Linear Algebra
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Lecture 9.3 Inner Product and Length

So, we defined the notion of an inner product with the idea of getting hold of a definition of length of a vector V. So, let us now begin by defining what is the length of a vector V in an inner product spacing.

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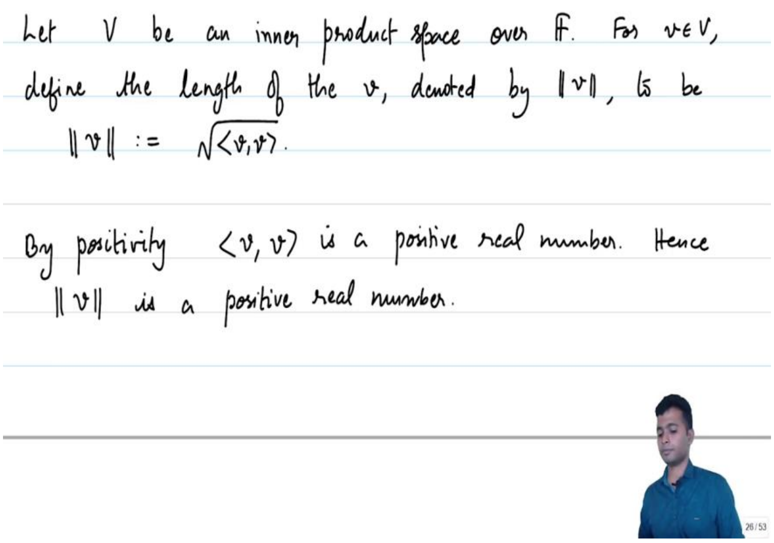

So, let V be an inner product space over F, for v in capital V define the length of the vector V denoted by length of v to be, define it to be the length of v is the square root of the inner product of v with itself. So, notice that by positivity irrespective of whether we consider a real vector space or a complex vector space by positivity inner product of v with itself is a positive real number. Hence the length of v is a positive real number, which is just the square root of the positive real number. Let us look at an example.

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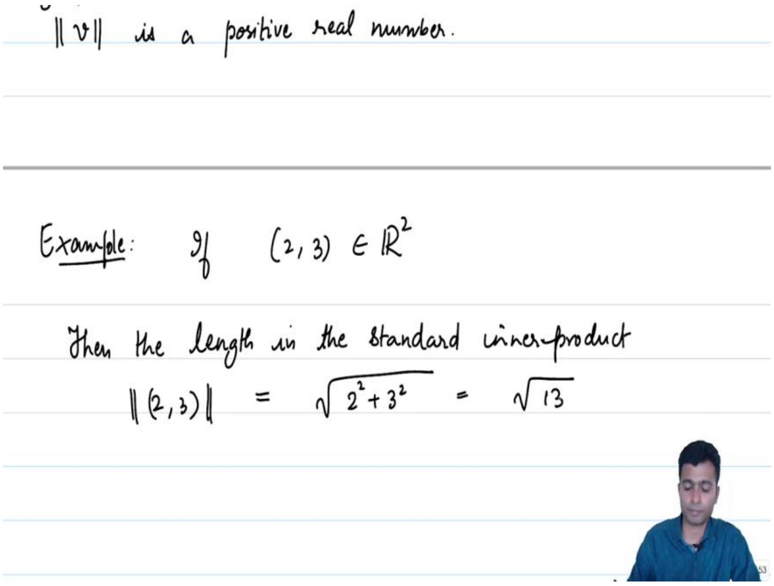

So, well I do not want to spend too much time behind examples you could go to each and every 1 of the examples of inner product spaces we have defined and take arbitrary elements and compute its length or getting used to this notion. So, in particular if say 2, 3 belongs to R2. Then what is going to be the length? Then the length in the standard inner product was look we call it as a just the dot product, so this is just going to be norm of 2 ,3 which will be equal to the square root of inner product of 2, 3 with itself which is 2 square plus 3 square which is equal to square root of 9 plus 4, 13.

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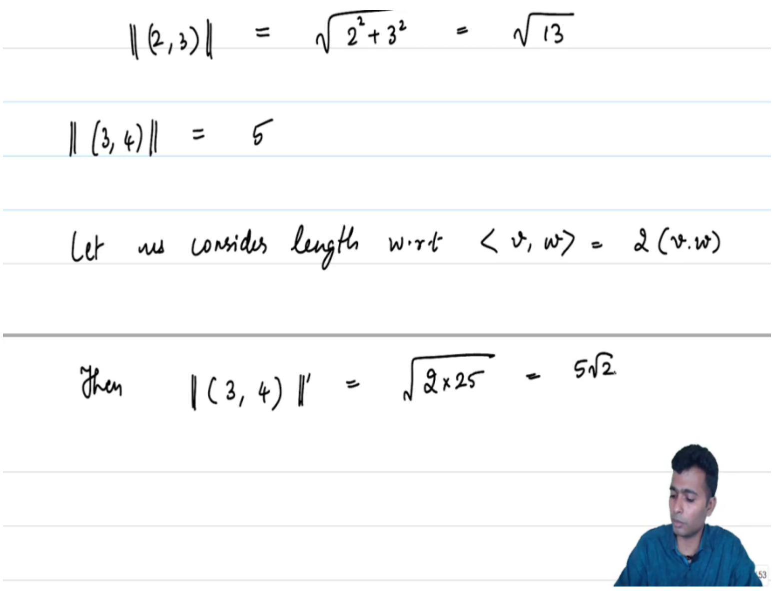

And how about the norm of the Pythagorean triplets 3,4, you can check that this is square root of 25 which is 5. But as you can see this is with respect to the standard inner product. What if we change the inner product to the other inner product we were? So other inner product on R2 that we were considering. So, let us consider length with respect to the inner product v, w to be equal to 2 times v dot w. Let us see what happens here. Then what is the length of 2,3 or let us not worry about 2,3, let us worry about 3,4. Let us put a prime here to denote that this is with respect to the non-standard inner product.

So, this is just going to be equal to square root of 2 times 25, which is equal to 5 root 2, which is not the same as the length that you obtain when you consider the length as defined using this standard inner product. So, the length is a notion which very much depends on the inner product and that is generally no standard notion of an inner product. So, yes so there is a notion of length which is very very heavily dependent on the inner product that is what you should keep in mind. So, the positivity of the inner product tells us that, the length of a non-zero vector cannot be zero.

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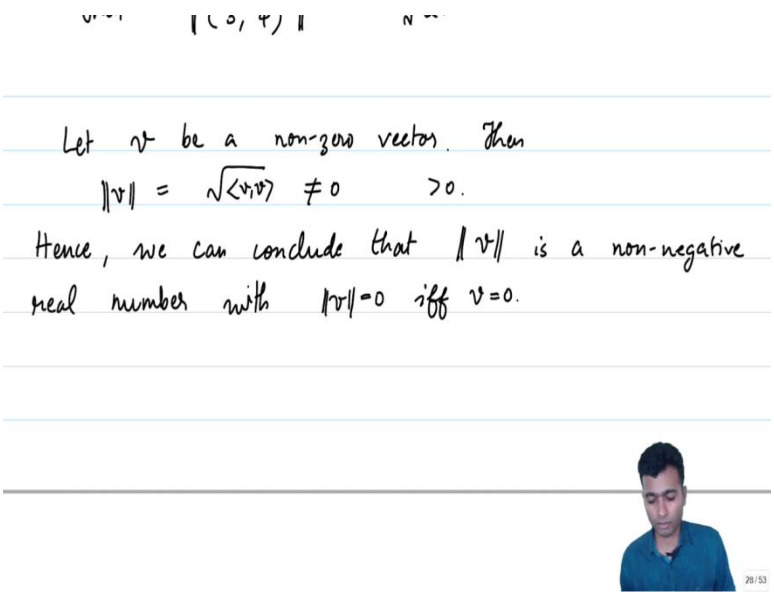

So, let v be a non-zero vector, then norm of v which is equal to the square root of v, v whatever is there inside the square root is a positive number, is not equal to 0 this is a positive number. So, hence we have that, we can conclude in particular norm of 0 is inner product of 0 with itself which is equal to 0 and therefore we can conclude that the length of any nonzero vector or any vector equal to 0 is if and only if v is equal to 0 in particular the length is a positive number.

Let us continue length of v is non-negative real number with norm of v is equal to 0 if and only if v is the 0 vector. How about the length of vector which has been dilated? So, let us c be some scalar.

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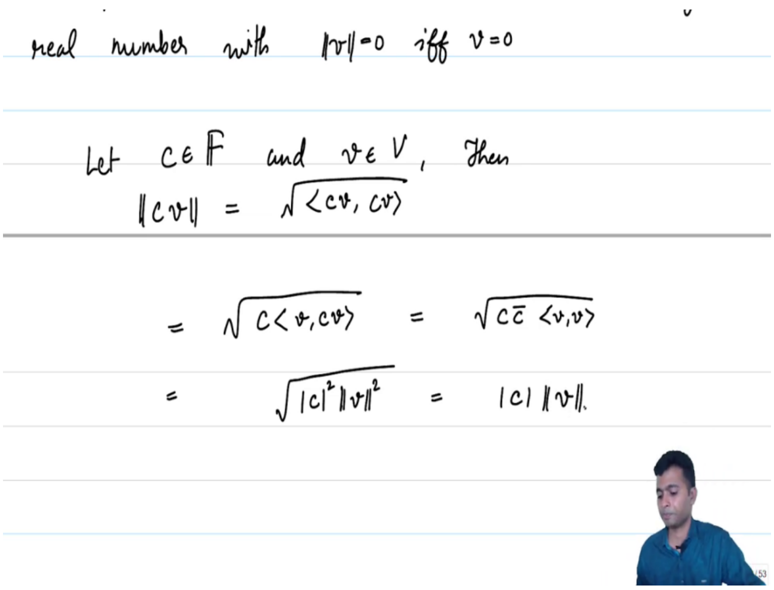

So, let c be in F. So, as you can see most of these conclusions are being drawn for vector spaces over F or in other words these are all conclusions which are true for both real vector spaces and complex vector spaces. So, let c be in F and v be a vector in capital V. What is the length of c times v, the length of c times v, this is square root of the inner product of cv with cv, but the linearity in the first property tells us that this square root of c times v, cv and we have already checked in the last video that the conjugate linear tells us that this is c, c bar v, v the inner product of v with itself.

But what is inner product of v with itself? Inner product of v with itself is, norm of v square and what is the product of complex number with its conjugate will just be the square of the absolute value. So, this is just going to be square root of mod e square times mod of v square which is mod of c times norm of v. So, in particular if you look at minus of v, the length of minus of v will be the same as the length of v. Length of 3 plus 4i times v will be 5 times length of v and so on. So, we have now discussed how we can obtain the length of a vector from an inner product. In some cases, we could get hold of the inner product from the length of the vector. So, let us see when that can be done.

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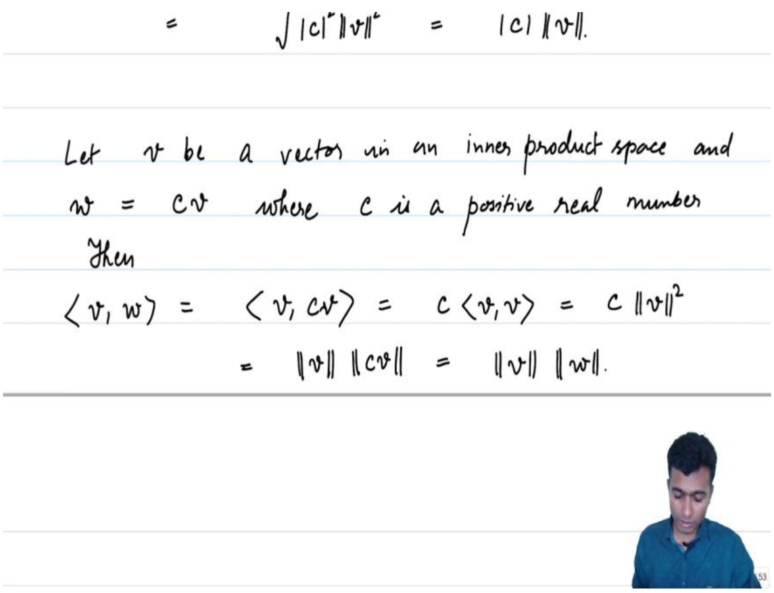

So, consider the case let v be a vector in an inner product space and small w be equal to c times v, where c is a positive real number, it is a parallel vector in some sense, so it is in the same directions just a dilation, positive real number. Let us see what happens to the inner product of v and w. We know that w is cv. So, this is just v times cv. But what is cv? cv is this is just c bar times v with inner product of v with itself, but c bar in our case is just c and because it is a real number, because it is a positive real number this is just inner product of v with itself which is equal to c times norm of v square.

But what is norm of v square? This is just equal to norm of v times norm of cv by whatever we have just checked, norm of, sorry I am saying norm many times but I mean length, the length of v times length of cv, but length of cv is just c times or absolute value of c times the length of v and absolute value of c here in this case is equal to c, because he is a positive real number and this is nothing but the length of v times the length of w. So, as noted the inner product of v and w here in this case is just the product of the inner product of v and w.

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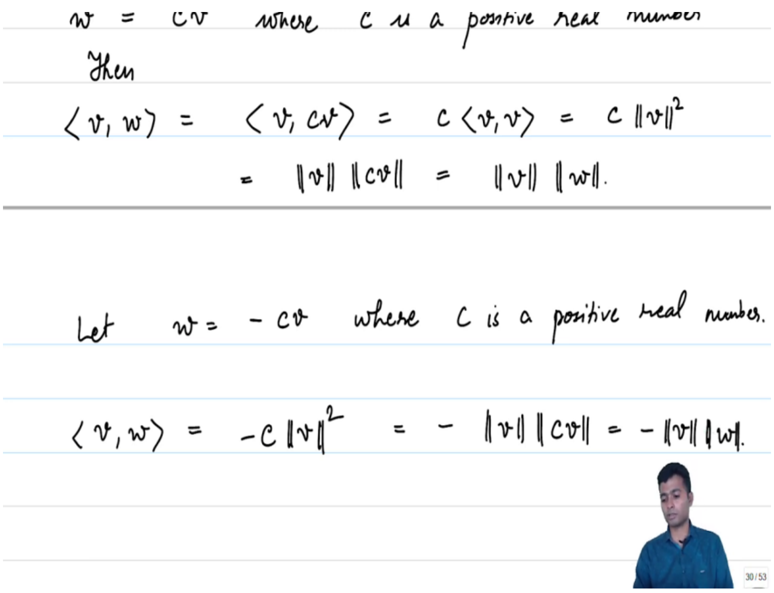

What if c is a negative real number? So, if c is a negative real number. Let us not put it this way. Let w be minus of cv, where c is a positive real number, that means it is a negative real number, times v, c is a positive real number. Now, what can we say about? What was written about? Where does things start changing? It will be the same till, let me say that, this is going to be the same till c times norm of v square minus of c times. This is just now going to be equal to minus of c times norm of v square and it is at this place where it will change.

This is just going to be minus of norm of v, or rather length of v times the length of cv, which is equal to minus of the length of v times the length of w, because absolute value of the negative of a number is, so absolute value of minus c is just c. So, this is precisely what we will be getting. That means in this case the inner product is the minus of the length of v times the length of w. So, in general the inner product of v and w is always between these 2 numbers. So, this is what is very famously called as the Cauchy Schwarz inequality.

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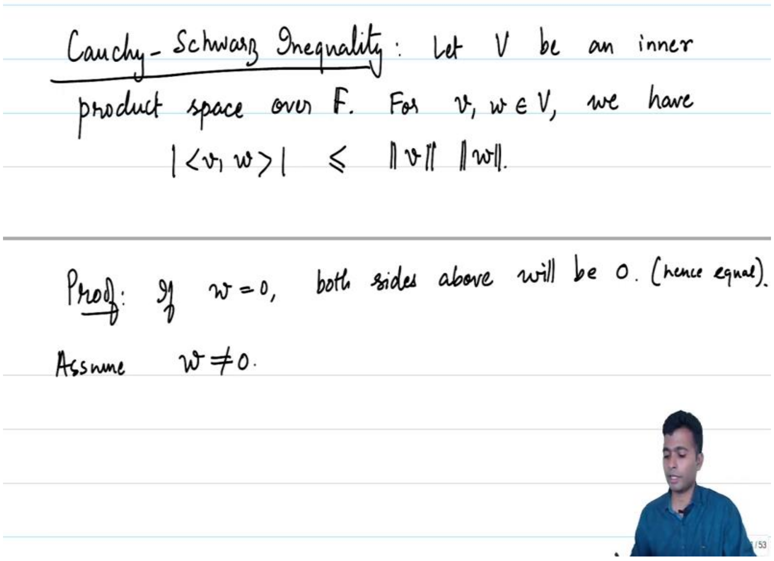

So, Cauchy Schwarz inequality, it is a very celebrated result in mathematics. So, let v be an inner product space over F. We are not saying this just for vector spaces over R, this can be concluded for vector spaces over c as well. For v, w in capital V, we have the absolute value of the inner product of v, w is less than or equal to, the norm of, so the length of v times the length of w. Let us look at the proof of the statement. So how does the proof go? So, if W is equal to 0, you can notice that both sides of the inequality will turn out to be 0.

Because the inner product of any vector with 0 is 0 and the length of the 0 vector is 0 as has already been noted. So, if w is equal to 0, both sides above will be 0. There is nothing to prove, it is just equality then. So, let us assume hence equal. So, yes in particular the inequality satisfied. So, assume that w is not equal to 0, we may assume without loss of generality that w is not equal to 0, The case when w equal to 0 is already being stated. Let us now invoke the positivity of the inner product on which vector.

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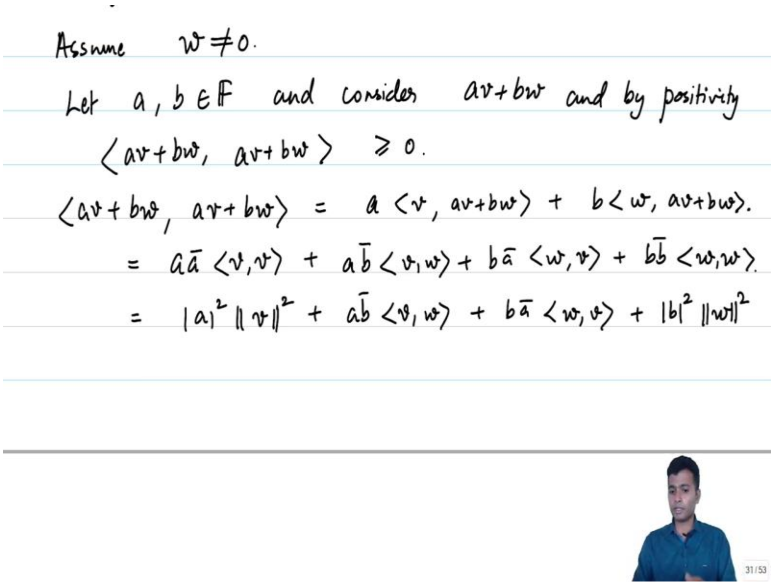

So, consider, so let a, and b be scalars, so let a, b be in F scalars and consider the vector av plus bw. Now this is a vector in the vector space v and by positivity of the inner product, so technically the third condition in the definition of an inner product av plus bw, av plus bw, inner product of this vector with itself, is greater than or equal to 0, with equality if the vector is 0. Let us see what is in the left-hand side here. av plus bw, you look at the inner product of this with itself.

By using the linearity this is just a times the inner product of v with av plus bw plus b times the inner product of w with av plus bw and pushing it more this will just be equal to aa bar, inner product of v with itself plus ab bar inner product of v with w plus ba bar inner product of w with v plus bb bar inner product of w with itself. So, what is the, a complex number with its conjugate? This is just mod a square and how about inner product of v with itself that is just the length of v with the product of the length of or the square of the length of v.

And then this will be just ab bar v, w plus ba bar w, v plus mod b square or rather length or mod b square times length of w square. This is precisely what we have. The inner product of av plus bw with itself. Now this is true for any scalars a and b. So, in particular if you pick our scalars a and b carefully, we can say quite a lot of things about what do you want to conclude.

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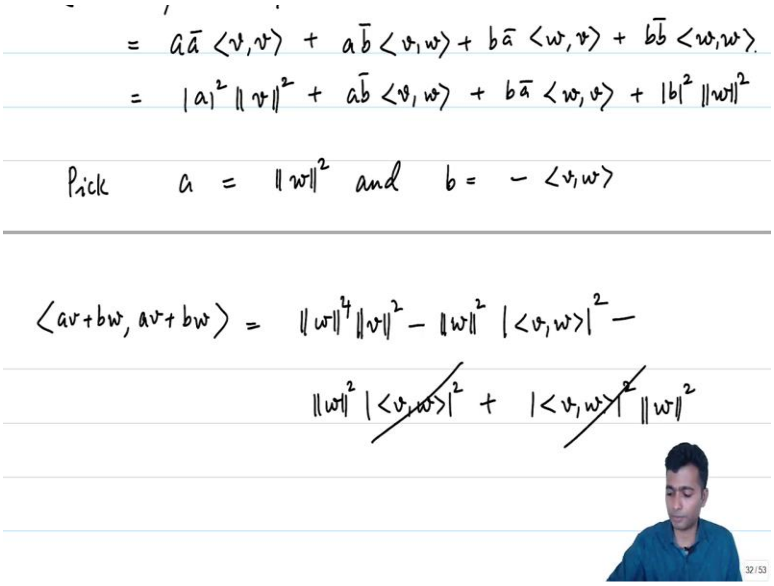

So, pick a to be equal to mod w square and let us see b to be equal to minus of v, w. So, pick a and b to be these 2 scalars. Notice that a is going to be a positive real number and b is going to be some scalar in F. It needs, if it is a complex vector space this could just be might just turn out to be some complex number. So, keep that in mind and this hence will tell us that av plus bw, inner product of that with itself is equal to the length of w to the power 4 times length of v square plus the length of w square times b bar.

b bar is just minus of v comma w the whole bar, which is just, so this is going to be minus and this will be just the inner product of v with itself square. What is the next number? This is a bar, a bar is just a, because a is a positive real number and plus length of w square times b is minus of v, w. So, let me change the sign here. So, this is going to be minus of v, w and w, v is just v, w bar. So, this is again going to be inner product of v, w square, mod of the absolute value of the inner product square.

Because we are multiplying a number with its conjugate and how but the last term, the last term will be absolute value of b square, absolute value of b square is just, inner product of, absolute value of inner product of v and w square and this times of norm w square and if you notice carefully, the last two terms can be cancelled out.

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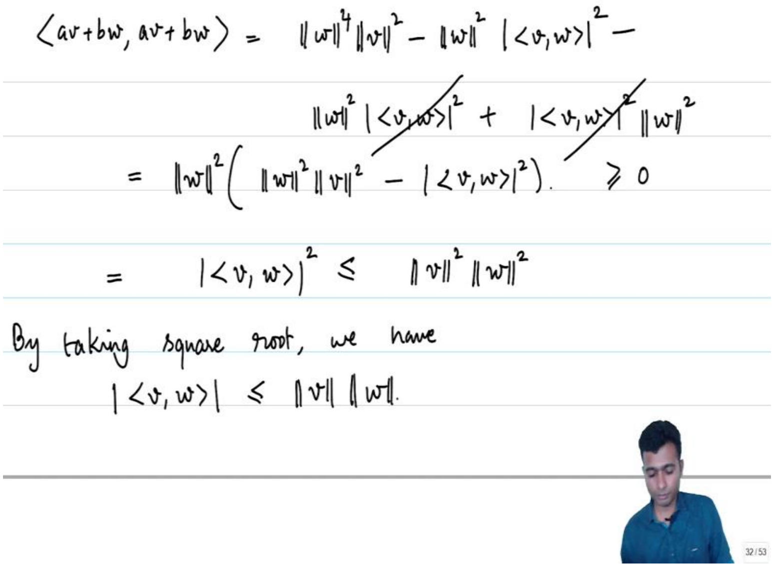

And what remains is which is equal to norm of w, sorry length of w square times, length of v square minus absolute value of inner product of v and w square, but by positivity we have just observed a few minutes back that this is greater than or equal to 0 and the length of w is a positive quantity length of w square, square of the length of w is also a positive quantity and therefore by multiplying by the reciprocal we get mod of absolute value of the inner product of v and w square is less than or equal to the length of v square times the length of w square.

Now all these are positive numbers. By taking square root, we have Cauchy Schwarz inequality which tells us that the absolute value of v, w is less than or equal to the length of v times the length of w and that is precisely what we were saying. If we look at w to be positive multiple, scalar multiple of v then this will be equal to the length of v times length of w and otherwise it will be minus if it is a negative multiple of v then it will, it is going to be a negative multiple minus of a length of v times length of w and every other case will be between that that is precisely what we have established.

In the case of complex vector spaces Cauchy Schwarz inequality tells us that the inner product of v and w it lies in a disc, centre at 0 and radius norm, length v times length w. That is precisely what the Cauchy Schwarz inequality says for complex vector spaces. That so we have already noticed that the length is not a linear object. So, we could consider the vector sum of 2 vectors and look at the length of that, this need not be equal to the sum of the length of these vectors.

For example, as already noted if you look at 1 0 0 1, if you look at the sum is 1 1 it has length root 2 with respect to the inner product obtained by the standard sorry the length obtained by its standard inner product. And whereas 1 0 and 0 1 each has length 1, so we do not really get the length, sum of the lengths of these 2 vectors is 2 but the length of the sum of these 2 vectors is root 2. So, we do not have linearity in when it comes to length. However, we have a bound on the length of the sum which is given by the very famous triangle inequality.

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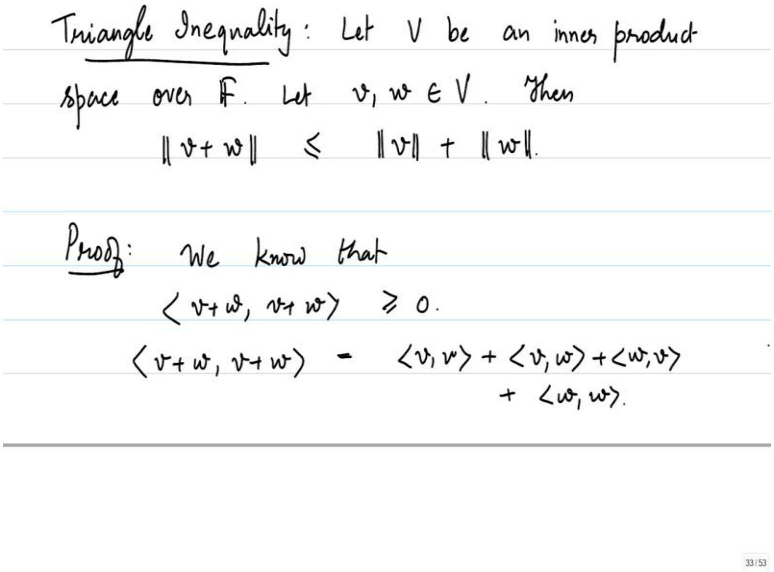

So, let us now prove what is classically known as the triangle inequality. So, let V be an inner product space and suppose we take 2 vectors both over F. So, this is true for both, real vector spaces and complex vector spaces. So let v, w be 2 vectors in the inner product space V, then the length of v plus w is bounded by the length of v plus the length of w. Let us give a proof of this statement.

This is a very classical triangle inequality. It just tells us that the length of one side of a triangle, in the case of our too, this is effectively telling us that the length of one side of a triangle will always be less than or equal to the sum of the lengths of the other 2 sides of the triangle and that is why it is called the triangle inequality. So, in the case of R2 it manifests in that manner.

So, let us give a proof of this. It is actually quite remarkable that, such an inequality is holding on any inner product space. So, let us consider again positivity of the inner product with the vector v plus w. So, we know that the inner product of v plus w with itself is greater than or equal to 0. But what is this? You know that this is inner product of v plus w with itself. This is just equal to inner product of v with itself plus inner product of v with w plus the inner product of w with v plus inner product of w with itself.

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$$\langle v+w, v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle.$$

$$\|v+w\|^2 = \|v\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} + \|w\|^2$$
$$= \|v\|^2 + 2\,\mathrm{Re}\left(\langle v,w\rangle\right) + \|w\|^2$$

(Exercise: Show that if $z = a+ib$, then $a \leq |z|$).

So, what is this? This is just the length of v square plus inner product of v, w plus, what is inner product of w, v, it is just inner product of v, w conjugate plus the length of w square by the very definition of length. So, what are we considering? This is inner product of v plus w. So, what are we considering, we are just considering the inner product of v plus w with itself? So that is nothing but the length of v plus w the whole square, is not it? So, what is this? This is equal to the norm of v square plus 2 times the real part.

So, this is just a plus ib plus a minus ib which is just going to be 2 times a, which is 2 times the real part of the inner product of v, w plus norm of w square. If, so a check for you an exercise, an exercise for you is to check that, so that if z is equal to a plus ib, then a is less than or equal to mod z. This is something which you should be able to do. Because what is mod z, mod z is square root of a square plus b square. Clearly it is greater than a.

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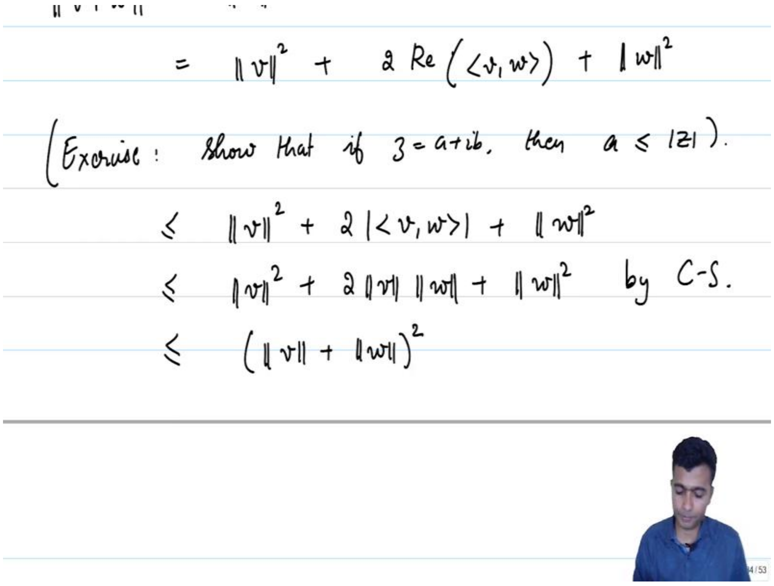

So, something is you should sit down and think about and with this you will be able to say that this is less than or equal to norm of b square plus 2 times the absolute value of the inner product of v with w plus the length of w square. But what do we know about the absolute value of the inner product of v and w? It has to be somewhere between 0, it will be in the disk with centre at 0 and radius norm b or length v times length w.

So, this is less than or equal to by Cauchy Schwarz 2 times the length of v plus the length of w plus the length of w square by Cauchy Schwarz, let me write just CS for Cauchy Schwarz. And now the right-hand side is just length of v plus length of w the whole square. So, recall what we have done.

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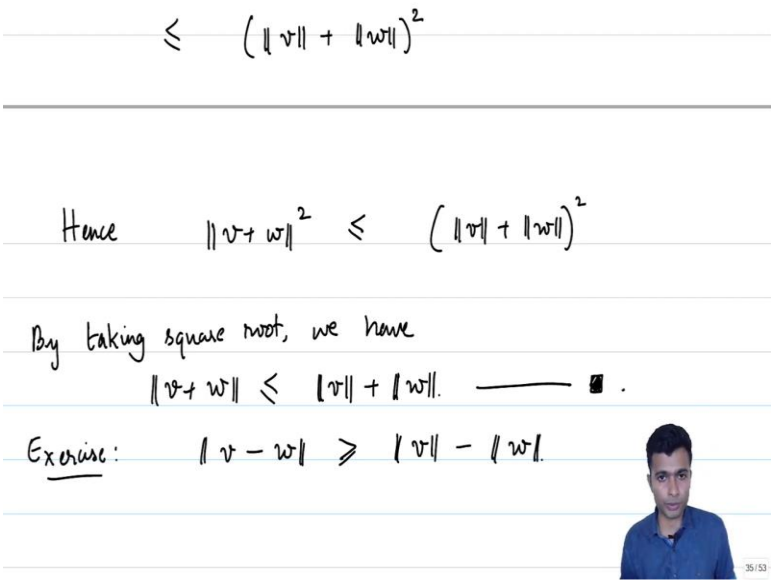

We have just shown that hence we have shown that norm of a sorry length of v plus w square is less than or equal to the length of v Plus length of w whole square. By taking square root, we have the length of v plus w is less than or equal to the length of v plus length of w and that is precisely what we had intended to prove. So, there are variants of the triangle inequality. So, let me just leave it as an exercise for you to show that exercise show that norm of sorry length of v minus w is greater than or equal to the length of v minus the length of w. This is an exercise for you to work out.

There are other variants which maybe we will see in the assignment. In next video we shall discuss the notion of the angle of a vector space, sorry angle between 2 vectors or well we might not see angle between 2 vectors however we will talk about orthogonality between 2 vectors. We will try to capture when 2 vectors are in some sense perpendicular to each other in an inner product space.