Linear Algebra
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Lecture 9.4 Orthogonality

So, from high school physics we know that if the dot product of 2 vectors is equal to 0, then the 2 vectors are perpendicular each other or orthogonal to each other. Let us now define a similar notion of orthogonality for a generally inner product.

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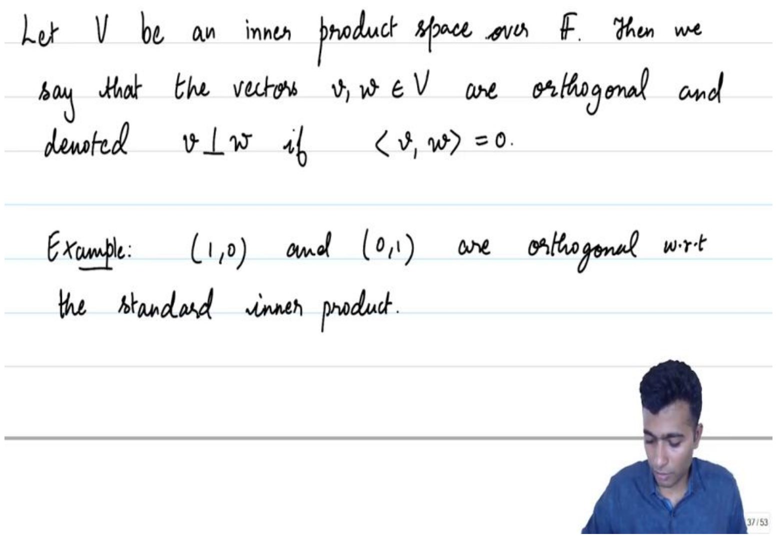

So, let V be an inner product space over R, so again over F sorry, our vector space could be, a vector space over R or vector space over C, does not matter, the definition holds. Then, we say that, the vectors v, w in capital V are orthogonal to each other are orthogonal and denoted v orthogonal w, if the inner product of v and w is 0. So, for example, if you look at 1,0 and 0,1, they are orthogonal with respect to the standard inner product. In fact, they are orthogonal even with respect to the nonstandard inner product, which we had defined earlier. It is just 2 times the standard inner product. Even with respect to that, these are the orthogonal vectors.

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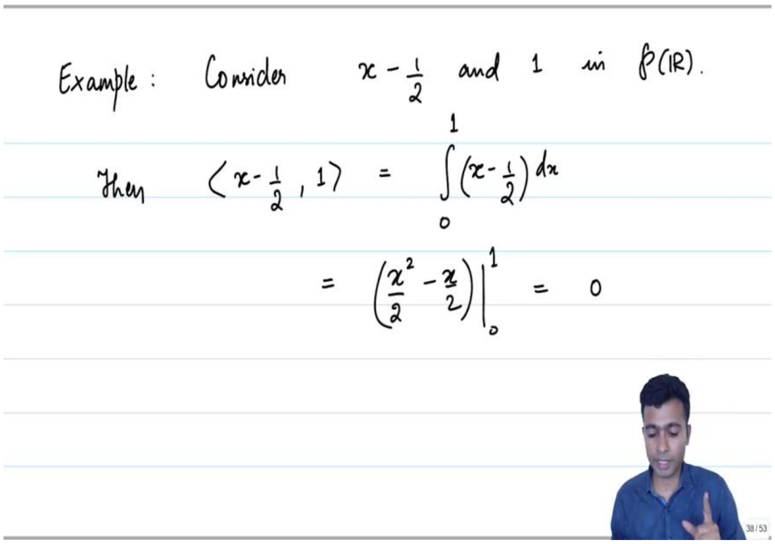

Let us take an example to depict that the orthogonality is a property which is very much dependent on what inner product we consider. So, let us consider P of R, so consider x minus 1 by 2 and 1 in P of R. Let us look at what would be the inner product of these 2. Then the inner product, so, let us look at, we had defined 2 different inner products. Let us look at the inner product of these 2 vectors with respect to both these inner products.

So the first o1 was x minus half and 1 was what, 0 to 1 x minus 1 by 2 times 1 bar dx and is this just was equal to integral 0 to 1, let me just write it down, x square by 2 evaluated from 0 to 1 minus x by 2 evaluated from 0 to 1.Actually it is minus x by 2, this evaluated from 0 to 1 which is equal to 0. So yes, x minus 1 by 2 and 1 are orthogonal with respect to this inner product.

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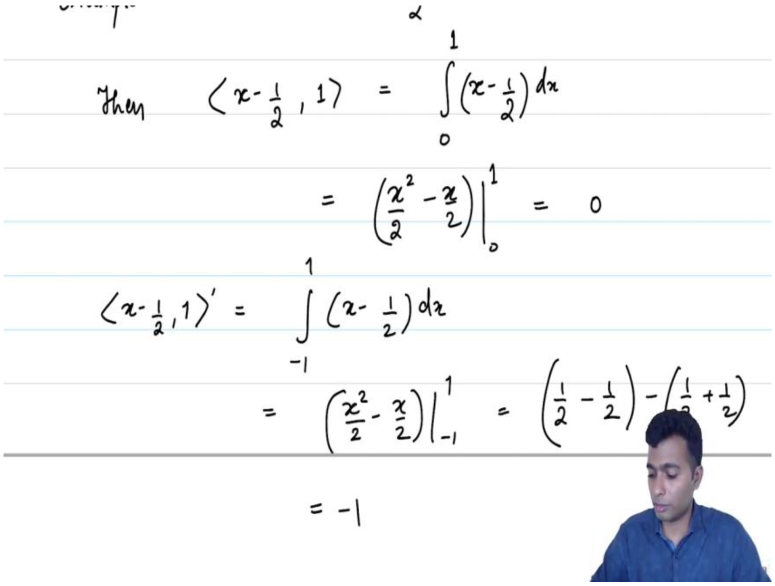

Let us see if it is orthogonal with respect to the other one. x minus half,1 let us put prime here to denote that it is different inner product. This is just in the integral minus 1 to 1 of x minus 1 by 2 dx, which is equal to x square by 2 minus x by 2, which is from minus 1 to 1, which is equal to 1 by 2 plus 1 by 2 minus, sorry, minus 1 by 2, minus 1 by 2 plus 1 by 2 which is equal to minus 1. It is not 0. So, these 2 vectors or these 2 polynomials are not orthogonal with respect to the second inner product.

So, the 2 vectors being orthogonal very much depends on which inner product we are considering. Let us look at 1 more example to show the other way. Let us consider 2 vectors or 2 polynomial which are inner product, orthogonal with respect to the second in a product inner but which is not with respect to the first 1.

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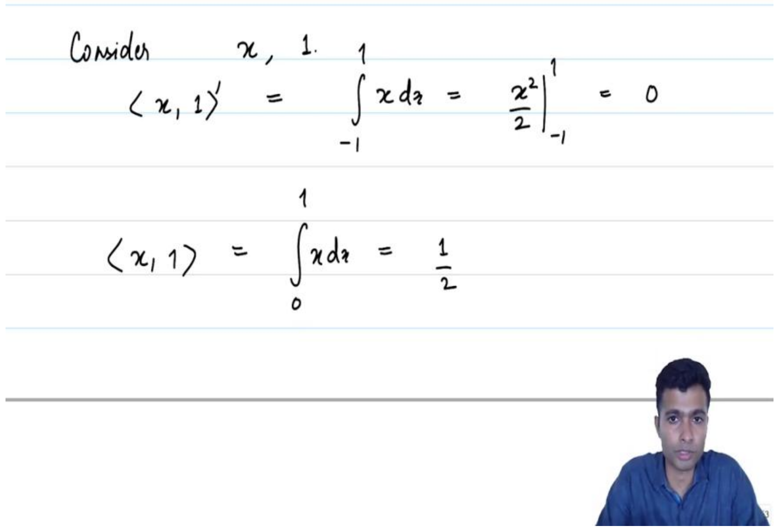

So, consider, yes, the polynomial x and 1 and what is the inner product of x with 1? This is just, let us put a prime here to see what happens for this. This is minus 1 to 1 x dx, which is equal to x square by 2, evaluator from minus 1 to 1 definite integral which is going to be 0. So yes, x and 1 are polynomial which are orthogonal to each other with respect to the second inner product. However, what is the inner product of x and 1 with respect to the first 1.

This is 0 to 1 x dx, which is just, let me just write it down and it is going to be half, so this is not equal to 0, so they are not orthogonal with respect to the first inner product. So, orthogonality is a relation which very heavily depends on which inner product we are considering. So being orthogonal is in some sense the other opposite extreme of being parallel. So, if you notice quasi shores, we had noticed that the quasi shores inequality.

If a vector w is a scalar multiple of a vector v then the inner product of v and w is, the absolute value of the inner product is equal to, the length of the, product of the length of the vector here. That is 1 extreme and the other extreme when it is a 0 and that is precisely when v and w are orthogonal to each other. So, the extremes are when it is parallel and when it is orthogonal to each other. So, orthogonality captures the other opposite extreme. Orthogonality is a symmetric property.

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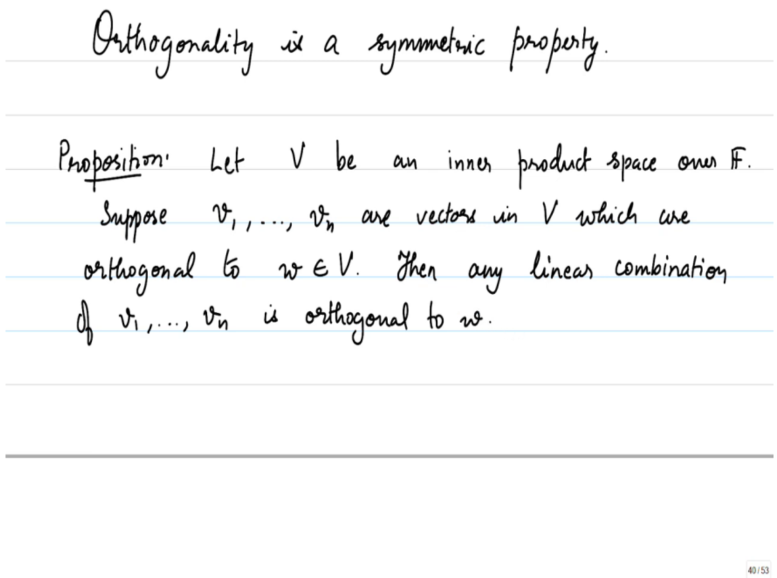

So, if v and w are orthogonal. So, let me just note that orthogonality is symmetric. If v is orthogonal to w, w is orthogonal to v. Orthogonality is a symmetric property. So, the next proposition, which we will prove tells is that orthogonality is a property which is preservative, if you consider a linear combination of vectors. So, if you write down and clarify what I just said. So, let V be some inner product space over F. So, it could be a real vector space, real inner product space or complex inner product space.

This theorem holds for both the cases. Let V be the inner product space over F. Suppose v1to vn are vectors in V which are orthogonal to another vector say small w in capital V. Suppose each of these vectors v1, v2 up to vn orthogonal to w then proposition tells us that any linear combination of v1, v2 up to vn will also be orthogonal to w. Then any linear combination of v1 to vn is orthogonal to w, small w. Let us give a proof of this proposition.

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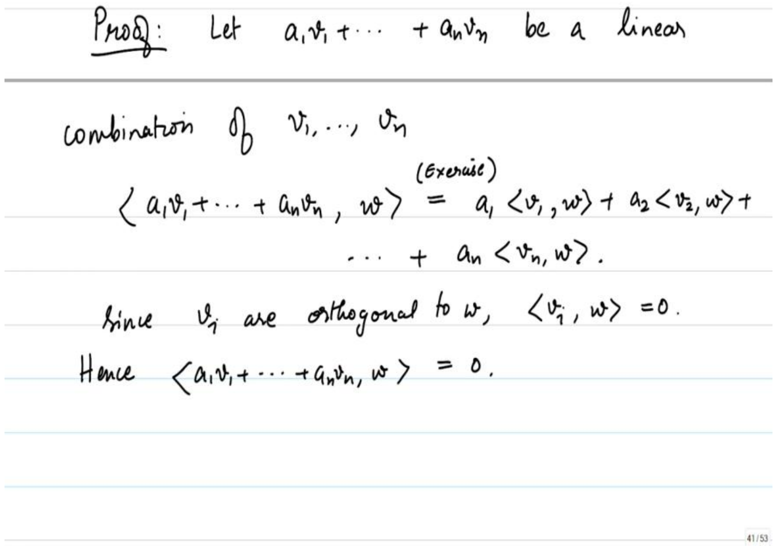

So, it is considered some linear combination of v1, v2 up to vn. So, let a1v1 plus up to anvn be a linear combination of v1 to vn. So here a1, a2 up to an are scalars. So, if it is a complex vector space, a1, a2 up to an are some complex numbers. If it is a real vector space, a1, a2 up to an will be some real number. So, what will be the inner product, so we would like to show that the vector a1v1 up to anvn is orthogonal to w, we would like to show that this is equal to 0.

But by linearity, by an induction argument on linearity, I will just leave it as an exercise for you to check, that this is a1 times the inner product v1 with w plus a2 times the inner product of v2 with w plus dot dot dot an times the inner product of vn with w. But what do we know about the vi’s? Each of the vi’s are orthogonal to w. Since, vi are orthogonal to w, vi, w, the inner product of this equal to 0 and hence a1v1 plus anvn, inner product with w will just turn out to be 0, 0 scalar.

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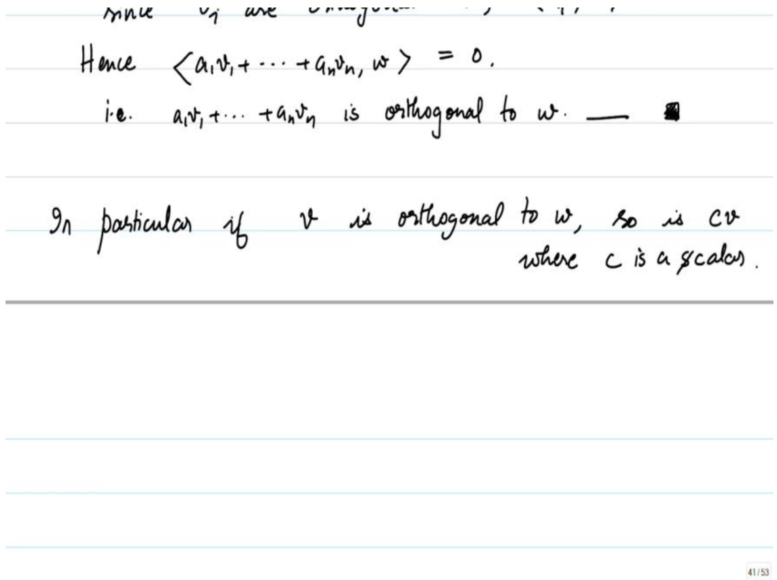

Hence, that is the definition a1v1 plus anvn is orthogonal to w. So, in particular if, so we approved the result, in particular if we take some vector v, which is orthogonal to w and if you look at any vector in the subspace spanned by v that will also be orthogonal to w, because it is a linear combination of v. So, in particular, let me just note that, if v is orthogonal to w, so is c times v, where c is a scalar and hence the entire subspace, any vector in the entire subspace will be orthogonal to w.

So, 1 of the first results that comes to our mind when we consider perpendicular vectors is the Pythagoras theorem. It is one of the oldest theorems, which has been known to mathematicians. We will now prove that a version of Pythagoras theorem is true in an inner product space. So, the celebrated Pythagoras theorem.

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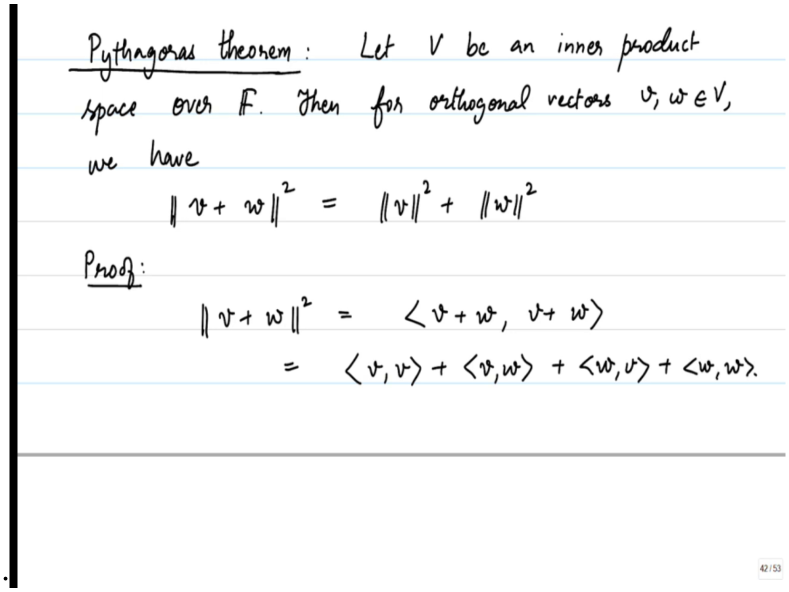

So, as you can see, all the famous classical theorems are getting generalized. So, this is clearly the right notion to look at. So, let V be an inner product space over F. So, either a real inner product space or complex inner product space, a Pythagoras theorem that we are going to state down holds. Then, for orthogonal vectors v, w in capital V, we have the length of v plus w square is equal to the length of v square plus the length of w square. So, the sum of the 2 sides of the triangle, right triangle is equal to the sum of the square of the 2 sides of the triangle is equal to the square of the hypotenuses.

Let us give a proof of the celebrated theorem, in the case of inner product space. So, what is the length of v plus w square? That is just the inner product of v plus w with itself, but our inner products is linear in both, the first variable and conjugating it in the second variable. Using that, this just going to be equal to v, v plus v, w plus w, v plus w, w. But what do we know about v and w? We know that v and w are orthogonal vectors.

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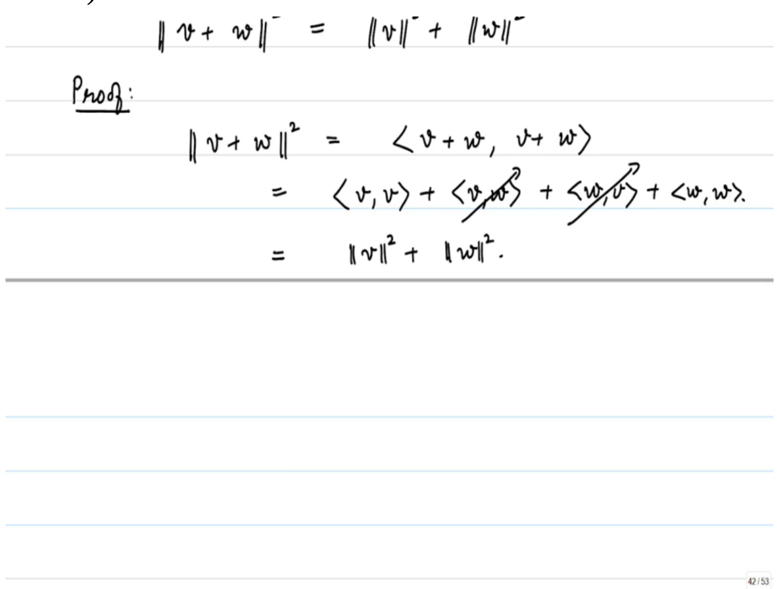

So, this inner product vanishes, this inner product vanishes, because orthogonality tells us that inner product of the v and w is equal to the inner product of w and v, which is equal to 0. What is this? This is nothing but the length of v square plus length of w square. So, the inner product and the properties there in uses a very compact and an elegant proof of the Pythagoras theorem. In fact, we can say a little more about, a more general version of the Pythagoras theorem. In fact, that can be proved. So, let me just write that down.

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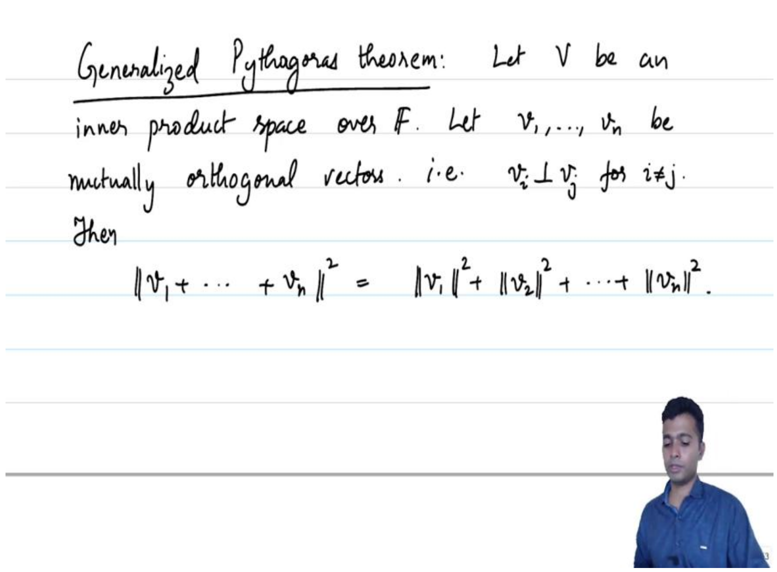

Let me call it generalized Pythagoras theorem. So, if so, let V be an inner product space over F and suppose v1, v2 up to vn are vectors which are mutually orthogonal. So, let v1 to vn be mutually orthogonal vectors. So, what does that mean? i.e. vi is orthogonal to vj for I naught equal to j. Of course, no vector can be orthogonal to itself. It is an exercise for you to check that no vector can be orthogonal to itself. So, let me have a look at what was the symbol used for orthogonality.

So, it has been introduced. So yes, v is orthogonal to itself is introduced by this particular symbol. So, vi is orthogonal to vj. So maybe after I write down the statement, I will give you an exercise to check that no vector can orthogonal to itself, in fact I just mentioned it. So then what does the generalized Pythagoras theorem tell us? It tells us that, if you look at the, sum of these vectors, look at the square of it is length, this is equal to square of length v1 square plus length of v2 square plus length of up to vn square. So, this is the generalized Pythagoras theorem. So how do we prove this particular theorem? We shall prove this theorem by induction.

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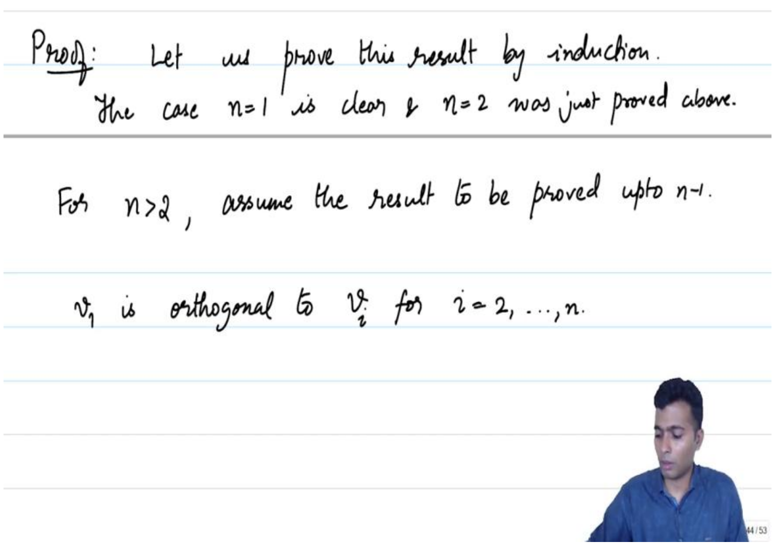

So, let us prove this result by induction. Now in a case when n is equal 1, there is nothing to prove. So, the case n is equal to 1 is clear and n is equal to 2 was just proved above. So, for n greater than 2, let us impose the induction argument. Assume the result to be proved up to n minus 1. Let us prove the result for in a case when n comes out. So, before we get into any of that, notice that v1 is orthogonal to vi for i greater than 1, so for i equal to 2,3, upto n. So, let me write is like that. And by what we just proved few minutes back. If you have a vector which is orthogonal to collection of vectors, then it will be orthogonal to any linear combination.

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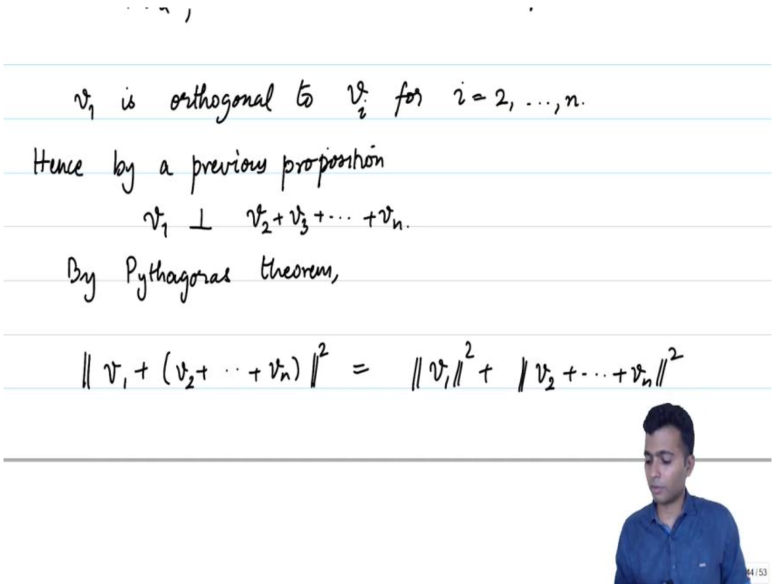

So, in particular, hence by a previous proposition, let me use the symbol for orthogonality, v1 is orthogonal to v2 plus v3 plus up to vn. Because it is orthogonal to each of v2, v3 and so on, that means now by Pythagoras theorem, not the generalized theorem, we are proving the generalized one, by Pythagoras theorem for 2 orthogonal vectors the length of v1 plus v2 plus up to vn whole square, this is nothing but the length of v1 square plus the length of v2 plus up to vn square.

But we have already proved, sorry, by induction I put this is, we have already assumed that the result is true for upto n minus 1 and if you consider v2, v3 upto vn there are only n minus 1, mutually disjoint vectors.

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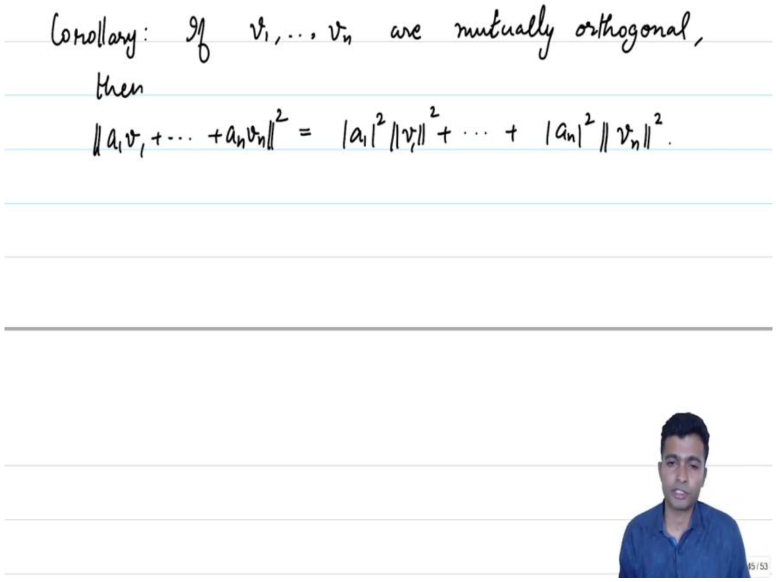

And by induction, this is equal to norm of v2 square or rather length of v2 square plus upto a length vn square by induction. So, this is by Pythagoras theorem, I have written it above. By Pythagoras theorem we have this first statement and by induction we have the second statement. And this precisely what we had set out to prove. So, we have a generalist Pythagoras theorem. So, as a corollary, if v1 to vn are mutually orthogonal, so I leave this as an excise for you, then the length of a1v1 plus anvn square this is equal to mod a1 square length v1 square plus dot dot dot mod an square length of vn square. So in the next week we shall try to get hold of a basis consisting of such orthogonal vectors.