Linear Algebra
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Lecture 37
Problem Session

So, this is a problem session, which is based on the fifth week of this course, the material covered in the fifth week of this course and as usual it is meant to supplement the questions that were given to you in the assignment. I hope you have thought about the problems in the assignment quite well. Let us begin by solving a problem on the change of basis matrix.

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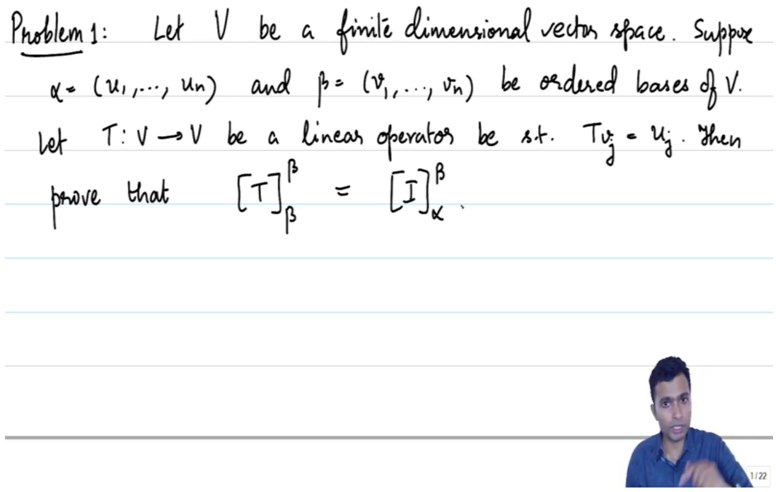

So, problem one, so let V be a finite dimensional vector space and suppose we are given two ordered bases. Suppose, alpha equal to say u1 to un and beta say v1 to vn be ordered bases of V and let T be a linear operator, be a linear operator, linear transformation with some V to itself which maps vj to uj. T of vj is equal to uj, then prove that the matrix of T from alpha to, matrix of T with respect to the bases beta is equal to the change of bases matrix from alpha to beta.

So, notice that the change of bases matrix from alpha to beta is the matrix of the identity linear transformation. So, this problem tells us that the change of base is matrix, is actually the matrix of the linear transformation, which is just mentioned in this problem.

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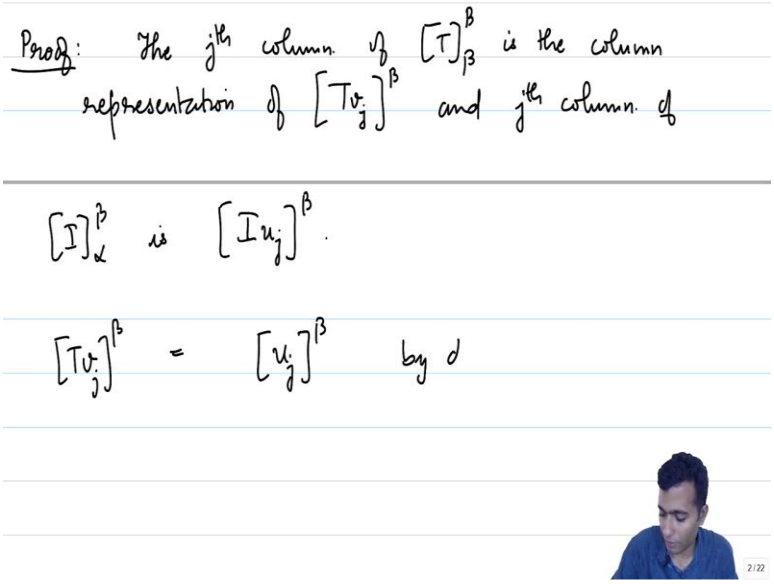

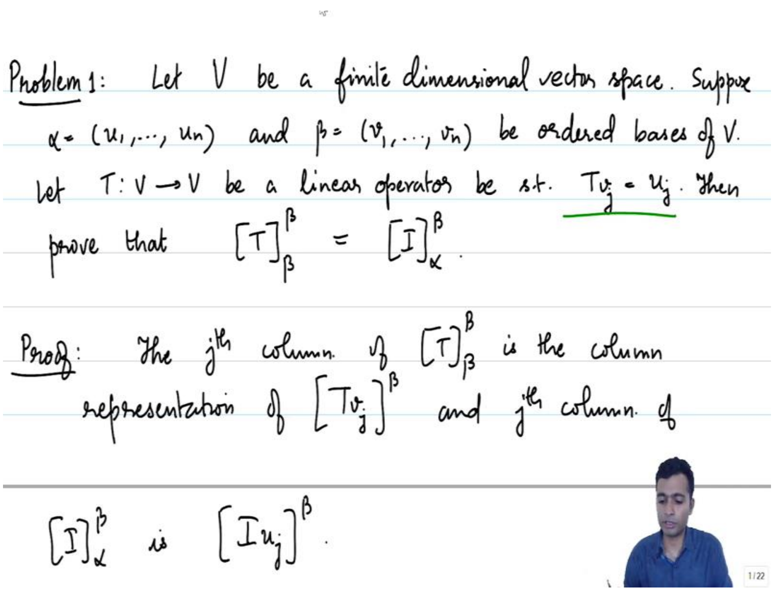

So, let us solve for this problem, let us prove this result. So, we shall solve this problem by proving that the columns of the matrix of T with respect to beta and the change of bases matrix from alpha to beta are the same. So, we will just proof that the columns are the same. Why is that the case? So, let us see, why is it enough? Because if all the columns are the same, then the matrices in particular are equal and what are the columns.

The columns are well understood, so the jth column of T beta beta is the column representation of the jth vector in beta, which is T of jth vector in beta, which is Tvj which is with respect to beta and the jth column of I alpha beta is I and jth, I of vjth vector in alpha which is uj and this is with respect to beta. So, let us see if these two columns are the same. What is Tvj beta, this is what we are interested in.

So, we will not explicitly compute this, we will just notice that Tvj by the very definition is uj beta by definition of T, just scroll up, let us just scroll up and observe which I am underlining in green is Tvj is equal to uj is the definition of T itself. So, notice that there is a unique such linear transformation which can be defined in this manner if you extend it as a linear transformation to T by one of the theorems we have proved earlier and by the very definition of T, T vj is uj and this turns out to be uj and the column representation of vj with respect to beta.

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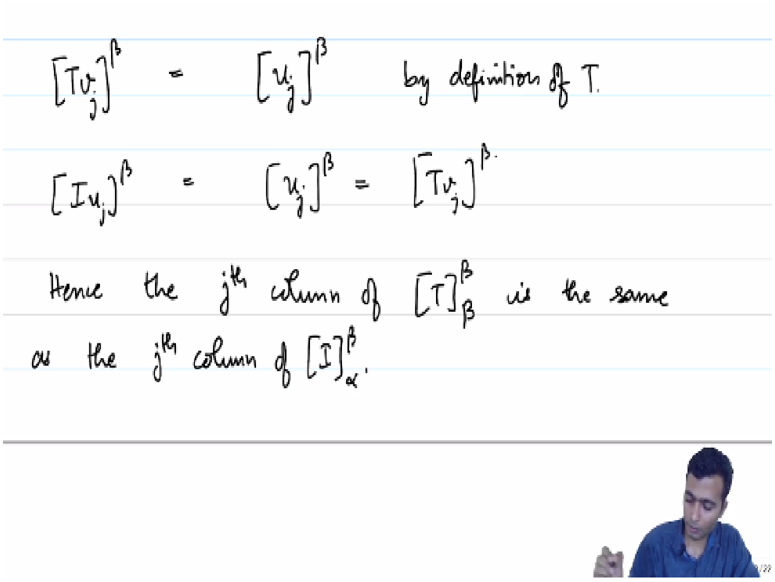

How about I uj with respect to beta, this I uj is the identity map on uj which is again uj with respect to beta, which is T vj with respect to beta. Hence the, hence the jth column of T beta beta is the same as the jth column of I alpha beta. So, the column that was picked was arbitrary.

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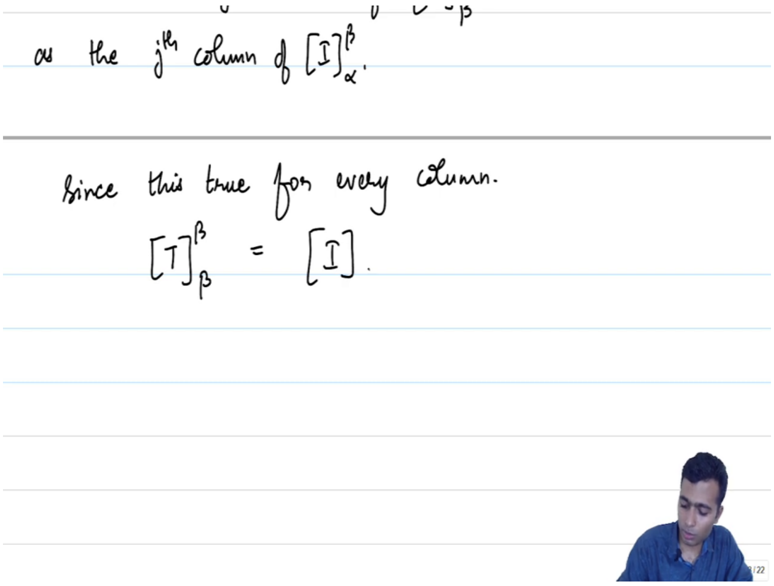

Since this is true for every column, we conclude that the matrices are the same. So, that is actually quite remarkable, because these two notions are fairly the same, that is what this problem tell us, the matrix of linear transformation if you have understood it, well every change of bases matrix is just this.

So, the next problem. So, the next problem deals with the notion of a graph of a linear transformation. So, this is a subset which is defined in the product space of V cross W where V is the domain and W is the codomain. So, let me write down the problem and give the relevant definitions there.

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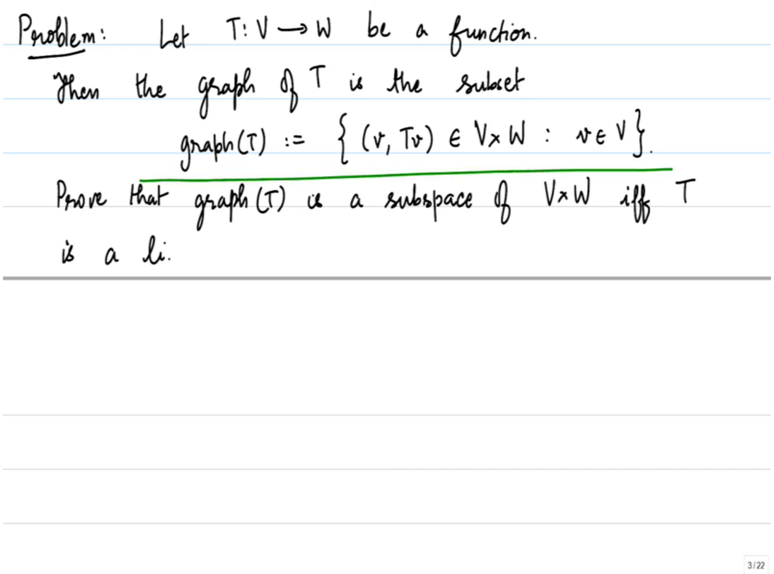

Problem: Let $T : V \longrightarrow W$ be a function.

Then the graph of T is the subset

$$\text{graph}(T) := \{ (v, Tv) \in V \times W : v \in V \}.$$

Prove that graph(T) is a subspace of $V \times W$ iff T is a linear transformation.

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Problem, so let T be a linear operator, linear map from linear transformation from V to W be a linear transformation. We define, then the graph of T is the subset, let me write it down graph of T is being defined to be the set of all v, Tv in V cross W such that v belongs to capital V.

So, notice that this is a subset of V cross W which is a product vector space, the vector space operations there are borrowed from or not borrowed, are obtained using the vector space operations in V and W and graph of T is now just a subset which is defined explicitly in this manner.

So, what does the problem us to do? The problem asks us to prove the following. Prove that graph of T is a subspace of V cross W, so is a subspace of V cross W, if and only if T is a linear map. So, I should maybe change the problem a bit, we did not have started with linear transformation be a function and still we can talk about the graph of the function T. So, T be some function.

Then we can talk about the graph of T in this manner, in the definition which I have just underlined in green. So the problems tell us that graph of T is in particular a subspace if and only the linear transfer, the map T, the function T is a linear transformation, since (())(08:32) is a characterization of when is a, another characterization of when a function is a linear transformation.

So, there are two parts to this problem, one is to show that if graph of T is a subspace, then T is a linear transformation and the converse that if T is a linear transformation, then graph of T is necessary a subspace. So, let us give a proof of the statement here.

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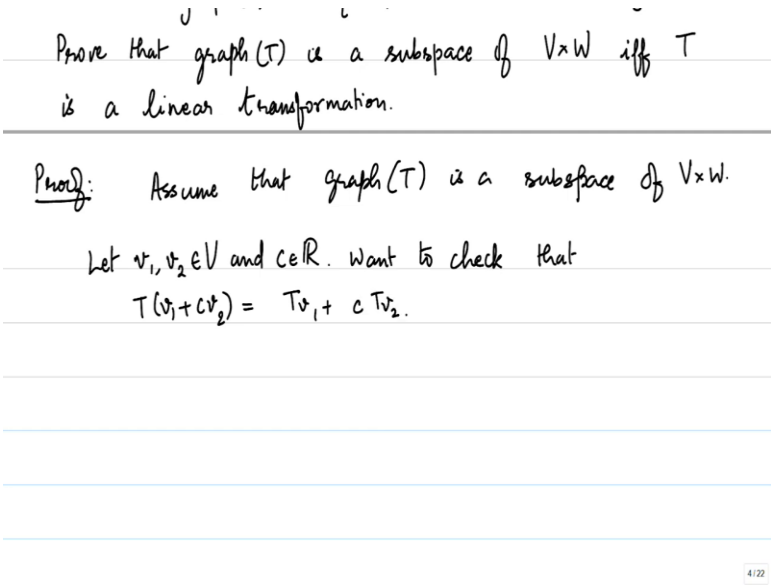

So, notice that graph of T being a subspace of V cross W is in the vector space operation of V cross W which, let me remind you is defined using the vector space operation in V and the vector space operations in W. So, what does, so let us prove the forward direction, assume that graph of T is a subspace of V cross W. So, we would like to now show that T is a linear transform.

So, let v1 and v2 be in capital V. So, we would like to see. So, want to check that T of v1 plus v2 is equal to T, maybe I should just do it in one go. So, let us change it a bit let v1, v2, V in capital be and c be in capital R the field of scalars, then T of cv, v1 plus cv, we would like to show that, this is equal to T v1 plus c times T v2, so this is our goal. Let us see, let us see what can be established.

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Let $v_1, v_2 \in V$ and $c \in \mathbb{R}$. Want to check that

$$T(v_1 + cv_2) = Tv_1 + cTv_2.$$

Since graph(T) is a subspace and

$(v_1, Tv_1) \in$ graph(T), $(v_2, Tv_2) \in$ graph(T)

we have

$$(v_1, Tv_1) + c(v_2, Tv_2) \in \text{graph}(T)$$

i.e. $$(v_1 + cv_2, Tv_1 + cTv_2) \in \text{graph}(T)$$

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So, notice that, notice that graph of T being a subspace. Since graph of T is a subspace and v1, Tv1 belongs to graph of T, v2, Tv2 belongs to graph of T. We have the following element also, since graph of T is a subspace, we have v1 Tv1 plus c times v2 Tv2 belongs to graph of T. This is exclusively because, graph of T is closed under scalar multiplication and vector addition.

So, because of that, this element should also be graph of T, but if you carefully look at it i.e v1 plus cv2 by the very definition of our vector addition in the product vector space. This is v1 plus cv12 and Tv1 plus c Tv2, this belongs to graph of T, but what is graph of T. Graph of T is the correction of all vectors of the type v, Tv, so that means there is some w.

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$$(v_1, Tv_1) + c(v_2, Tv_2) \in \text{graph}(T)$$

i.e. $(v_1 + cv_2, \; Tv_1 + cTv_2) \in \text{graph}(T)$

By the defn of graph(T), $\exists\, w \in V$ s.t

$$(v_1 + cv_2, \; Tv_1 + cTv_2) = (w, Tw).$$

$$\Rightarrow \quad v_1 + cv_2 = w$$

$$Tv_1 + cTv_2 = Tw$$

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By the definition of graph of T, there exist some w such that where w in capital V, such that v1 plus cv2, Tv1 plus c Tv2, this vector is equal to w, Tw, cause every element in graph of T is of this type w, Tw. But what does it mean or two elements to be equal in the product vector space this means that v1 plus c v2 is equal to w and Tv1 plus c Tv2 is equal to Tw.

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$$Tv_1 + cTv_2 = Tw$$

But $\Rightarrow$ $Tv_1 + cTv_2 = T(v_1 + cv_2)$ by $(*)$.

Conversely, Let (v_1, Tv_1) and $(v_2, Tv_2) \in \text{graph}(T)$.

Then $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, \; Tv_1 + Tv_2)$

$$= (v_1 + v_2, \; T(v_1 + v_2))$$ Since T-linear transformation.

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But what was w? W was v1 plus cv2, but this implies Tv1 plus c Tv2 is equal to T of from star, v1 plus c v2 by star and this is precisely what we had attempted to prove. The converse is extremely similar and argument, I will just quickly write the steps for you. So, what is it that we have to show conversely, we assume that T is a linear transformation and we would like to show that graph of T is a subspace.

So, I will just show the vector addition, so let v1 Tv1 and v2 Tv2 be in graph of T. We will show that the vector addition of these is also an element in graph of T scalar, multiplication I will leave it to you as an exercise, it is extremely similar or very similar to how we will prove this. So, what is v1 then, v1 Tv1 plus v2 Tv2 is equal to v1 plus v2 Tv1 plus Tv2, but what do we know about T, T is a linear transformation.

Since T is a linear transformation, so this I will not write it like this. This is equal to v1 plus v2, the first element remains untouched, what is Tv1 plus Tv2? Tv1 plus Tv2 is equal to T of v1 plus v2 because T is a linear transformation, since T is a linear transformation. But what does that mean?

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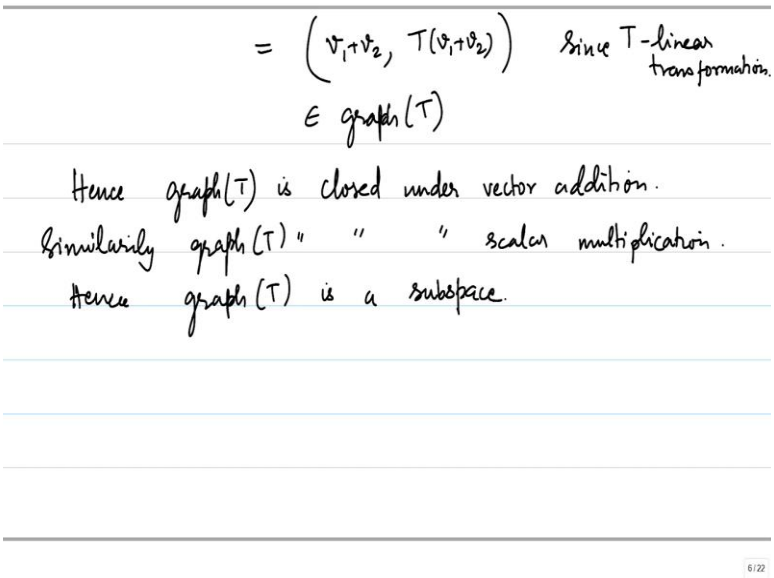

This just tells us that, this is an element of graph of T as well, because this is of the type w, Tw, where w is v1 plus v2, so this is in graph of T. So, graph of T is closed under vector addition. Similarly, graph of T is also closed under vector, scalar multiplication. I will leave that as an exercise for you and therefore it is a subspace.

So, the next problem is another problem which explores the notion of product vector spaces. This actually is a problem that the problem which we will be doing next, will be a problem which actually borrows a lot of notions which we have developed so far. So, let us see what the problem is.

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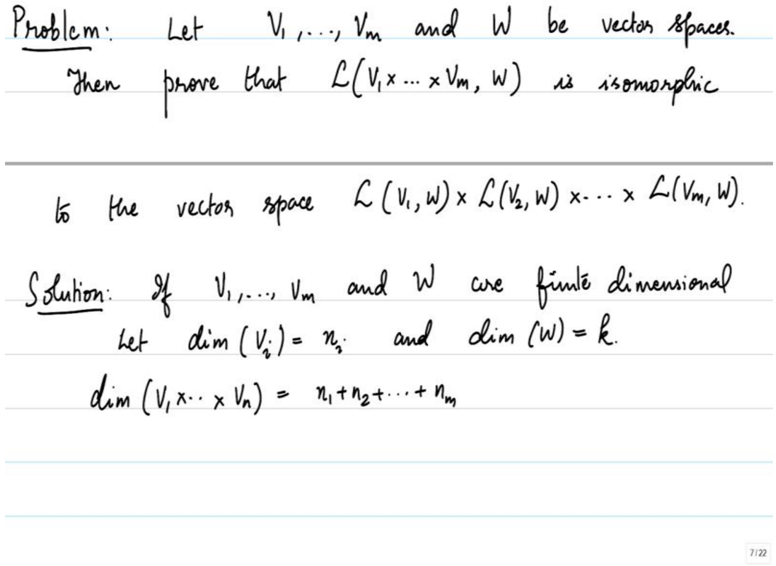

So, let V1 to Vm and W be vector spaces then prove that, so recall that L of V, W is the space of all linear transformations from V to W. So, here we are going to talk about L of V1 cross up to Vm, W. So, this is the vector space of all linear transformations from the vector space V1 cross V2 up to Vm into W.

Now this is a vector space we already know that the linear, connection of all linear transformations from a vector space to another vector spaces, vector space in itself. The problem tells us that this vector space is isomorphic to the vector space which is L of V1, W, which is in particular a vector space product L of V2, W up to L of Vm, W.

So, let us try to prove this problem, proof rather. To prove this result, my first observation would be that if V1, V2 up to Vm and W if they were finite dimensional vector spaces then the proof is extremely straight forward. So, let us just note that case first. If V1, V2 up to Vm and W are finite dimensional then notice that dimension of V1, so let each of these dimensions, let dimension of Vi be equal to say ni and dimension of W be equal to something like say k, then the first observation would be that V1 cross V2 up to Vn as dimension n1 plus n2 plus, up to nm.

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Solution: If $V_1, \ldots, V_m$ and W are finite dimensional

Let $\dim(V_i) = n_i$ and $\dim(W) = k$.

$$\dim(V_1 \times \cdots \times V_m) = n_1 + n_2 + \cdots + n_m$$

$$\& \dim\big(\mathcal{L}(V_1 \times \cdots \times V_m, W)\big) = (n_1 + \cdots + n_m)k$$

$$\dim\big(\mathcal{L}(V_j, W)\big) = n_j k$$

$$\dim\big(\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)\big) = n_1 k + n_2 k + \cdots + n_m k.$$

$$= \dim\big(\mathcal{L}(V_1 \times \cdots \times V_m, W)\big)$$

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And what is the dimension of and dimension of L of V1 cross V, I think I have made a mistake here it is m Vm, W. This vector space will have dimension, if you go back to the material from your previous weeks, this will be n1 plus n2 up to nm times k.

Now what is dimension of L of Vj cross W that again by our very, one of the theorems, that by one of the theorems we proved earlier, this is going to be nj times k and dimension of L of Vj rather V1 W cross L of Vm W. This will be sum of all these dimensions, which will be n1 times k plus n2 times k plus up to nm times k, which is equal to the dimension of L of V1 cross up to Vm, W.

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$$\dim\big(\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)\big) = n_1 k + n_2 k + \cdots + n_m k.$$

$$= \dim\big(\mathcal{L}(V_1 \times \cdots \times V_m, W)\big)$$

Hence they are isomorphic.

Define $\Phi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$

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Solution: If $V_1, \ldots, V_m$ and W are finite dimensional

let $\dim(V_i) = n_i$ and $\dim(W) = k$.

$$\dim(V_1 \times \cdots \times V_m) = n_1 + n_2 + \cdots + n_m$$

$$\& \quad \dim\left(\mathcal{L}(V_1 \times \cdots \times V_m, W)\right) = (n_1 + \cdots + n_m)k$$

$$\dim\left(\mathcal{L}(V_j, W)\right) = n_j k$$

$$\dim\left(\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)\right) = n_1 k + n_2 k + \cdots + n_m k.$$

$$= \dim\left(\mathcal{L}(V_1 \times \cdots \times V_m, W)\right)$$

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So, we have now two vector spaces which have the same dimension and by one of the theorems, we have proved earlier, they have to isomorphic, hence they are isomorphic. But if you notice the problem statement of the problem carefully, there is no assumption on whether V1, V2, Vm or W any of them are finite dimensional that assumption is not given to us. Hence this is a proof which will work only in a very special case when they are finite dimensional.

Let us give a proof of, when let us give a proof of the problem without putting in the extra assumption that these are finite dimensional vector spaces. So, what we will do is, in this case we cannot invoke some trick or some theorem powerful theorem and say that yes, they are isomorphic, because the dimensions are same. Now we will try to explicitly construct an isomorphism between two vector spaces.

So, let us define a map phi from L of V1, W cross this particular product vector space Vm, W. Notice that each of these are vector spaces and the product vector spaces define in the usual manner. From this let us define a map into L of V1 cross up to Vm W. So, remember that these are not the most straight forward vector spaces that we look at. These are vector spaces of linear transformation.

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Define $\Phi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$

$$\Phi(T_1, \dots, T_m) = T$$

where $T(v_1, \dots, v_m) := T_1 v_1 + T_2 v_2 + \cdots + T_m v_m.$

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Define $\Phi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$

$$\Phi(T_1, \dots, T_m) = T$$

where $T(v_1, \dots, v_m) := T_1 v_1 + T_2 v_2 + \cdots + T_m v_m.$

$$T\big((v_1, \dots, v_m) + (u_1, \dots, u_m)\big) = T(v_1 + u_1, \dots, v_m + u_m)$$

$$= T_1(v_1 + u_1) + \cdots + T_m(v_m + u_m)$$

$$= T_1 v_1 + T_1 u_1 + \cdots + T_m v_m + T_m u_m$$

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So, we will have to talk about what is phi of a T1, T2 up to Tm where Tis are linear transformation from Vi to W and what should it give us. It should give us some linear transformations from V1 cross V2 up to Vm and into W. So, what will this T will be, what will this T be, T will have domain v1 cross v2 to up to vm. So, a typical element will be v1, v2 up to vm.

So, we would like to define what T of v1, v2 up to vm is. So, we would like to define what T of v1, v2 up to vm. So, we will define this to be, let us see T1, v1 which makes sense plus T2 v2 plus Tm vm. So, suppose we define our T in this manner. So, well barrage of questions, plenty of questions start coming up right now. The first question is capital phi linear, first of all is T, this capital T which I am now underlining in green.

Is this a linear transformation at all, is this a linear map at all. So, that is actually a simple check I will not venture or maybe I should do it. So, let us say suppose T of v1 to vm plus u1 to um is considered by the definition of our vector space operations on the product space, this is just going to be equal to T of v1 plus u1 vm plus um maybe I do not have to write it like this, let us see anyway.

This object which is equal to T1 of v1 plus u1 plus up to Tm of vm plus um which is equal to T1 v1 plus T1 u1, why? Because, T1 is a linear transformation, similarly T2 is a linear transformation. Similarly, Tm is a linear transformation. Notice that all these make sense, because um belongs to vm and Tm is a linear transformation from vm to capital W. So, um is in capital Vm, so this makes sense.

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where $T(v_1, \dots, v_m) := T_1 v_1 + T_2 v_2 + \cdots + T_m v_m$.

$$T\big((v_1, \dots, v_m) + (u_1, \dots, u_m)\big) = T(v_1 + u_1, \dots, v_m + u_m)$$
$$= T_1(v_1 + u_1) + \cdots + T_m(v_m + u_m)$$
$$= T_1 v_1 + T_1 u_1 + \cdots + T_m v_m + T_m u_m$$
$$= (T_1 v_1 + \cdots + T_m v_m) + (T_1 u_1 + \cdots + T_m u_m)$$
$$= T(v_1, \dots, v_m) + T(u_1, \dots, u_m)$$

$|||^{ly}$ $\quad T(c(v_1, \dots, v_m)) = c\,T(v_1, \dots, v_m)$

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$= \dim(\mathcal{L}(V_1 \times \cdots \times V_m, W))$

Hence they are isomorphic.

Define $\Phi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$

$$\Phi(T_1, \cdots, T_m) = \boxed{T}$$

where $\boxed{T(v_1, \ldots, v_m) := T_1 v_1 + T_2 v_2 + \cdots + T_m v_m.}$

$$T((v_1, \ldots, v_m) + (u_1, \ldots, u_m)) = T(v_1 + u_1, \ldots, v_m + u_m)$$
$$= T_1(v_1 + u_1) + \cdots + T_m(v_m + u_m)$$
$$= T_1 v_1 + T_1 u_1 + \cdots + T_m v_m + T_m u_m$$

Now regrouping these are all vectors in W, so regrouping this is T1v1 plus T2v2 plus, as up to Tm vm by using the vector space additions properties. This is sum of these two vectors in W, but by the very definition this is T of v1 to vm plus T of u1 to um, so yes, we do have that the map T respects the vector addition. Similarly, you can show that T of c times v1 to vm, this is equal to c times T of v1 to vm.

So, what we just defined is indeed a linear map, T is indeed a linear map. So, the object we have defined here this T which we have defined as the following in this box that is a linear transformations. So, capital phi take element in the vector space L of v1, w cross L of v2, w up to L of vm, w and gives us some elements in L of v1 cross v2 up to vm, w. Now the question is this map phi, a linear map and then the next question would be is this phi an isomorphism.

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$|||^{ly}$ $T(c(v_1,\dots,v_m)) = c\,T(v_1,\dots,v_m)$

<u>Claim</u>: Φ is a linear transformation.

Let $(T_1,\dots,T_m)$ & $(S_1,\dots,S_m) \in \mathcal{L}(V_1,W)\times\cdots\times\mathcal{L}(V_m,W)$.

$$\Phi\big((T_1,\dots,T_m)+(S_1,\dots,S_m)\big) = \Phi\big((T_1+S_1, T_2+S_2,\dots, T_m+S_m)\big)$$

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Let us see phi the next thing to check is that, claim. Phi is a linear transformation. Notice that we are just proved that phi takes some element in the domain and gives, indeed gives us some element in the range. Now the step is to show that it is a linear transformation. So, for checking that, we take two vectors look at its sum and see whether that is equal to the sum of the images. So, let T1 to Tm and S1 to Sm be vectors in L of V1, W cross up to L of Vm, W.

Now capital phi of T1 up to Tm plus S1 up to Sm, this is what we are interested in. What is this? This is by definition equal to phi of, by the vector addition which is defined here, this is T1 plus S1 normally we should put a bracket, T2 plus S2 plus up to Tm and there is no plus sign, Tm plus Sm. This is what our phi would be.

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$$\Phi(T_1+S_1,\dots,T_m+S_m)(v_1,\dots,v_m) =$$

$$(T_1+S_1)v_1 + \cdots + (T_m+S_m)v_m$$

$$= T_1v_1 + S_1v_1 + \cdots + T_mv_m + S_mv_m$$

$$= (T_1v_1 + \cdots + T_mv_m) + (S_1v_1 + \cdots + S_mv_m)$$

$$= \Phi(T_1,\dots,T_m)(v_1,\dots,v_m) + \Phi(S_1,\dots,S_m)(v_1,\dots,v_m)$$

$$= \big(\Phi(T_1,\dots,T_m) + \Phi(S_1,\dots,S_m)\big)(v_1,\dots,v_m)$$

$$\forall\ (v_1,\dots,v_m) \in V_1 \times \cdots \times V_m.$$

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So, what is phi of T1 plus S1, up to Tm plus Sm on v1, v2 up to vm, where v1, v2 up to vm belongs to capital V1 cross capital V2 up to capital Vm. Let us see what this object is, by definition this is equal to T1 plus S1 on v1 plus up to Tm plus Sm on vm. So, notice that we have defined the notion of the sum of two linear transformations from the same domain to the same range.

So, T1 plus S1 by definition this is equal to T1 v1 plus S1 v1. Similarly, this is all, we have to split in a similar manner, finally the term would be Tm vm plus Sm vm. Now these are all vectors in W. I can group the relevant terms T2 v2 and so on Tm vm all terms like this and then S1 v1 plus up to Sm vm. So, we have grouped all these terms and what is this, this is basically phi of T1 to Tm on v1 to vm plus phi of S1 to Sm on v1 to vm.

Now yet again by the definition of some of linear transformations, this is phi of T1 to Tm plus phi of S1 to Sm on v1 to vm. So, I believe that you have kept track of which addition is happening where or you should maybe go through these steps once again and notice that this is for all v1 to vm in capital V1 cross up to capital Vm. But what does it mean for two functions to be equal on every element on the domain.

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$$\Rightarrow \quad \Phi\big((T_1,\dots,T_m)+(S_1,\dots,S_m)\big) = \Phi(T_1,\dots,T_m) + \Phi(S_1,\dots,S_m)$$

$$\text{Illy} \quad \Phi\big(c(T_1,\dots,T_m)\big) = c\,\Phi(T_1,\dots,T_m)$$

Hence Φ is a linear transformation.

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This means that the functions are the same, phi of T1 to Tm plus S1 to Sm is equal to phi of T1 to Tm plus phi of S1 to Sm. So, we are now established the additivity property as well. I will leave it to you to check in a very similar manner that phi of c times T1 to Tm is equal to c times phi of T1 to Tm.

So, notice that we have just established that this capital phi is a linear transformation. So, we have done one half of our job to prove that these two vectors spaces are indeed isomorphic. We have shown that, we have a linear transformation which is a good candidate of course, we have some work remaining. We will now establish that our capital phi is both injective and surjective.

(Refer Slide Time: 31:45)

Hence Φ is a linear transformation.

Claim: Φ is bijective.

Injectivity: We shall show that $Null(\Phi) = \{0\}$.

Let $(T_1, \ldots, T_m) \in Null(\Phi)$

$\Rightarrow \quad \Phi(T_1, \ldots, T_m) = 0 \quad$ (i.e. $0 : V_1 \times \cdots \times V_m \longrightarrow W$).

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So, the next claim, capital phi is bijective. Let me put it in one word and show that, this is a linear transformation which is both injective and surjective. If we manage to prove this claim where one of theorems, we would proved earlier we would have establish that capital phi is an isomorphism. So, let us see, injectivity let us first establish injectivity. When is a linear transformation injective, it is injective if and only if the null space is 0.

So, if you notice this particular problem is using almost a very big chunk of the theory that we have developed so far. So, I would request you to carefully look at the problem again once again after you go through this. So, injectivity we shall show that the null space of phi is the 0 vector. What is a 0 vector? 0 vector is 0, 0, 0 n times where 0 is the 0 linear transformation of the, the ith coordinate will give you the 0 linear transformation on vm, that is what the 0 on the right here will be. So, what is the meaning of something being in the null of phi.

So, let T1 to Tm be in the null space of phi. This means that phi of T1 to Tm, this is the 0 map. So, recall that this is the 0 map from V1 cross V2 up to Vm into W that is what this means. Let us try to see what this means.

(Refer Slide Time: 33:48)

Let $(T_1, \dots, T_m) \in \text{Null}(\Phi)$

$$\Rightarrow \quad \Phi(T_1, \dots, T_m) = 0 \quad (\text{i.e. } 0 : V_1 \times \cdots \times V_m \longrightarrow W).$$

i.e for any $v_1 \in V_1$

$$\Phi(T_1, \dots, T_m)(v_1, 0, \dots, 0) = 0$$

$$\Rightarrow \quad T_1 v_1 + 0 + \quad + 0 = 0$$

$$\Rightarrow \quad T_1 v_1 = 0$$

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This means that, i.e for vi in or let me put it this way, for v1 in capital V1 capital phi of, where is the T1, T1 to Tm of v1, 0, 0 this is the 0 map acting on this, which is the 0 in W, what is that? This is 0 in W. Now, what is this, the left hand side is just T1 v1 plus T2 of 0 which is 0 plus Tm of 0 which is 0, is equal to the 0, all these are vectors in W which implies T1 v1 is equal to 0. This is for all v, so for that is for any v1, this is true.

(Refer Slide Time: 34:45)

$$\Phi(T_1, \dots, T_m)(v_1, 0, \dots, 0) = 0$$

$$\Rightarrow \quad T_1 v_1 + 0 + \quad + 0 = 0$$

$$\Rightarrow \quad T_1 v_1 = 0$$

$$\Rightarrow \quad T_1 \equiv 0$$

lly $\quad T_j \equiv 0 \quad \forall \quad 1 \le j \le m.$

Therefore $\quad \text{null}(\Phi) = \{0\}.$

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Which implies T1 is identically equal to the 0 linear transformation. Similarly, we conclude that Tj is identically equal to 0 for all 1 less than or equal to j less than or equal to m and therefore null of phi is just the 0 0, 0, 0. So, that is what we have established, so hence T is injective.

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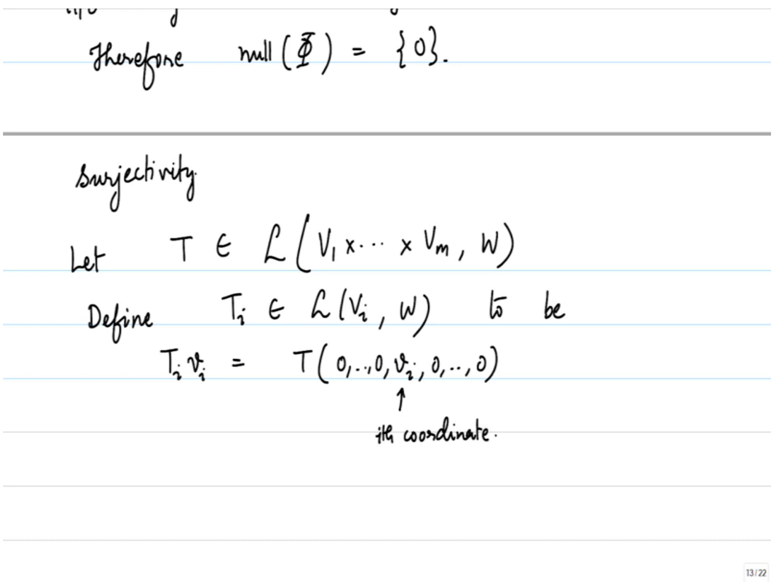

How about surjectivity, so let T be a vector, T be a vector in L of V1 cross V2 up to Vm, W that means it is a map from V1 cross, linear transform from V1 cross V2 up to Vm into W. So, we would like to get hold of a candidate in the domain which is L of V1, W up to L of Vm. W which is map to this T that is our goal.

So, what we will do is define, Ti in L of Vi, W to be Ti of v or let me put a vi or does not matter vi to be equal to T of 0,0,0, vi, 0, 0 where this is in the ith coordinate. You should check that this Ti is indeed linear transformation, so that I will leave it as an exercise.

(Refer Slide Time: 36:49)

Let $T \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$

Define $T_i \in \mathcal{L}(V_i, W)$ to be

$$\underline{T_i v_i = T(0, \ldots, 0, v_i, 0, \ldots, 0)}$$

ith coordinate.

Check that T_i is a linear transformation.

$$\Phi(T_1, \ldots, T_m)(v_1, \ldots, v_m)$$
$$= T_1 v_1 + \cdots + T_m v_m$$

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Define $\Phi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$

$$\Phi(T_1, \ldots, T_m) = T$$

where $\underline{T(v_1, \ldots, v_m) := T_1 v_1 + T_2 v_2 + \cdots + T_m v_m.}$

$$T\big((v_1, \ldots, v_m) + (u_1, \ldots, u_m)\big) = T(v_1 + u_1, \ldots, v_m + u_m)$$
$$= T_1(v_1 + u_1) + \cdots + T_m(v_m + u_m)$$
$$= T_1 v_1 + T_1 u_1 + \cdots + T_m v_m + T_m u_m$$
$$= (T_1 v_1 + \cdots + T_m v_m) + (T_1 u_1 + \cdots + T_m u_m)$$
$$= T(v_1, \ldots, v_m) + T(u_1, \ldots, u_m)$$

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Check that Ti is a linear transformation. Now let us see what is phi of, so this is for all i equal to 1 to 2 up to m. So, what is phi of T1 to Tm, so we got T1, T2 up to Tm like that. We would like to see what is, we would like to see what is a phi of T1, T2 up to Tm acting on some vector v1 up to vm. Now this by definition is T1 vi plus T2 v2 up to Tm vm by the very definition of what our phi is.

So, let me just show you what phi is which we have defined long time back, which we have used many, many times already. See this is our definition of phi by this definition we have, our phi of T1 to Tm on v1 to vm is the vector in w, but what is T1 of v1, let me remind you what Ti of vi was this is our Ti of vi. So, what we in particular T1 of v1.

(Refer Slide Time: 38:07)

$$= T_1 v_1 + \cdots + T_m v_m$$

$$= T(v_1, 0, \ldots, 0) + T(0, v_2, 0 \ldots, 0) + \cdots + T(0, \ldots, 0, v_m)$$

$$= T(v_1, v_2, \ldots, v_m)$$

Hence Φ is surjective

$\Rightarrow$ Φ is an isomorphism.

Define $T_i \in \mathcal{L}(V_i, W)$ to be

$$T_i v_i = T(0, \ldots, 0, \underset{\uparrow}{v_i}, 0, \ldots, 0)$$

ith coordinate.

Check that T_i is a linear transformation.

$$\Phi(T_1, \ldots, T_m)(v_1, \ldots, v_m)$$

$$= T_1 v_1 + \cdots + T_m v_m$$

$$= T(v_1, 0, \ldots, 0) + T(0, v_2, 0 \ldots, 0) + \cdots + T(0, \ldots, 0, v_m)$$

$$= T(v_1, v_2, \ldots, v_m)$$

Problem: Let $V_1, \ldots, V_m$ and W be vector spaces. Then prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ is isomorphic

to the vector space $\mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W) \times \cdots \times \mathcal{L}(V_m, W)$.

Solution: If $V_1, \ldots, V_m$ and W are finite dimensional

Let $\dim(V_i) = n_i$ and $\dim(W) = k$.

$\dim(V_1 \times \cdots \times V_m) = n_1 + n_2 + \cdots + n_m$

& $\dim(\mathcal{L}(V_1 \times \cdots \times V_m, W)) = (n_1 + \cdots + n_m)k$

This just will be T of v1, 0, 0, 0. What will be T2 of V that will be T of 0, v2, 0, 0, 0 and how about the last one Tm of vm that will be T of 0, 0, 0, 0 and the last one m. It precisely equal to T of v1, v2 up to vm because T is a linear transformation. So, yes, so we have indeed obtained a vector T1, T2 up to Tm in L of v1 w v2 cross L of v2 w cross up to L of vm w which maps to T. So, hence phi is surjective, which establishes that phi is an isomorphism.

So, we have proved the fairly long problem here to show that the two vector spaces, this is the problem which I just kept here, this is the vector spaces involved this and this are isomorphic, irrespective of whether it is finite dimensional or infinite dimensional, the result holds. So, let us now next prove a problem on affined spaces of a vector space.

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Problem 4: Let V be a vector space. Let $v, v' \in V$ and $\mathcal{U}, W$ be subspaces of V s.t $v + \mathcal{U} = v' + W$. Then prove that $\mathcal{U} = W$.

Proof: Since $v + \mathcal{U} = v' + W$

$\exists\ w \in W$ s.t

$v + 0 = v' + w$

$\Rightarrow (v - v') = w \in W$.

So, problem 4, I guess. So, let V should I, let me not impose the condition of V being a finite dimensional vector space as of now. So, let V be a vector space we will add a finite dimensional if needed, I do not think this problem needs to be a finite dimensional. So, let V be a vector space and v, v prime be vectors in capital V. So, let me do one thing let v, v prime be in capital V and u, w be subspaces of capital V such that v plus u is equal to v prime plus W.

So, the affine subset of U by translating it with v, is the same as the affine subset obtained by translating w by v prime. Suppose this is the case, then prove that u is equal to W. So, there is a fair amount of rigidity involved here if, this condition of v plus capital U is equal to v prime plus capital W is imposed then u and capital U and capital W should necessarily be the same subspace.

So, let us give a proof on this statement. So, what do we have to check here, so the first thing to notice is that, since v plus capital U is equal to v prime plus capital W, there exist a w in capital W such that v plus 0 which is in particular vector in capital U is equal to v prime plus small w. But what does this mean, this implies that v minus v prime is equal to w, which is in capital W.

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$$\Rightarrow \quad (v - v') = w \in W.$$

Let $u \in U$. Enough to show that $u \in W$.

$\exists \; w' \in W$ s.t.

$$v + u = v' + w'$$

$$\Rightarrow \quad u = -(v - v') + w' \in W$$

$$\Rightarrow \quad U \subseteq W$$

Ill^y $W \subseteq U$ and hence $U = W$.

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Problem 4. Let V be a vector space. Let $v, v' \in V$ and U, W be subspaces of V s.t $v + U = v' + W$. Then prove that $U = W$.

Proof: Since $v + U = v' + W$

$\exists\ w \in W$ s.t

$$v + 0 = v' + w$$

$$\Rightarrow (v - v') = w \in W.$$

Let $u \in U$. Enough to show that $u \in W$.

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So, now let u be in capital U, we will show that this small u is an element in capital W as well. Establishing that capital U is contained in capital W. If we do that, it will be a symmetric argument, by a very similar argument w will also be contained in capital U and hence they will be equal. So, it will be done, so enough to show, so enough to show that u is in capital W if we prove this then, as I said capital U will be contained in capital W.

So, let us see, if u is in capital U, again let us revisit the hypothesis we know that v plus capital U is the same as v prime plus capital W. So, that means, there exist w prime in capital W, such that v plus small u is equal to v prime plus w prime. But that would imply u is equal to v minus of v minus v prime plus w prime.

But notice that v minus v prime is in capital W minus of v minus v prime is in capital W and W prime is also in capital W, which means that this sum is in capital W, which implies that u is contained in capital W. Similarly, by a very similar argument W is contained in u and hence u is equal to W which we had set out to prove, so that completes the problem. So, the next problem again is, a problem which involves questions spaces. So, let me just write it down and then solve it.

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$|||^{ly}$ $W \subseteq U$ and hence $U = W$.

<u>Problem</u>: Let V be a finite dimensional vector space and U be a subspace of V. Let $(v_1 + U, \ldots, v_n + U)$ be a basis of V/U. Further let $(u_1, \ldots, u_m)$ be an ordered basis of U. Then $(v_1, \ldots, v_n, u_1, \ldots, u_m)$ is an ordered basis of V.

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So, let V be a finite dimensional vector space and capital U be a subspace of V. Let the ordered set, v1 plus capital U, v2 plus capital U, vn plus capital U, what is that, these are affine sets and these are elements in V mod u. Let this be a basis of V mod u. So, in particular V mod u is being given to be an end dimensional vector space and we are given a very specifically a basis of V mod u.

Further, let u1 to um be an ordered basis both are ordered basis by the way, basis of capital U. The problem is to show that then v1 to vn, U1 to Um is an ordered basis of capital V. So, in particular if we have one ordered basis of V mod u and an ordered basis of capital U then we will be able to get hold of a basis of capital V. So let us prove this statement. So, let me give a proof of this.

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Proof: We know that

$$\dim(V/U) = \dim(V) - \dim(U)$$

$$\Rightarrow \dim(V) = \dim(V/U) + \dim(U).$$

$$\Rightarrow \dim(V) = n + m.$$

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$$\Rightarrow U \subseteq W$$

$|||^{ly}$ $W \subseteq U$ and hence $U = W$.

Problem: Let V be a finite dimensional vector space and U be a subspace of V. Let $(v_1 + U, \ldots, v_n + U)$ be a basis of V/U. Further let $(u_1, \ldots, u_m)$ be an ordered basis of U. Then $(v_1, \ldots, v_n, u_1, \ldots, u_m)$ is an ordered basis of V.

Proof: We know that

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So, in order to look at a proof of this, let me remind you that there is a very nice statement about the dimensional theorem, in this case, special dimension theorem in this case for quotient vector spaces, you go back to your lectures and see, that we know that, dimension of V mod u is equal to the dimension of V minus V dimension of U.

So, in particular dimension of V is equal to dimension of V mod u plus dimension of U. Now we know exactly what these numbers are, dimension of V mod u is equal to n and dimension of u is equal to m by the very hypothesis. So, this implies that dimension of V is equal to n plus m. So, notice that v1, v1 plus u up to vn plus u is a basis there. Similarly, u1 u2 up to un is a capital U, un1, un2 up to um is basis of capital U.

(Refer Slide Time: 47:44)

$\Rightarrow \dim(V) = \dim(V/U) + \dim(U).$

$\Rightarrow \dim(V) = n + m.$

Consider $\beta = (v_1, .., v_n, u_1, .., u_m).$

Enough to show that β is linearly independent.

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So, what we get is that, consider this set consider beta to be equal to v1 to vn, u1 to um. This is a set consisting of precisely, n plus m (())(48:03). So, if we manage to prove that this is say linearly independent or if we manage to prove that this is a spanning set. If one of them is solved, one of them is proved, then we get the other for free by one of the consequences of the replacement theorem.

So, we will show that this is linearly, so we will show that this is linearly independent. So, enough to show to prove that this is a basis enough to show that beta is linearly independent. So, let us try to establish that, so let us take one linear combination which is equal to the 0 map.

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Enough to show that β is linearly independent.

Let $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{R}$ be s.t.

$$a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m = 0$$

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So, let a1 to an and b1 to bm be scalars. So, these are as of now real vector spaces be such that a1 v1 plus an vn plus b1 u1 plus bm um is equal to the 0 vector in capital V. Now let me remind you that we have a very special map from V into V mod u.

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$$a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m = 0$$

Recall that $\pi : V \to V/U$ given by

$\pi(v) = v + U$ is a linear transformation.

Then $\pi(a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_mu_m) = 0$

$\Rightarrow a_1(v_1 + U) + \cdots + a_n(v_n + U) + b_1(u_1 + U) + \cdots + b_m(u_m + U) =$

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Recall that pi from V into V mod u given by pi of small v is equal to small v plus capital U this is a linear transformation. So, let us apply pi to the vector on top, then pi of a1 v1 plus an vn plus b1 u1 plus up to bm um, this will be equal to pi of 0, which is equal to the 0 vector in V mod u. But what is this 0 vector in V mod u? It is 0 plus capital U and pi is a linear transformation. In particular this implies that a1 times v1 plus capital U plus up to an times vn plus capital U plus b1 times u1 plus capital U plus up to bm times um plus capital U is the 0 vector.

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Then $\pi(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = 0$

$$\Rightarrow a_1(v_1+U) + \cdots + a_n(v_n+U) + b_1\overset{0}{\cancel{(u_1+U)}} + \cdots + b_m\overset{0}{\cancel{(u_m+U)}} = 0$$

$$\Rightarrow a_1(v_1+U) + \cdots + a_n(v_n+U) = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_n = 0 \quad \left(\because (v_1+U, \ldots, v_n+U) \text{ is a basis of } V/U\right).$$

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Let $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{F}$ be s.t.

$$\underline{a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m} = 0 \longrightarrow$$

Recall that $\pi : V \to V/U$ given by $\pi(v) = v + U$ is a linear transformation.

Then $\pi(a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_m u_m) = 0$

$$\Rightarrow a_1(v_1+U) + \cdots + a_n(v_n+U) + b_1\overset{0}{\cancel{(u_1+U)}} + \cdots + b_m\overset{0}{\cancel{(u_m+U)}} = 0$$

$$\Rightarrow a_1(v_1+U) + \cdots + a_n(v_n+U) = 0$$

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But let me now remind you that u1, u2 up to um, they are basis elements in capital U. So, in particular they are an element of capital U and if you look at the affine subset here with these elements, they are all the same as the 0 element there. So, all these are 0 vectors in V mod u and what we are left with is a1 times v1 plus capital U plus up to an times vn plus capital U is the 0 vector in V mod u. But what were v1 plus u, v2 plus u up to vn plus u. They were, they formed a, basis of V mod u. So, this implies that a1 is equal to a2 up to an is equal to 0.

Since v1 plus capital U vn plus capital U is a basis in particular linearly independent of capital V mod u. So, because of this each of these are 0. We are done with half, so let us revisit our equation to begin with here. Here what happened now is that if this, if such an equation is 0 we have established that a1, a2 up to an should be necessarily 0, that is what we have established just now. So, let us see, the implication of this.

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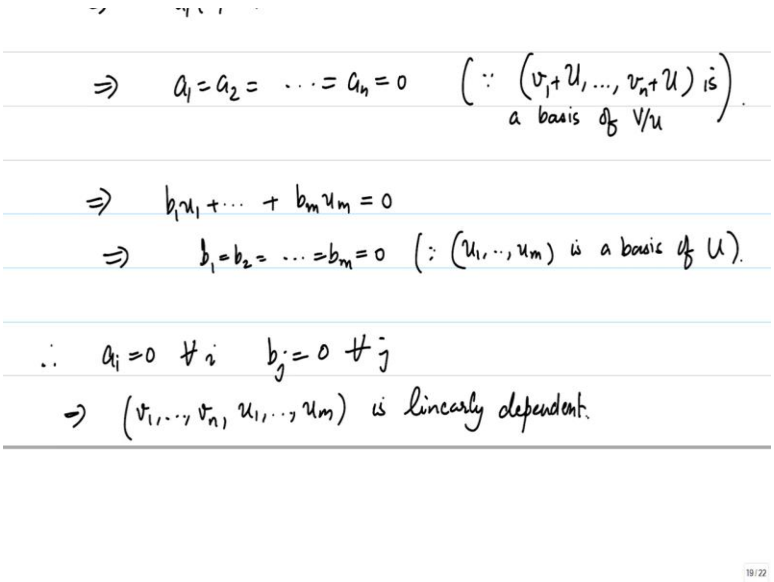

So if you put a1, a2 up to an is equal to 0 this implies b1 u1 plus up to bm um is equal to 0, but un u2 up to um is in particular a basis of capital U, they are linearly independent which implies b1 is equal to b2 up to bm is also equal to 0. Since, u1 to um is a basis of u. So, therefore ai is equal to 0 for all I, bj is equal to 0 for all j, where i is from 1 to n and j is from 1 to m and this implies that v1 to vm, vn and u1 to um is linearly independent. Now a linearly independent set of size n plus m in a vector space of dimension n plus m should necessarily be a spanning set as well.

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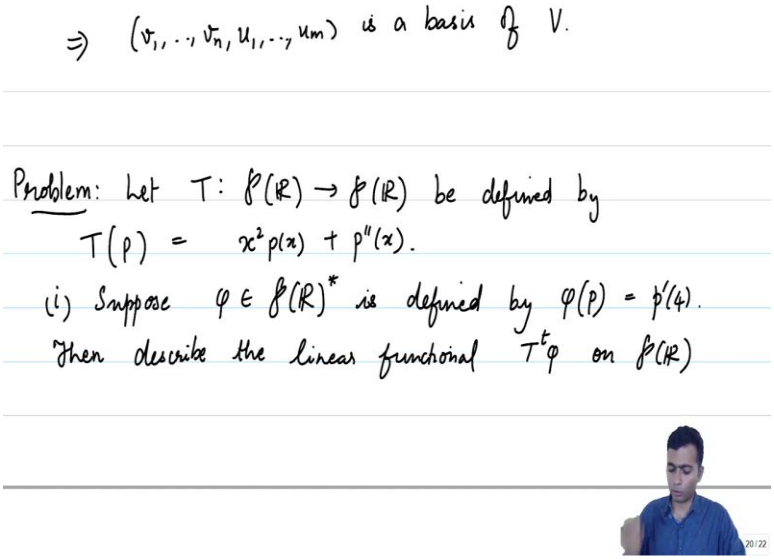

This implies that v1 to vn and u1 to um together is a basis of V and that is precisely what we had set out to prove. So, the next problem we will deal with dual spaces, the transpose of linear transformation. So, let us see what the next problem is. So, let us start with a linear transformation T from this space of polynomial over R into the space of polynomials over R. So, let T from p of R to p of R be defined by T of p is equal to x square times p of x square times p of x plus p double prime of x.

So, notice that x square times p of x is a linear transformation, p double prime of x. So, T of p going to x square times p of x is a linear transformation, x of p going to p double prime of x is also a linear transform, if you add to linear transformation, it is also a linear transformation. So yes, it is very easy to check that, whatever we have just written down is indeed a linear transformation.

So. suppose we have given such a linear transformation. Suppose, so the first part of this problem says that suppose, phi is an element in the dual space of p of R. Recall that the dual space of p of R consists of all linear functionals on p of R. They are linear maps from p of R into R. This is a vector space over R, so the linear functionals in particular will form a vector space which is denoted as p of R star.

So, let us pick one element phi which is in p of R star, which is defined by phi of p is equal to let us say p prime of 4, then the problem demands that we find out, what is T transpose of phi is, then describe let me just reword it, describe the linear functional rather T transpose only, linear function it is not a functional, T transpose of phi. So, before I describe what this will be, let me yeah this is actually going to be a functional.

So, where is T from? T is from p of R into, p of R T transpose will be from p of R star into p of R star. So, phi is an element in p of R star T transpose of phi will also be in element of p of R star. So, that is the first problem, so this is a linear functional in particular of p of R. So, I could have written in functional here no problem and what about the second problem.

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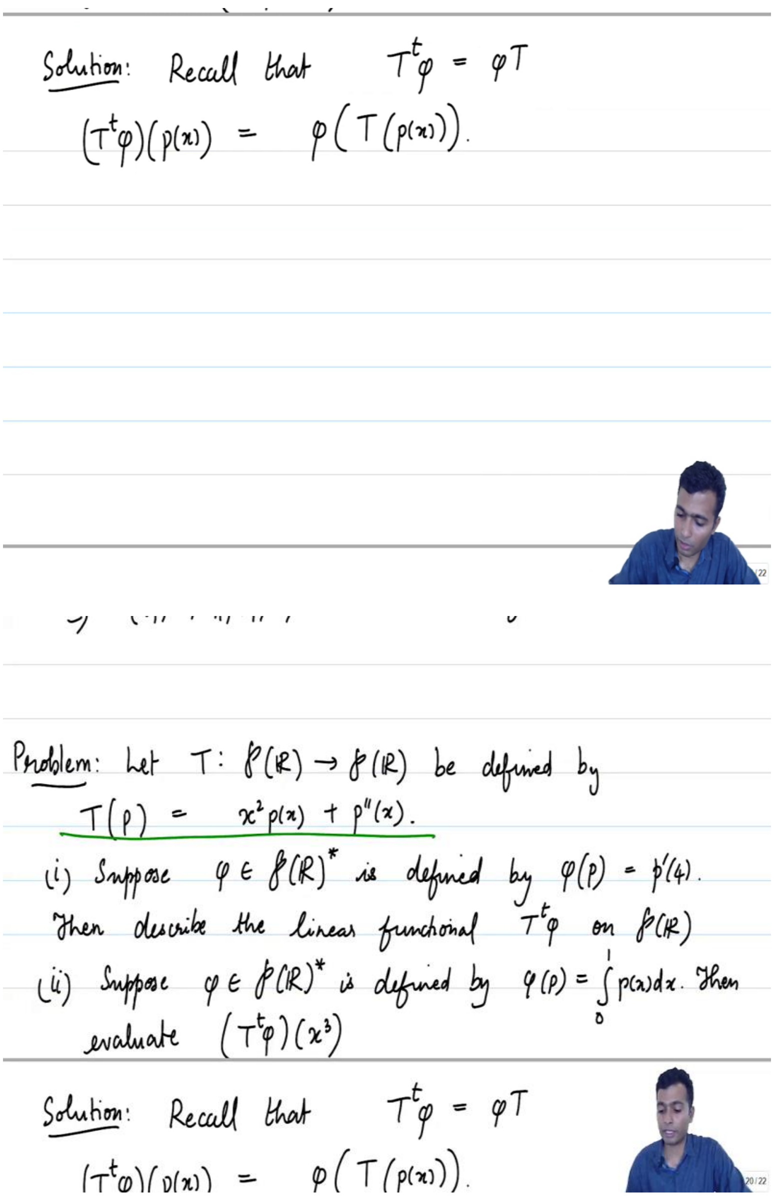

Suppose phi be in p or R star is defined by phi of p as being the integral from 0 to 1 of p of x dx. Then evaluate T Transpose phi, so notice that T transpose phi will be just like in the previous case that will be linear functional. So, in particular it can act on some polynomial let us say, what is the value when evaluated at T of x cubes, this is the (())(58:06), so this number is what we have to evaluate.

So, you should get back some element in R. So, this will clearly be some element in R. So, let us solve for it. So, let us recall what is the definition of T transpose, recall that T transpose phi is just defined to be phi T phi composed with T. So, T transpose phi at a point say p at a polynomial p of x will just be equal to phi of T of p of x. This is precisely what our definition tells us, but what is our T let us recall what our T is I am underlying it in green right now. T is defined as x square times p of x plus p double prime of x.

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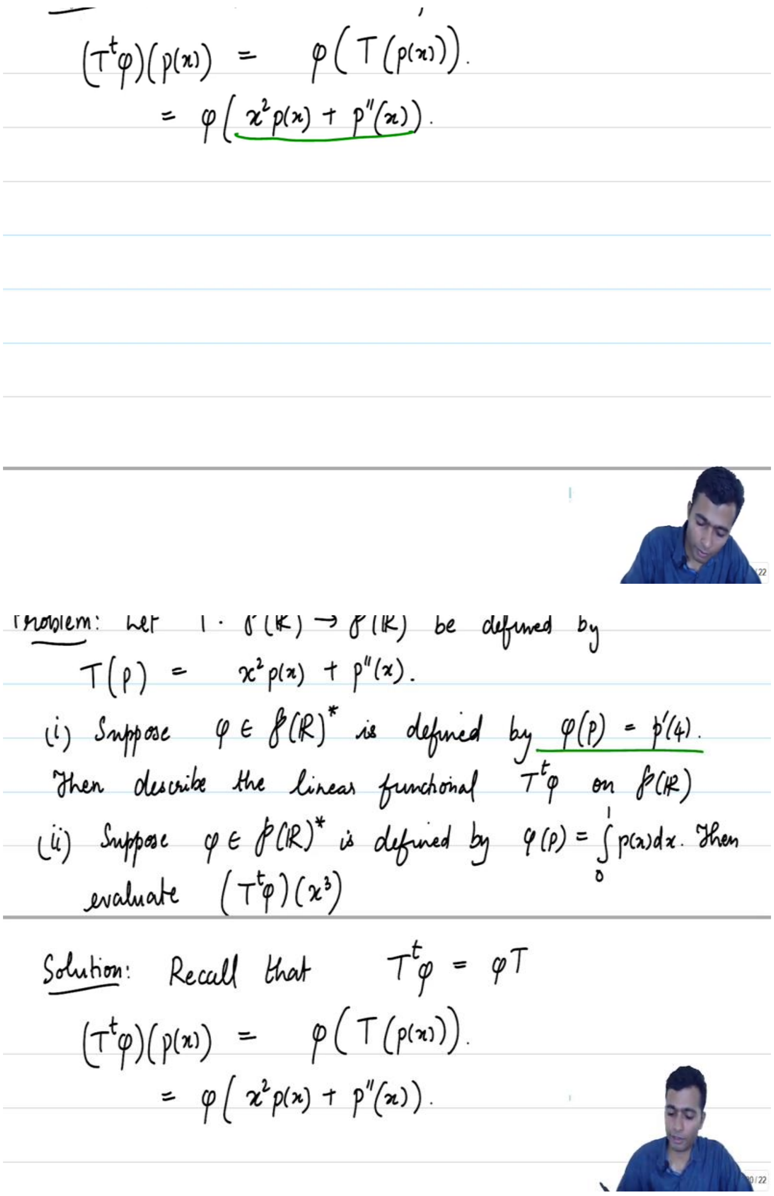

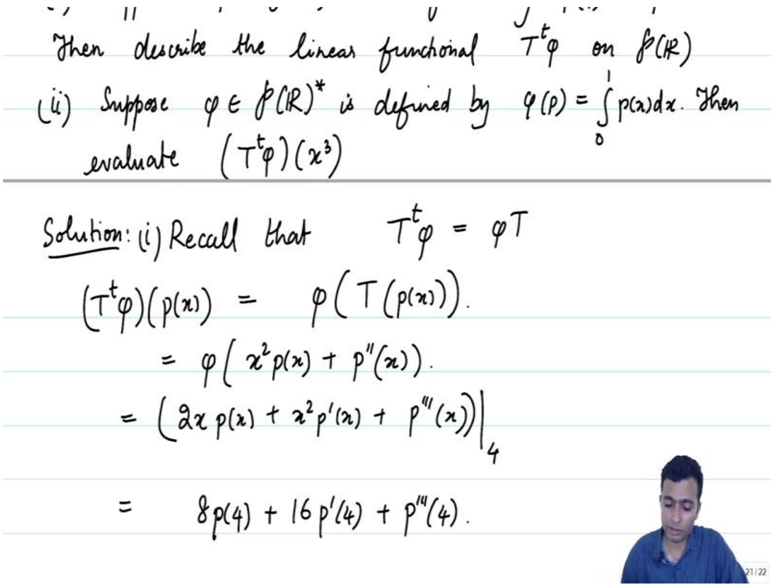

So, this is just going to be equal to phi of x square times p of x plus T double prime of x, then what is our phi let me now again underline it in green to show you what that is that is just the evaluation at 4 of the derivative of the polynomial. So, this polynomial which is, in the brackets, which again I am underlying it in green, that will be the derivative of that, is to be evaluated at 4.

So, that will just turn out to be just equal to, let me just quickly write it down, that is going to be 2x p of x plus x square times p prime of x plus p triple prime of x. This evaluated at 4 which is going to be equal to 8 times p of x plus 4 square is 16 times p, this is not p of x, this is p of 4, this is going to be p prime of 4 plus p triple prime of 4. This is precisely what T transpose phi of a polynomial p will look like. So, this is precisely the functional T transpose phi. So, that solves this problem. So, let us look at the second one, the second one is to show that, is to evaluate what is T transpose phi at x cube.

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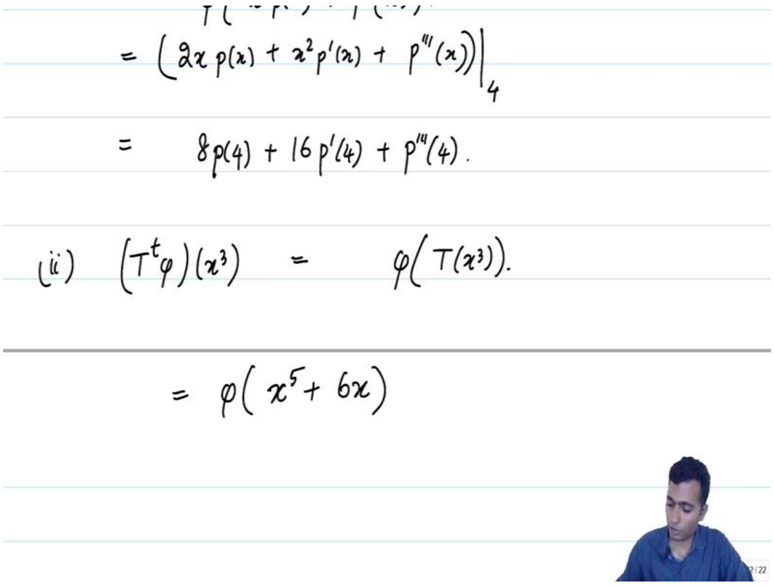

We are just going to, so we are just going to mimic, what we just did, so that is just going to be T transpose phi at x cube, this is what we would like to evaluate, this is nothing but phi of T of x cube. But we know precisely what T of x cube is this is equal to phi of x square times x to the power 3, which is x to the power 5 plus T double prime, which is 3 into 2, 6 times and what is phi of this.

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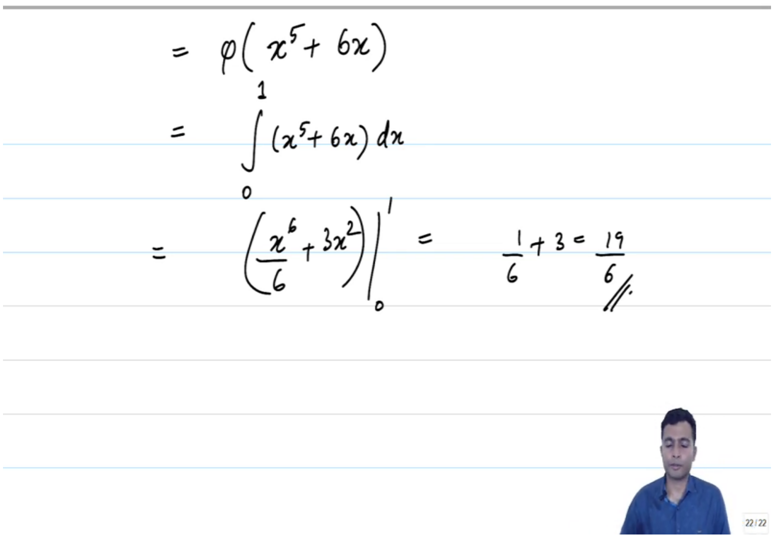

I hope I am not making a mistake here this is just going to be 0 to 1 x to the power 5 plus 6 x dx which is equal to x to the power 6 by 6 plus x square 3x square evaluated at, evaluated from 0 to 1, which is 1 by 6 plus 3, which is 19 by 6, this is the correct answer. Let me stop here.