, the problem is to find then find the eigenvalues and the corresponding eigenvectors of T.

So, let us look at, so if you notice in the standard basis, the matrix of T will be a matrix which is given by 1 in every entry. So, that is something which you should go back and check. So, if the problem is to be rephrased in the language of matrices. So, this will be the statement A be the matrix which n cross n matrix where every entry is 1 and then prove or rather find the eigenvalues and eigen corresponding eigenvectors of this particular matrix.

So, for us the language of linear transformations and the matrices have been already noted to be the same through identifying a basis. So, it is the same for us. So, let us go ahead and see what the solution to this problem is. So, what is an eigenvalue? An eigenvalue is scalar lambda, such that there exist some nonzero vector v, where Tv is equal to lambda times v. So, let lambda be an eigenvalue, then what does this mean?

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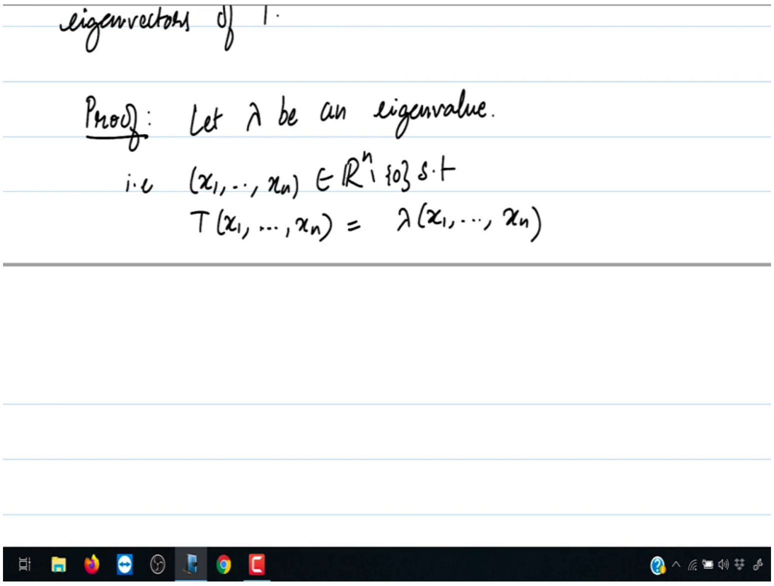

eigenvectors of T.

Proof: Let λ be an eigenvalue.

i.e $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ s.t

$$T(x_1, \dots, x_n) = \lambda(x_1, \dots, x_n)$$

This mean i.e. lambda is an eigenvalues means that there exist x1 to xn in Rn, such that T of x1. So, this is Rn minus, let me just put 0 to denote the zero element 0, 0, 0 in Rn. It should not be a zero vector, the 0 vector is avoided. So, this will be lambda times x1 to xn. Let us see what this becomes, what is this going to manifest as.

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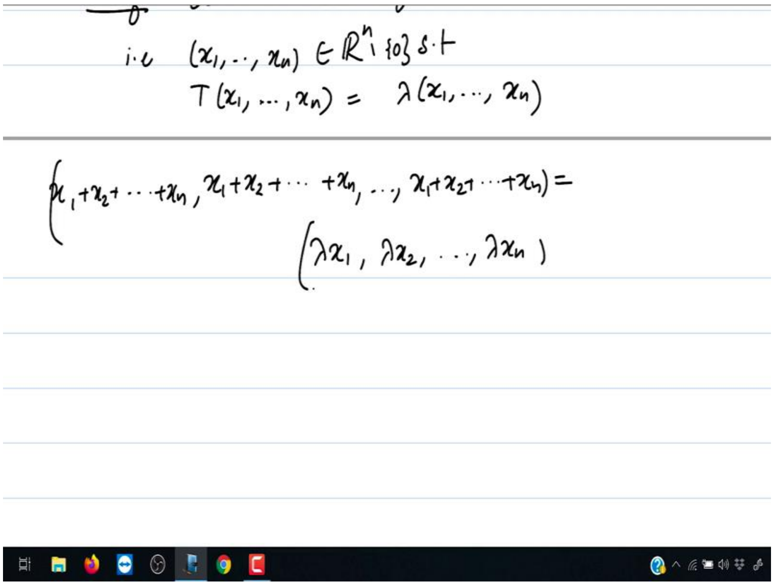

i.e $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ s.t

$$T(x_1, \dots, x_n) = \lambda(x_1, \dots, x_n)$$

$$(x_1+x_2+\cdots+x_n, x_1+x_2+\cdots+x_n, \dots, x_1+x_2+\cdots+x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

T of x1 to xn is x1 plus x2 plus up to xn, x1 plus x2 plus up to xn. This vector which will be equal to lambda x1 lambda x2, so on up to lambda xn. What does that mean?

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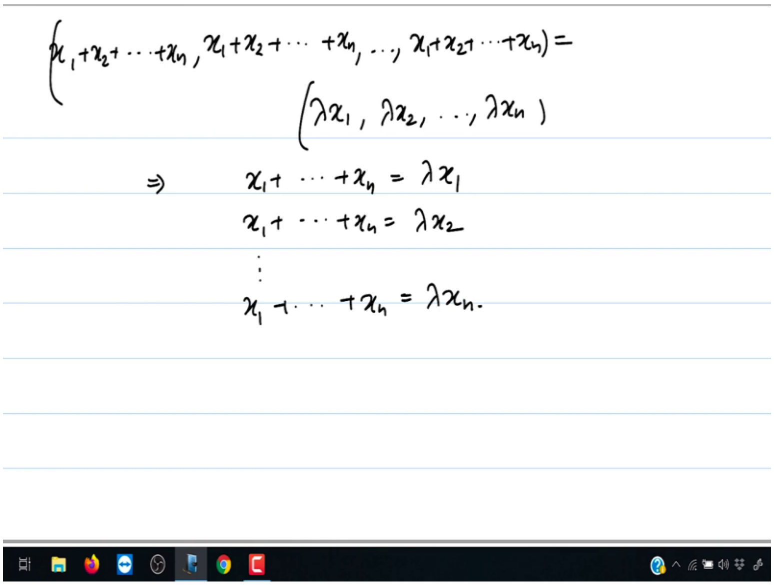

$$(x_1+x_2+\cdots+x_n, x_1+x_2+\cdots+x_n, \ldots, x_1+x_2+\cdots+x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

$$\Rightarrow \quad x_1+\cdots+x_n = \lambda x_1$$
$$x_1+\cdots+x_n = \lambda x_2$$
$$\vdots$$
$$x_1+\cdots+x_n = \lambda x_n.$$

This means that we have these equations, it implies x1 plus up to xn is equal to lambda x1, x1 plus x2 plus up to xn is equal to lambda x2, x1 there will be n such equations and this will be the, this will be the impact.

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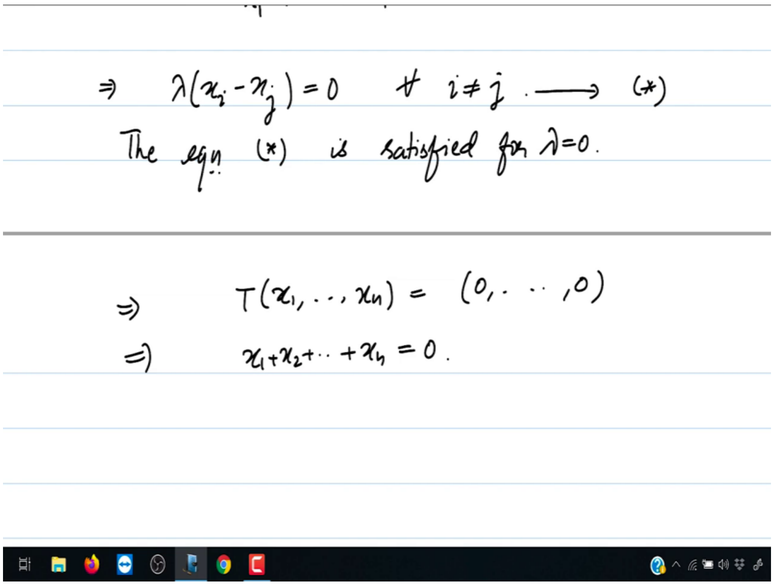

$$\Rightarrow \quad \lambda(x_i - x_j) = 0 \quad \forall\ i \neq j \longrightarrow (*)$$

The eqn (*) is satisfied for $\lambda = 0$.

$$\Rightarrow \quad T(x_1, \ldots, x_n) = (0, \ldots, 0)$$
$$\Rightarrow \quad x_1+x_2+\cdots+x_n = 0.$$

This implies that lambda times xi minus xj is equal to 0, for all i not equal to j. So, there are two possibilities here, if lambda is equal to 0 then, this equation is satisfied. So, let me write it like that, the equation, the equation star is satisfied for lambda is equal to 0, but what does that mean? This means that, This means that T of x1 to xn is equal to 0 times this, which means that x1 plus x2 plus up to xn is equal to 0. This is precisely what it means.

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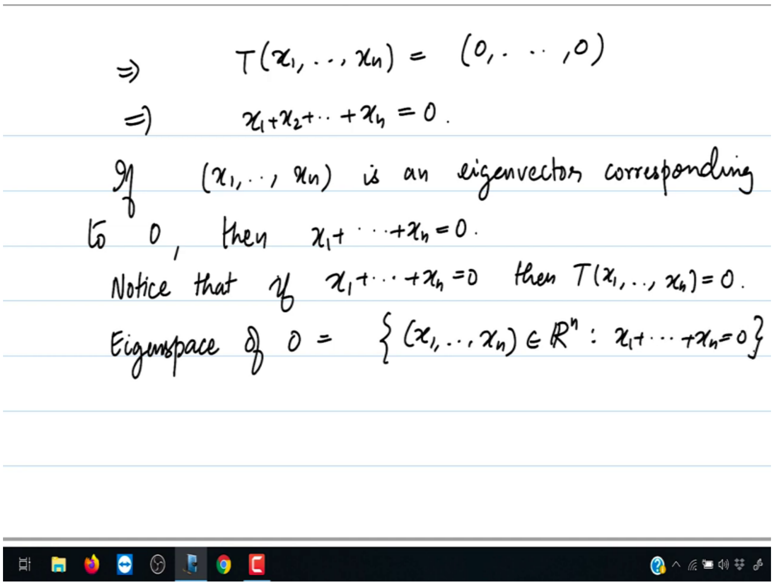

$\Rightarrow \quad T(x_1,\dots,x_n) = (0,\dots,0)$

$\Rightarrow \quad x_1+x_2+\dots+x_n = 0.$

If $(x_1,\dots,x_n)$ is an eigenvector corresponding to 0, then $x_1+\dots+x_n=0$.

Notice that if $x_1+\dots+x_n=0$ then $T(x_1,\dots,x_n)=0$.

Eigenspace of $0 = \{(x_1,\dots,x_n)\in \mathbb{R}^n : x_1+\dots+x_n=0\}$

So, if x1 to xn is an eigenvector of corresponding to 0 or in this case, it is just going to be, the null space of T, then x1 plus x2 plus up to xn is 0. I will leave it as an exercise for you to check that, notice that or check that if x1 plus x2 plus up to xn is 0, then T of x1, x2 up to xn is equal to 0. So, which is 0 times x1, x2 up to xn. So, the eigenspace, let me use that word eigenspace of 0 is the set of all x1 to xn in Rn, such that x1 plus x2 plus up to xn is equal to 0. We know that this is an n minus 1 dimensional vector space. So, that is when lambda is equal to 0.

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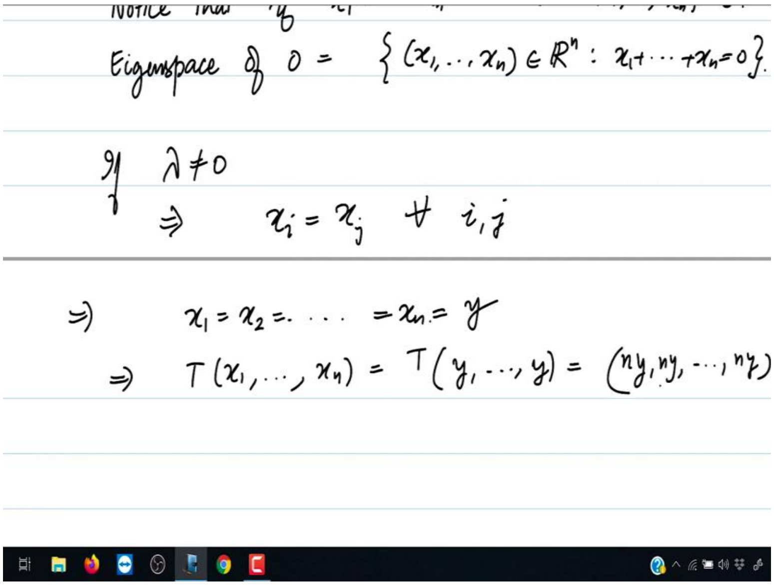

Eigenspace of $0 = \{(x_1,\dots,x_n)\in \mathbb{R}^n : x_1+\dots+x_n=0\}.$

If $\lambda \neq 0$

$\Rightarrow \quad x_i = x_j \quad \forall\, i,j$

$\Rightarrow \quad x_1 = x_2 = \dots = x_n = y$

$\Rightarrow \quad T(x_1,\dots,x_n) = T(y,\dots,y) = (ny,ny,\dots,ny)$

If lambda is not equal to 0, let us see what happens then. This would imply that xi is equal to xj for all i, j. This implies that x1 is equal to x2 is equal to up to xn and what does that imply. This implies that T of x1 to xn is equal to T of x, x, x let us call this equal to say y, this is just equal to T of y, y which is equal to y1 plus y1 plus y1 so on, this is going to be ny, ny up to ny where n is the dimension or Rn.

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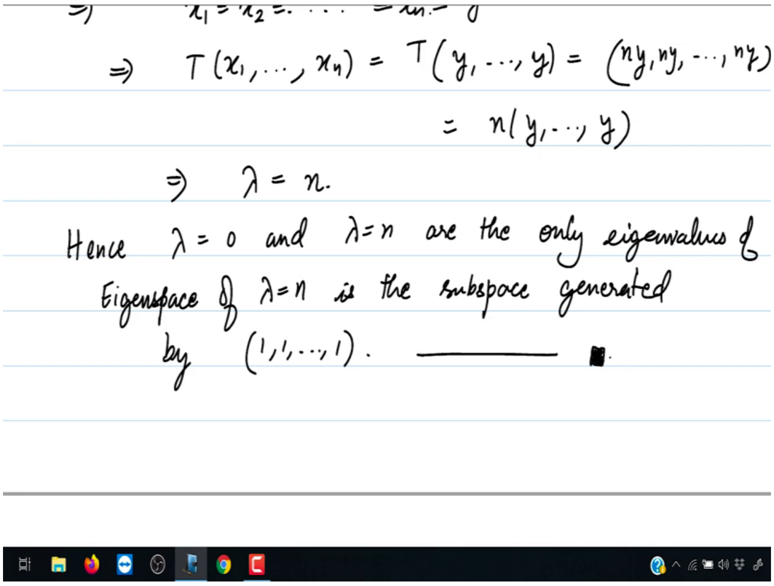

$\Rightarrow \quad x_1 = x_2 = \cdots = x_n = y$

$\Rightarrow \quad T(x_1, \ldots, x_n) = T(y, \ldots, y) = (ny, ny, \ldots, ny)$

$= n(y, \ldots, y)$

$\Rightarrow \quad \lambda = n.$

Hence $\lambda = 0$ and $\lambda = n$ are the only eigenvalues of

Eigenspace of $\lambda = n$ is the subspace generated by $(1, 1, \ldots, 1)$. ——— ■.

Which is equal to n times y, y, y, y. So, this gives that lambda is equal to n, so if lambda is not equal to 0, lambda is supposed to be equal to n. So, hence lambda equal to 0 and lambda is equal to n are the only eigenvalues of T. What about the eigenspace? Eigenspace of lambda is equal to n is given by, is the subspace generated by 1, 1, 1.

So I will leave that again for you to check its straightforward to see that 1, 1, 1 the 1, 1, 1, the vector 1, 1, 1 is in the eigenspace and we already check that if something is in the eigenspace it will force the coordinates to be equal and hence in the vector space generated by 1, 1, 1. So yes the eigenspace of 1 is the same as the subspace generated by 1, 1, 1. That we complete the. So, I would ask you to think about whether this given matrix T or given linear transform diagonalizable, that is a job for you to think about.

Observe that there are two eigenvalues here check what is the dimension of the eigenspace corresponding to 0 get hold of some vectors there, look at what is the eigenspace corresponding to 1 get hold of a vector and think about whether this particular linear transformation is diagonalizable. The next problem we will deal with the eigenvalues and eigenspaces in infinite dimensional vector spaces. So, this is an exercise or this is a problem to exhibit that these notions are studied in infinite dimensional vector space as well. So, let us look at what the problem is.

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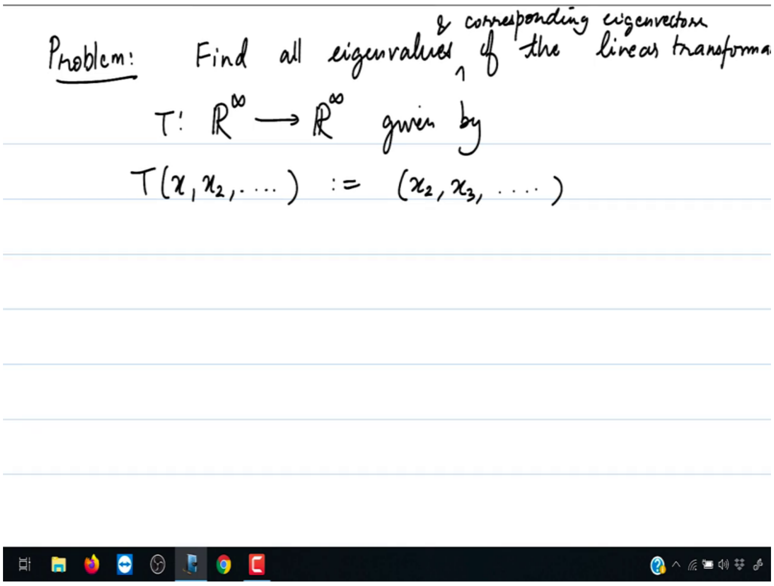

Problem: Find all eigenvalues & corresponding eigenvectors of the linear transforma

$T: \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by

$T(x_1, x_2, \dots) := (x_2, x_3, \dots)$

So, find all eigenvalues of the linear transformation T from R infinity to R infinity given by this is the left shift operator. If you recall what that is T of x1, x2 dot, dot, dot. So, R infinity was the space or the vector space of all sequences x1, x2 and so on and what was this map let me again define it for you, this is just x2, x3 and so on to find all eigenvalues and the corresponding eigenvectors. So, maybe I will put it here and corresponding eigenvectors. So, what will be an eigenvalue here like.

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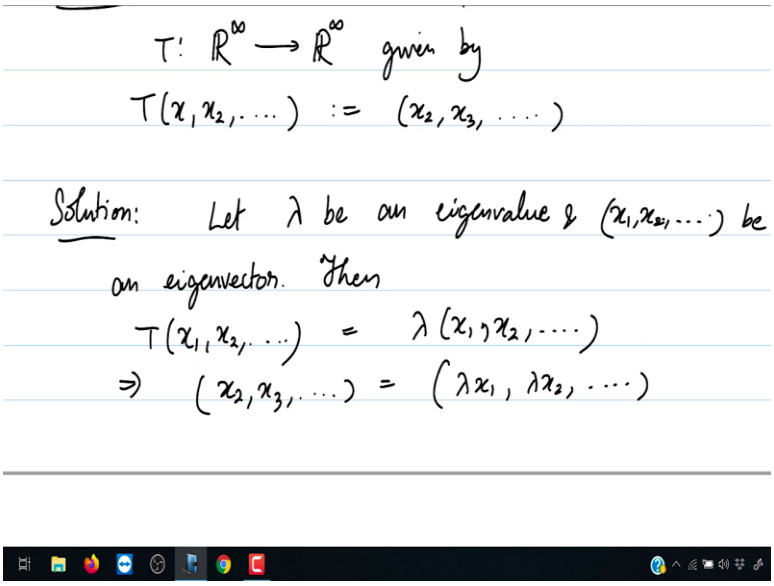

$T: \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by

$T(x_1, x_2, \dots) := (x_2, x_3, \dots)$

Solution: Let λ be an eigenvalue & $(x_1, x_2, \dots)$ be an eigenvector. Then

$$T(x_1, x_2, \dots) = \lambda (x_1, x_2, \dots)$$

$$\Rightarrow (x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \dots)$$

So, the eigenvalue for solution. So, let lambda be an eigenvalue, let try to see what will be the properties that lambda will satisfy. Then and x1 to x2 be an eigenvector. Let us see what that means, then T of x1, x2 and so on is equal to lambda times x1, x2 dot, dot, dot that precisely what it means. But what is T of x1, x2 and so on this is, this implies that x2, x3 so on is equal to lambda times x1. So, I will put it inside, this is lambda x1, lambda x2 and so on.

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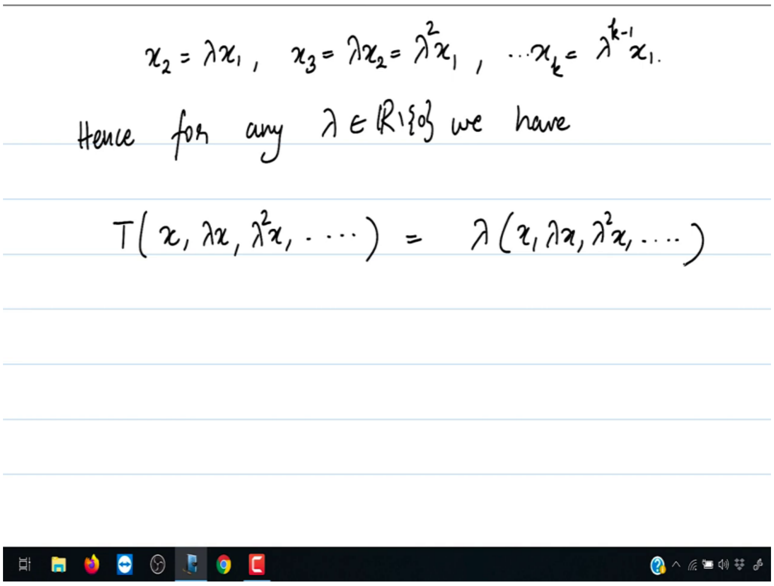

$$x_2 = \lambda x_1, \quad x_3 = \lambda x_2 = \lambda^2 x_1, \quad \dots x_k = \lambda^{k-1} x_1.$$

Hence for any $\lambda \in \mathbb{R}\setminus\{0\}$ we have

$$T(x, \lambda x, \lambda^2 x, \cdots) = \lambda(x, \lambda x, \lambda^2 x, \dots)$$

And we do a component wise equating of the terms here, to obtain x2 is equal to lambda x1, x3 is equal to lambda x2, which is equal to lambda square x1. Similarly, xn or rather xk is equal to lambda to the power k minus 1 times x1. So, this forces the eigenvector to, for any lambda hence, for any lambda in R, we have so lambda in R let me put R minus 0. We have T of let us see x1, x lambda x, lambda square x and so on is equal to lambda times x, lambda x, lambda square x and so on.

So, every real number hence turns out to be an eigenvalue for this particular linear operator T and what will be the eigenspace corresponding to the lambda here that will be the subspace generated by these elements.

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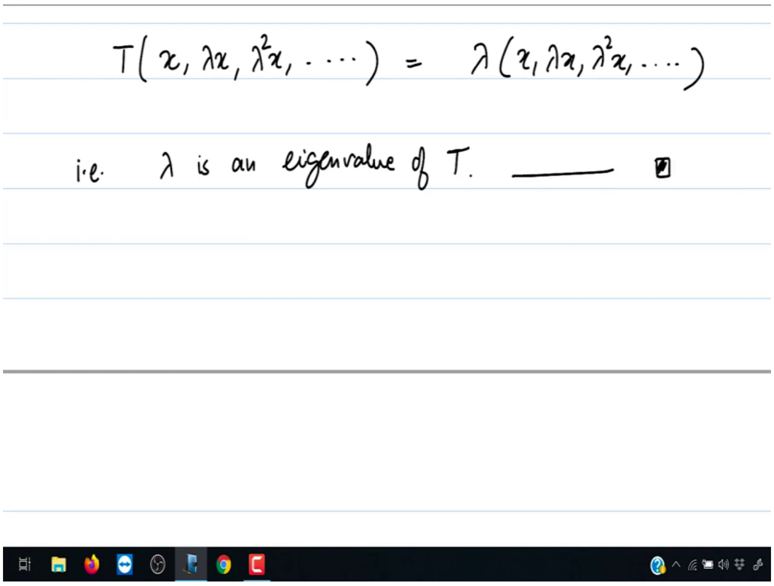
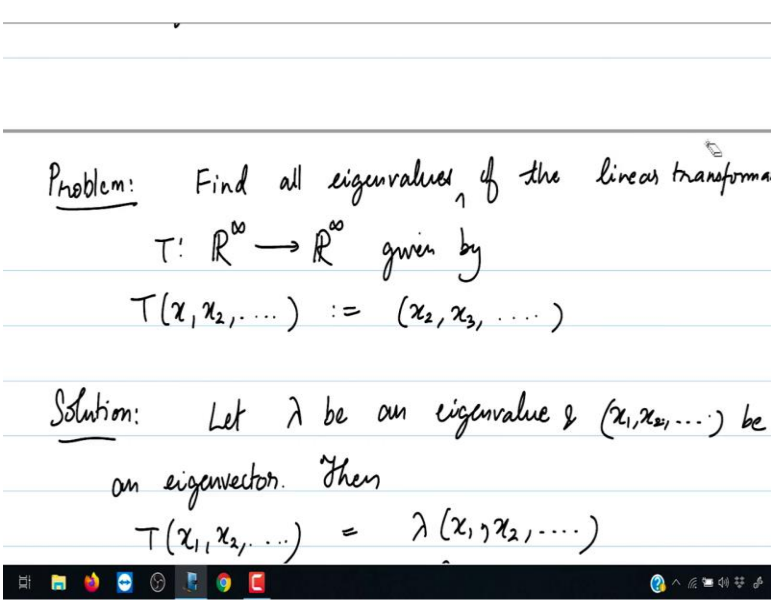


$$T(x, \lambda x, \lambda^2 x, \dots) = \lambda(x, \lambda x, \lambda^2 x, \dots)$$

i.e. λ is an eigenvalue of T. ——— ∎



Problem: Find all eigenvalues, of the linear transforma

$T: \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by

$$T(x_1, x_2, \dots) := (x_2, x_3, \dots)$$

Solution: Let λ be an eigenvalue & $(x_1, x_2, \dots)$ be an eigenvector. Then

$$T(x_1, x_2, \dots) = \lambda(x_1, x_2, \dots)$$

So, the eigenspace i.e lambda is an eigenvalue of T and the let me not talk about eigenspaces, that is something which maybe I should drop from the problem as well. We will just be contained with putting the eigenvalues here. So, every real number hence turns out to be eigenvalue of T. So, the final problem is going to deal with the eigenvalues of the linear transformation on the n cross n matrices which sends a matrix A to its transpose.

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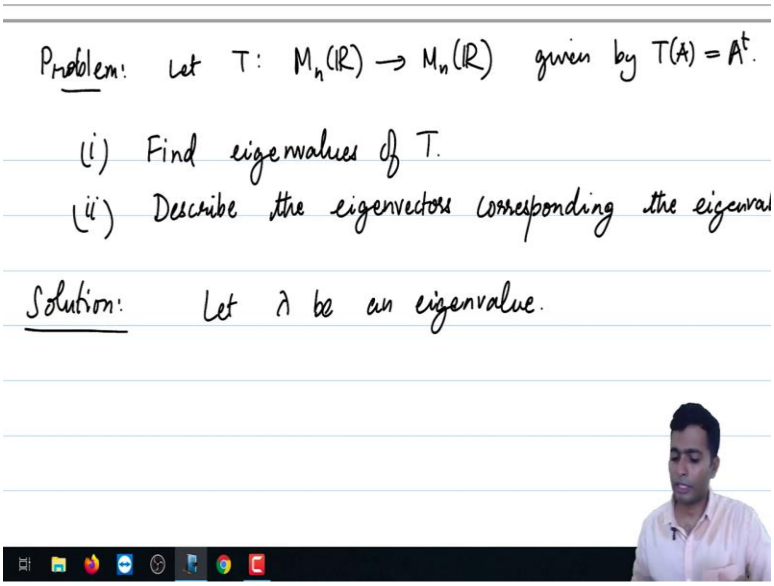

Problem: Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $T(A) = A^t$.

(i) Find eigenvalues of T.

(ii) Describe the eigenvectors corresponding the eigenval

Solution: Let λ be an eigenvalue.

So, let so problem I will put a, the new problem in a new page. So, let T be a map from Mn of R to Mn of R given by T of A is equal to A transpose. So, I will leave it to you to check that this map actually is a linear map. So, the first problem is to find eigenvalues of T. The second part of the problem is to describe the eigenvectors corresponding to the eigenvalues just like in previous case.

Let us look at a solution. So, what is the map T doing? The map T is taking a matrix and sending it to its transpose. So, we have to find the eigenvalues of T. So, let lambda in R or let lambda be an eigenvalue. We do not know yet whether it is an eigenvalue or not, let lambda be an eigenvalue.

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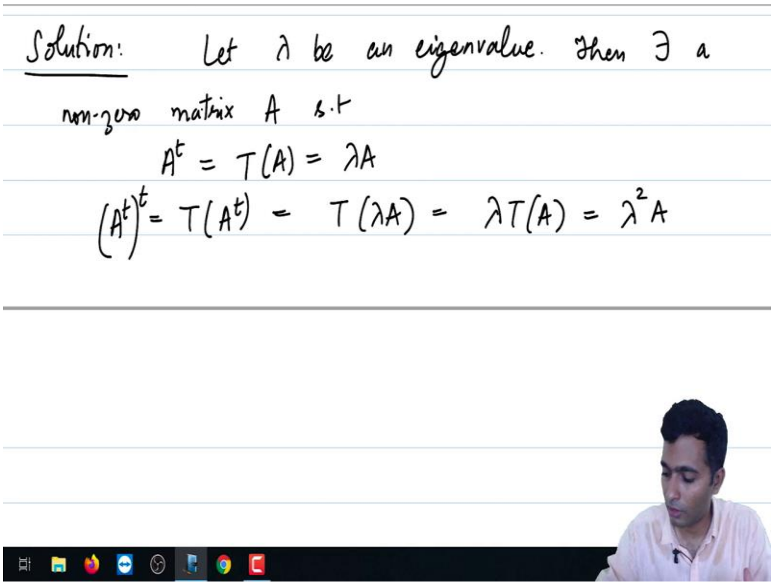

Solution: Let λ be an eigenvalue. Then $\exists$ a non-zero matrix A s.t

$$A^t = T(A) = \lambda A$$

$$(A^t)^t = T(A^t) = T(\lambda A) = \lambda T(A) = \lambda^2 A$$

Then, there exist a nonzero matrix, that means at least one of the entries is nonzero capital A, such that, T A is equal to lambda times A. But what is T A, T A is just equal to A transpose. Let us do one thing, let us apply T A once more then, T to A transpose then T of A transpose will be equal to T of lambda A, which will be lambda times T of A. But recall that to begin with A was an eigenvector corresponding to lambda.

So, this is going to be lambda square times A and what is going to be T of A transpose, this is going to be A transpose the transpose of that matrix. So, what is the transpose of the transpose of a matrix? I will give you back the original matrix that we begin.

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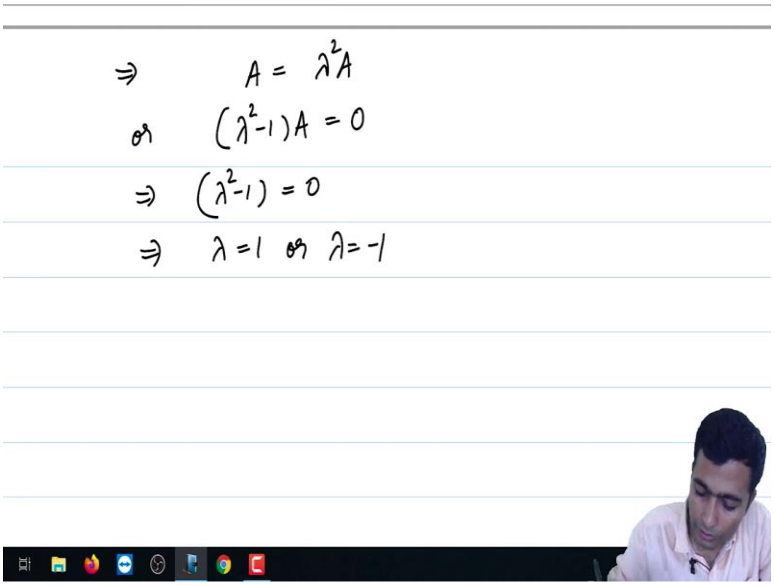

$$\Rightarrow \quad A = \lambda^2 A$$

$$\text{or} \quad (\lambda^2 - 1)A = 0$$

$$\Rightarrow \quad (\lambda^2 - 1) = 0$$

$$\Rightarrow \quad \lambda = 1 \text{ or } \lambda = -1$$

This implies that A is equal to lambda square times capital A or lambda square minus 1 times capital A is equal to 0. But by equating it to 0 component wise, what we will be able to conclude is that lambda square minus 1 is equal to 0 and this forces lambda to be equal to 1 or lambda to be equal to minus 1. So, the only possible eigenvalues of the operator T here will be 1 or minus 1. Let us see what happens when lambda is 1 and when happens when lambda is minus 1.

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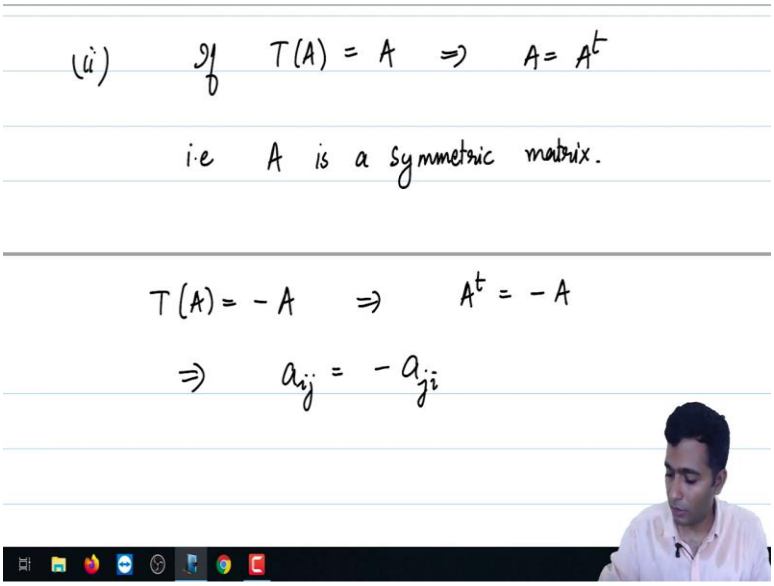

(ii) If $T(A) = A \Rightarrow A = A^t$

i.e A is a symmetric matrix.

$T(A) = -A \Rightarrow A^t = -A$

$\Rightarrow a_{ij} = -a_{ji}$

So, if T of A is equal to 1 times A. This implies that A is equal to A transpose, which is i.e this means that, the matrix is a symmetric matrix, this is what is called as a symmetric matrix and what will happen when T of A is equal to minus of A lambda being minus 1 here. This implies that A transpose is equal to minus of A.

So, this implies that aij is equal to minus of aji. So, this will turn out to be the eigenvectors corresponding to minus 1. So, it is a good exercise to sit down and check what the basis for the eigenspace corresponding to 1 will be and what will be the eigenspace corresponding to minus 1 will be. Let me stop here.