Linear Algebra
Inner Product Spaces
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So, till now we have discussed in detail, what we can say about various properties of vector spaces using its vector addition and scalar multiplication. For example, we define what a basis is what a linear transformation are, and many properties of these objects were derived. And the early operations that we used in most of in all these were the vector addition and the scalar multiplication of argument vector space.

Next, we would like to ask the following question like in say R2, in R2 if we are given a vector, say 1, 0 or 1, 1. And if he asked what is the length of this vector? We can say that this is square root of 1 square plus 1 square, which is square root of 2, we would like to ask a similar question in a arbitrary vector space. So, for example, if a polynomial x square plus 1 is given in P2 of R, can we say that can we ask the following question. What is the length of this polynomial x square plus 1 in P2 of R?

Unfortunately, a vector space is not equipped to answer such geometrical questions. So, the operations which we have cannot be used to talk about a natural notion of length or angle between vectors like we could do in R2 and in order to do that, we will introduce an upgraded notion of a vector space which is called as an inner product space. So, to do that, let us look back into R2 or R3 and look at what are the, what is that we know about the length in R2.

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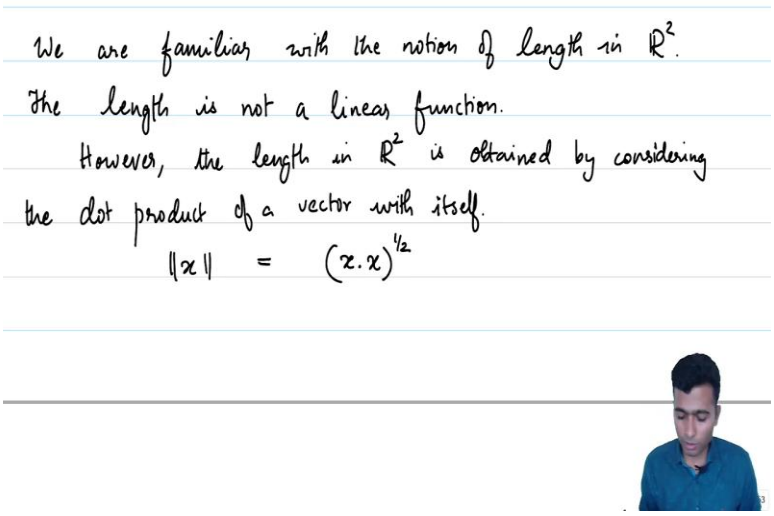

So, we are familiar with the notion of say length in R2. The first drawback of the notion of length is that it is not linear. So, this the length, the length is not a linear function, it's not a linear function. So, in other words, if you take say 1, 0 and 0, 1 both are vectors of length 1, if you add you get 1 comma 1, which is a vector of length root 2, which is not equal to the length of 1, 0 plus length of 0, 1, which should have been 2 right.

So, the length is not a linear function and considering vector spaces for so long it is, it is a desirable thing to look at things which will turn out to be linear and immediately we will be led to thinking about what uses the notion of length in R2. The length in, however, the length in say R2. In fact, in even in Rn and let us just focus on R2 is obtained by the dot product between 2 vectors, is obtained by considering the dot product of a vector with itself. So, now of x is just equal to x dot x to the power 1 by 2 if you are already familiar with the notion of dot product, if not we will be introducing it shortly.

So, there is no assumptions which will be made. If you are already familiar with it, you might be also familiar with the fact that the length of a vector can be realized as the dot product of a vector with itself and then you take the square, it is immediate that are, it is, it is actually quite easy to check that dot product is linear in both the variables.

So, the dot product is not just linear. Some other properties are also satisfied by the dot product in R2. And just like in the previous case, when we define vector spaces. We noted that mathematicians have studied these objects for many, many years and they have identified those properties of the dot product, which are desirable and which needs to be generalized to the more general setting of an abstract vector space.

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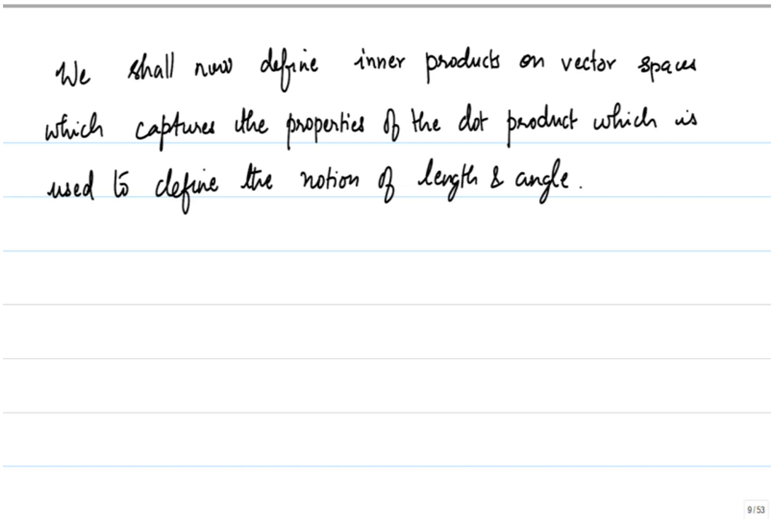

Similarly, we will define now the notion of an inner product, which is generalizing those properties of the dot product which is familiar in R2. So, we will be defining v, so we shall, so let us note it. We shall now define inner product, inner products on vector spaces which captures the properties of the dot product, which is used to define the notion of, in fact both length and angle.

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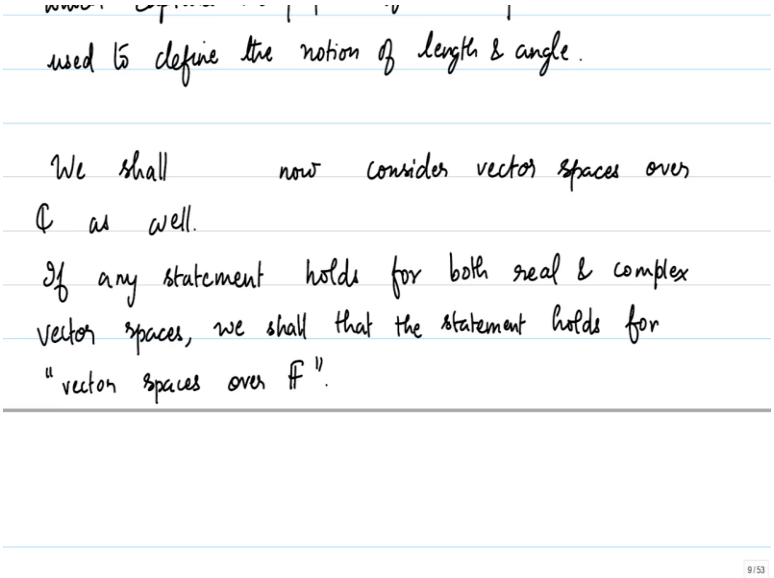

So, we are going to define inner product on vector spaces. And the first thing to keep in mind is that from now on we shall not just consider vector spaces over the field of scale has been R, we shall also consider vector spaces over complex numbers. So, let me note that we shall also now consider vector spaces over C as well. There is no also needed here. We shall now consider vector spaces over C as well and we will write.

So, if definition, if any statement holds for both vector, real and complex vector spaces. We shall say that we shall say that the statement was for vector spaces over F, holds for let me put it in quotes, vectors spaces over F. So, this would mean that this particular statement, if a statement is being made with vector spaces over F, it means that you consider vector spaces over R or vector spaces over C, a definition or a theorem or a proposition is going to be true for either of the cases. Okay.

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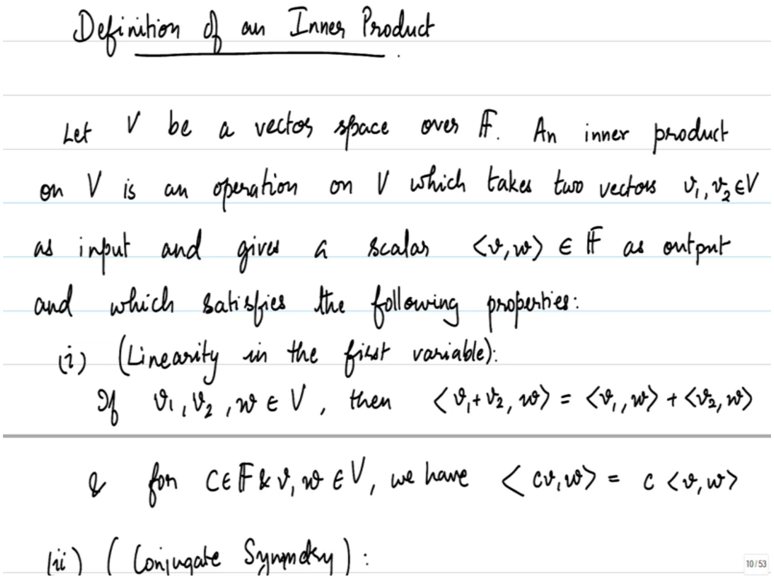

So, let us now start with the definition of an inner product. Definition of an inner product. So, let V be a vector space over F over the field of scalars being either R or C. So, let me write it as vector space over F. An inner product on V is an operation as you can see, we are introducing more structure. It is an operation on V which takes 2 vectors as input, takes 2 vectors v comma w as input and gives us a scalar as output and gives a scalar, so v comma w, which will be in F as output and which satisfies the following properties.

So, what are the properties that are satisfied by the inner product? The first one is linearity in the first variable. What does that mean? So, if v1, v2 and w are elements in capital V, then the inner product of v1 plus v2 and w this is equal to the inner product of v1 with w plus inner product of v2 with w. And for C in F, v comma w in capital V this is and, not confuse you.

And we have the inner product of c, v with w is equal to the scalar c times the scalar which is given by the inner product of v and w. So, that is the first property that we would like the inner product to satisfy, what is that, that we, if you are already familiar with the dot product, notice that this is something which is already satisfied by the dot product. How about the second property? So, this is called linearity of the first variable.

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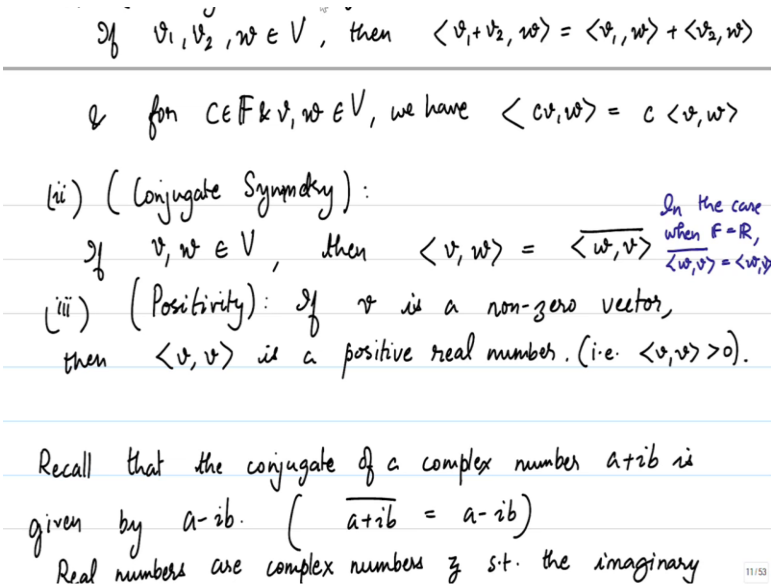

The second one is called Conjugate Symmetry. What does this? So, if v comma w are vectors in capital V then the inner product of v comma w is equal to the conjugate of the inner product of w and v. So, a digression, so maybe I will come back to this. So, come back to this property after I finish off with the third property. Positivity. So, if v is a non zero vector, then the inner product of v with itself is a positive real number.

This is the first time I am mentioning something as real or complex, here irrespective of what F is, inner product of v with itself is a positive real number. We will write it in short as inner product of P with itself is greater than 0, this is the notation for. All right. So, what is meant by the complex conjugate recall? Recall that the conjugate of a complex number, A plus ib is given by a minus ib. So, a plus ib bar is equal to a minus ib. So, what does this mean in the case that our complex number is a real number.

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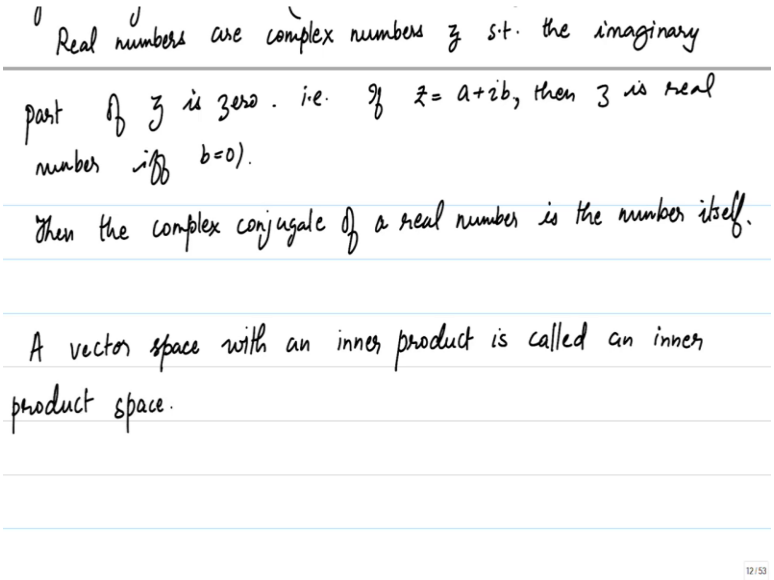

So, what is what are real numbers, real numbers are complex numbers with the imaginary part equal to 0. Complex numbers z such that the imaginary part of z is 0, i e, if z is equal to a plus ib, then z is a real number, if and only if b is equal to 0. Right? So, what is now the complex conjugate of a real number? Then the complex conjugate of a real number is itself. Because the imaginary part is 0. So, b 0 minus b is also 0, right. The complex conjugate of a real number is the number itself.

All right, now let us get back to the definition of an inner product. So, inner, inner product is an operation on the vector space. Till now, our vector space V has two operations, an addition operation and a scalar operation. We are now defining a new operation which takes 2 vectors and gives you back a scalar. So, if you are working on vector space over R an inner, inner product will take 2 vectors and give you back a real number. If you are working on a vector space over the complex numbers, the inner product will take 2 vectors and give you back a complex number.

These, this inner product satisfies the following three conditions. The first one is linearity. So, v1 plus v2 inner product with w is the inner product of v1 with w, added to the inner product of v2 with w, what is that the right-hand side, which I have just underlined by green here. The addition is of scalars in F. So, it is either if F is R, those are 2 real numbers getting added and if F is C, those are 2 complex numbers that are getting added.

And the left-hand side however, it is the inner product of just 2 vectors. One of them however, is obtained using the addition operation in a vector space. Similarly, the case when you look at C times V for some scalars C in F, right. So, the first statement is quite straight forward for either real vector spaces or complex vector spaces. In the case of 2 which is conjugate symmetry, you can notice that there is a conjugate involved and we just noted that if it is a real number.

If this is a vector space over R, our inner product will also be a real number and we just noted that the conjugate of a real number is equal to the number itself. So, let me just note that, let me use a dark blue to say that in the case, when F is equal to R, w, v bar will just turn out to be equal to w, v.

So, conjugate symmetry is just symmetry in the case when we are working over TL vector spaces. And how about the third property. The third property says that irrespective of whether it is a real vector space, or whether it is a complex vector space, if you look at the inner product of a vector v with itself, it should necessarily be a positive real number.

So, even the case when our vector space is a complex vector space if you take any vector v and look at the inner product of v with itself, if v is non zero it should necessarily be a positive real number. All right, so, an inner product, a vector space, which has an inner product on it is called an inner product space. So, let me give that definition as well. Vector space with an inner product is called an inner product space, Okay.

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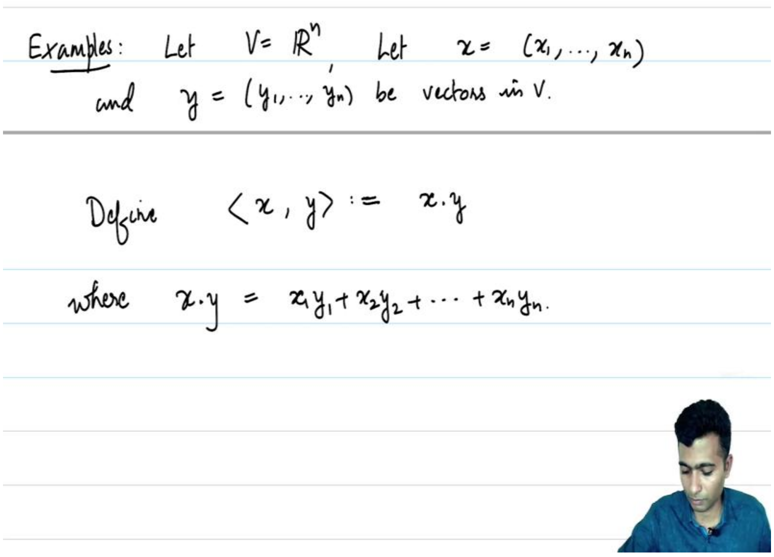

Let us now look at examples. So, the first example should naturally be the dot product. So, let V be equal to Rn, let us define the dot product. So, let x equal to x1 to xn, and y be equal to y2 to yn, be vectors in V. And the inner product of, define inner product of x with y to be or this is the, this is the same as define it to be x dot y, where x dot y is nothing but x1, y1 plus x2, y2 plus xn, yn. So, this is the classical dot product in R n and well is this an inner product? Obviously, the answer should be yes. So, we have to check for the following three properties, we have to check for linearity in the first variable, conjugate symmetry or in this case it as a real vector space. So, symmetry and positivity.

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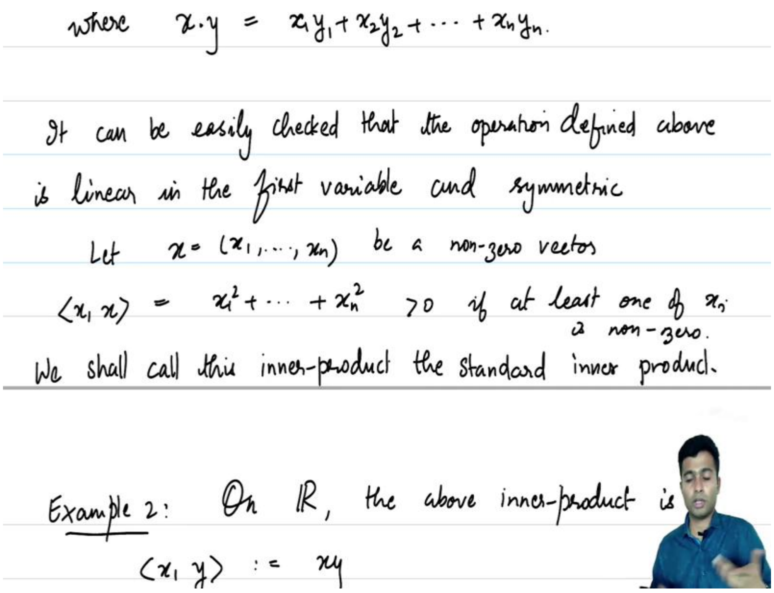

So, checking that it is, it is easily checked it can be easily checked that it is linear in the first variable, easily checked, which I leave it as an exercise that the operation defined above is linear in the first variable. Just take 2 vectors x and x prime. Look at x plus x prime, look at the inner product of x plus x prime with y, it will just turn out to be x1 plus x1 prime times, y1 plus x2 plus x2 prime times, y2 and so on, use the distributivity and all that and it is very easily check that it will turn out to be x in the product of x with y plus the inner product of x prime with y.

Very similarly, you can look, also check for real number times a vector, when you look at the inner product of that with another vector y, then it is going to be the real number times in a product of the vector with y. So, I will leave that as an exercise. How about the second property? The second property is symmetry. In this case, conjugate symmetry is the symmetry here because it is a real vector space. In fact, it is quite easy. So, to check that as well. So, this is a linear in the first variable and symmetric because the product.

So, what will be y dot, inner product of y comma x that will be y1, x1 plus y2, x2 up to yn, xn which is the same as x1, y1 plus x2, y2 plus xn, yn which is the inner product with x and y, inner product of x with y. So, write so, these two properties are easily verified. How about positivity? So, let x equal to x1 up to xn be a non zero vector that means, that at least one of the coordinates is non zero. And what is the inner product of x with itself. This is equal to x1 square plus up to xn square by very definition. But each of these are non-negative numbers and at least one of them has to be positive it is a square, right?

So, it has to be a positive number always. And if xi is not zero, it will be a strictly positive number, it will not be zero, which is greater than zero if at least 1 of xi is non-zero. So yes, so in the case of a non-zero vector in Rn, it is a, the inner product of the vector with itself is a positive number. All right, so this is indeed a inner product on Rn. But having said that, we could define other inner products on R, Rn as well. So, let me just give you another example, before okay, so, before that what is it going to manifest as on the vector space R.

So, let us just go into a second example. So, on R the above inner product, so let us see what happens there, on R inner product of x comma y will just be x times y. This is what is going to be the inner product, it is just going to be the multiplication of those 2 vectors, Okay. So, let us look at a different inner product on Rn. So, a different inner product. So, let me now change the notation. So, let me put up prime, prime on Rn. So, define the following. So, for x comma, for x equal to x1 to xn, and say y is equal to y1 to yn.

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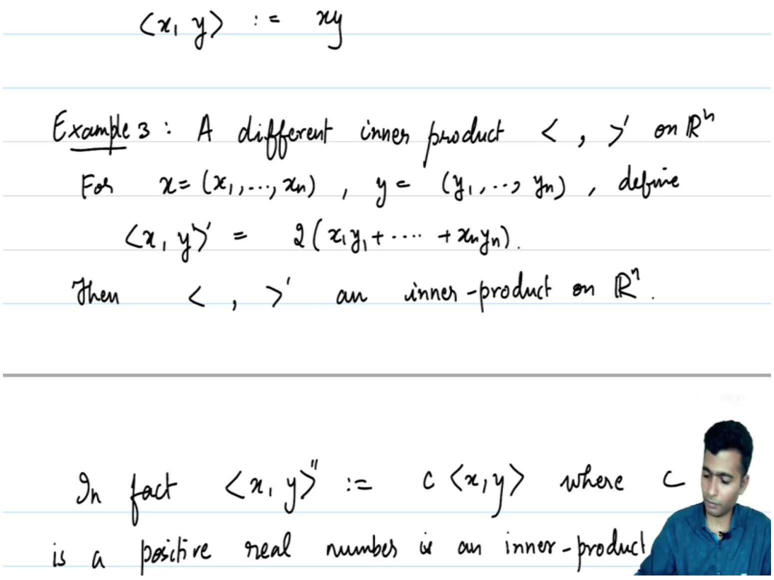

Define the inner product, the new inner product of x with y to be equal to 2 times x1, y1 plus up to xn, yn. And I will leave it to you to check that then this is also an inner product on Rn. In fact, in fact, x comma y defined as c times the inner product of x. So, let me put a double prime here, c times x comma y, where c is a positive real number is an inner product on Rn. So, as you can see, one could define multiple inner products on the same vector space.

Of course, this vector, this particular inner product is a bit special. So, let us do one thing. We shall call this inner product the standard inner product, we shall call this inner product the standard inner product. Of course, there was nothing special, however, about this inner product, because why is it better than say, this inner product, there is nothing, no reason to say that one is better than the other.

So, there is no natural choice. However, the analog is with the idea of a basis, maybe. If you look at Rn we have a standard basis, but that is not the unique basis which we can have on Rn, there could be uncountably many of them, right. Similarly, we could talk about many, many, many, many inner products on Rn itself, but yes one of them is particularly handy to use and that we will call as the standard inner product.

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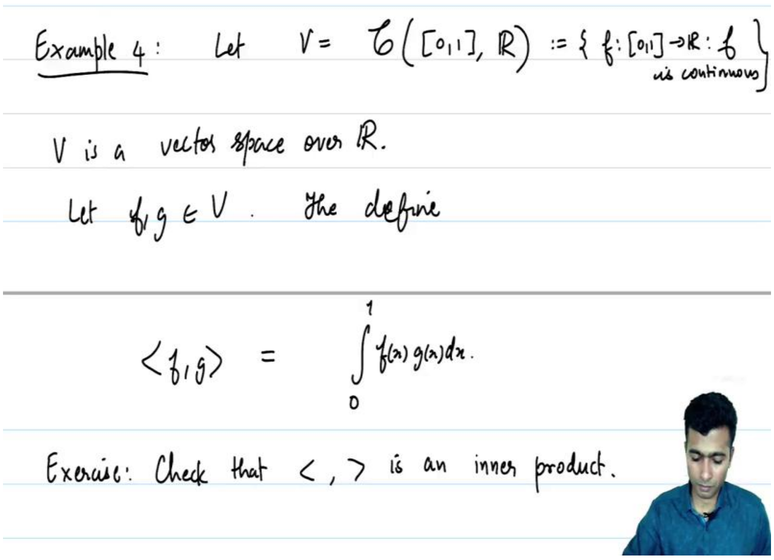

Okay, so let us look at other examples. Let us switch example, example three, example four is on, what could be a good example now. So, let us consider, let V be equal to C 0,1 to R so this time let me be very careful not going to casually write C 0, 1 this is C 0, 1 to R, which is the set of all f from 0,1 to R such that f is continuous. So, let us consider continuous functions, real valued continuous functions from 0, 1 to R.

This is a vector space over R. So, V is a vector space over R. So, again as you can see, we are slowly keeping track of what is our field of scalars, we are not casual with it anymore. Earlier, we used to only consider the field of scalars while we gave proofs, but please do keep in mind that all those results are true for vector spaces over complex numbers as well. Right? We will come to complex vector spaces here in a while, in a few minutes. But let us now focus on some of these examples of real vector spaces.

We would like to now define a, define an inner product on V. So, let us define, so let f comma g be functions on C 0, in functions on 0, 1. real valued functions on 0, 1 which are continuous, then define the inner product of f and g to be equal to integral from 0 to 1, fg, f of x, g of x, dx. f of x times g of x times dx. So, look at the integral of f times g from 0 to 1 can be easily checked again that the, all the properties in fact can be checked. So, I will leave it as an exercise.

Check that this, whatever we just defined, so as you can see, I am doing some abuse of notation. Every inner product, I am just defining by using some same notation of the bracket, but we are defining different inner products. In fact, we are defining different inner products on different vector spaces. So, there is some abuse of notation happening. However, let me assume that you are now mature enough to see that a particular inner product is on a particular vector space. So, this is an inner product.

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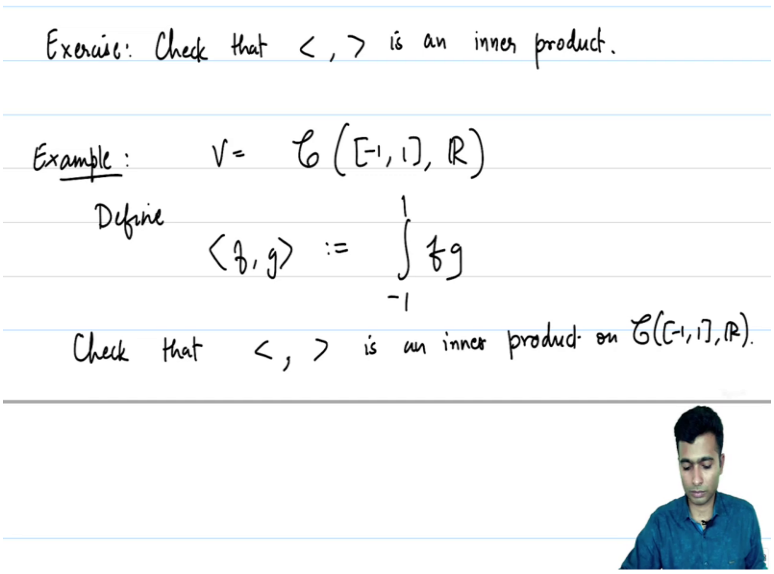

Let us look at another example. This time, I would like to consider V to be C minus 1, 1 into R, again real valued functions from minus 1 to 1 and define the inner product of f comma g. Now, we just defined it to be the integral from minus 1 to 1, f, g. Let me just write it as f g for f of x, g of x, dx and check that this is an inner product on C minus 1, 1 R. What is this number of examples? Example 5. Okay.

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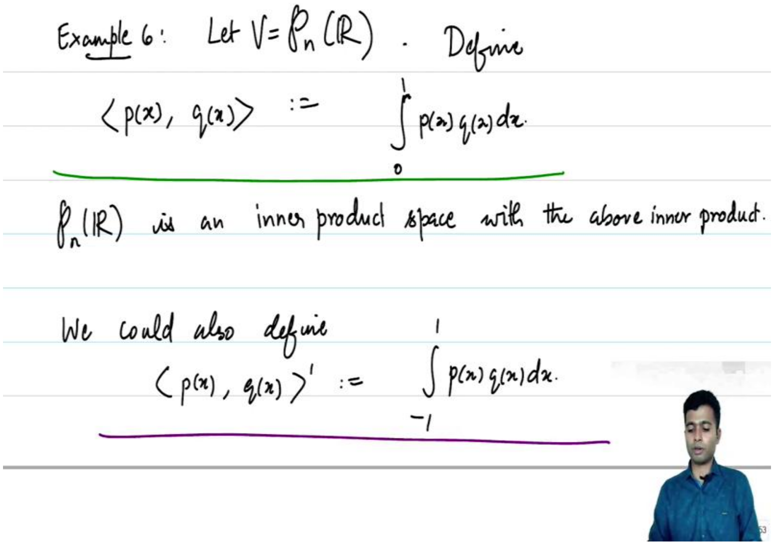

Let us now look at example 6, again this is also an example with the field of scalars being R. Oh yes, let us consider Pn of R, let V be equal to Pn of R, we would like to now define p of x, q of x. So define, oh yes, so recall that Pn of R is sitting inside P of R which is sitting inside C 0, 1 R for example, so, you can define 0 to 1 P of x, q of x, dx, this is a, this is an inner product, which is actually borrowed from C 0, 1 R, right. So, so, in particular. So, so, let me go slow.

If you take 2 polynomials p of x and q of x, both p of x and q of x are in particular functions from 0, 1 to R continuous functions from 0, 1 to R. Therefore, we can talk about the integral from 0 to 1of p of x, q of x, dx. And we have already checked that all the properties are getting satisfied all the three properties for, so, I will actually not venture too much into it positivity is a property, which is not that straightforward, but I will assume that once you do a real analysis course, you should be able to see the positivity of this operator, not operator positivity of this inner product.

And once we have established the 3 properties for C 0, 1, it is easy to see that with the same definition Pn of R will also turn out to be an inner product. So, Pn of R is an inner product space with the about inner product. But then if you could consider it as a continuous function from 0, 1 to R you could also consider it not because of that, it is also a continuous function from minus 1 comma 1 to R, right. So, we could also define, inner product of p of x comma q of x, let us put a prime here to denote that this is a different inner product, which is minus 1 to 1 p of x, q of x, dx.

So, we are now given two different inner products to Pn of R. So, this is one inner product and this is another inner product. So, we have now given 2 different inner products to Pn of R and there is no natural way to say that one is better than the other. This is precisely what I was pointing out a few minutes back, there is no one natural notion of an inner product, which can be given to p of R. So, I did not write p of R, p of R also.

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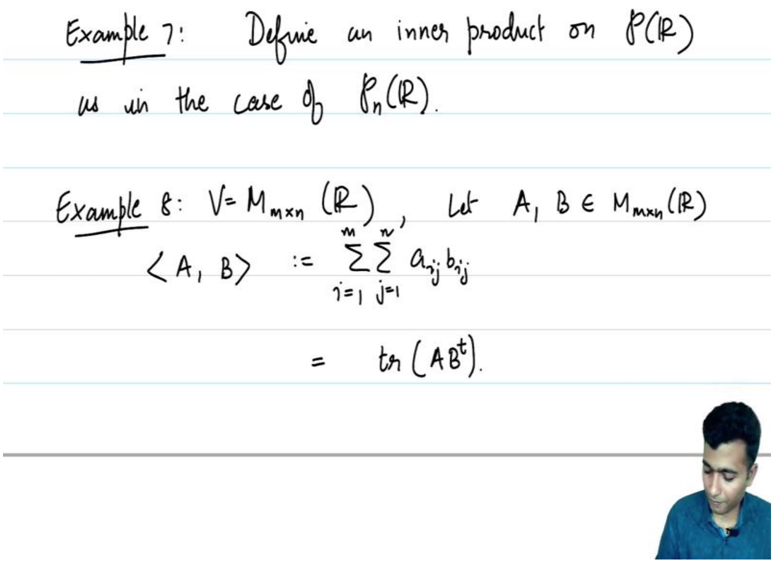

So, example 7 p of R can be given a very similar inner product. Define an inner product on p of R as in the case of Pn of R and make it into an inner product space. So, corresponding to this inner product, we can define an inner product on p of R. Corresponding to this inner product, we could define another inner product on p of R.

So, we could define more but yeah, let us defer that to some other time. So, there are many, many inner products which we could define on many of these in vector spaces, including p of R, there are many, in p of R there are many inner products, which you could define, many interesting inner products in fact. All right, so, let us one, look at one more example.

So, in M n of m cross, yeah m cross n of R. Let A comma B be matrices of size m cross n and we would like to define the inner product of A and B, which will just turn out to be equal to the sum of aij, bij, where i is going to run from 1 to m and j is going to run from 1 to n. And in alternate way it can be, or this is actually nothing but the trace of the matrix A times B transpose.

Again, I leave it as an exercise for you to check, it is actually very similar to the way we defined inner product in Rn. And it is very similar to check that the three properties of an inner product are satisfied here. Okay, so we have done quite a lot with V inner products let us spend some time on complex inner product spaces, or rather inner products on vector spaces over C. So, let us look at the next example.

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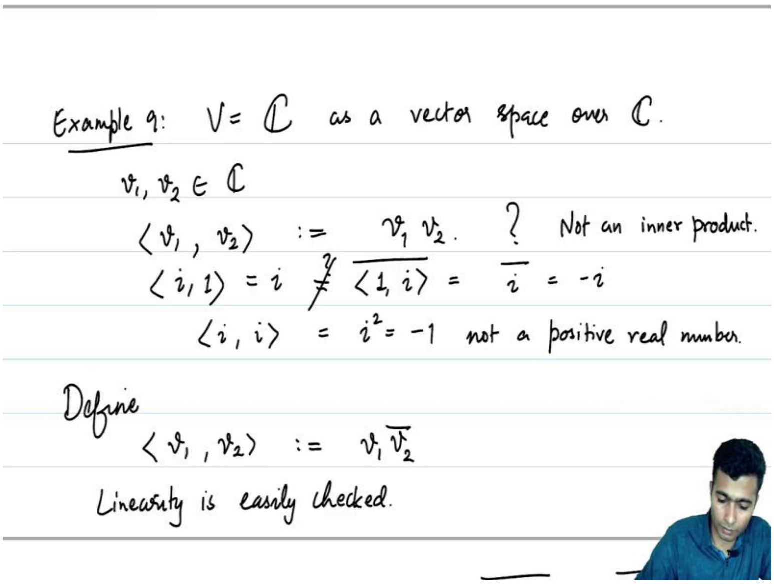

Next example being the simplest one. V, V equal to C as a vector space over C. And let v1, v2 be 2 complex numbers we would like to define, v1 comma v2. Okay, so what if we define it to be v1 times v2, will this be an inner product? Will this be an inner product on C? Well, I mean it is thought, we will tell you that this will not work because for example, the positivity will not get satisfied. If you look at i comma i, this is just going to be i squared, which is equal to minus 1 which is not a positive real number. Right?

It is not going to be positive real number. It is not going to be a conjugate symmetric either. So, if you say for example, look at let us look at i comma 1 that just going to be equal to i. And ideally it should have been equal, let me put a question mark. Is it equal to 1 comma i? bar, but what is 1 comma i bar? This is going to be bar of 1 times i which is equal to minus i if you notice.

So, this is not equal, i is not equal to minus i, so, this is not equal to the conjugate. So, this is neither conjugate symmetric, now, the properties are not getting satisfied. So, certainly there is something wrong with this, we cannot have this definition. So, not a good definition. Not a, not, does not satisfy, it is not an inner product, let me just write in that way, it is not an inner product.

But if you slightly tweak it, if you define the inner product of v1 and v2 to be the, define it to be v1 times the conjugate of v2, then it is a good exercise for you to check or maybe let us go through it. Let me not spend time behind the linearity, linearity is easily checked. How about conjugate symmetry? Let us look at v1, we know what v1, v2 is, let us look at what v2 comma v1 is, v2 comma v1 by definition is equal to v2 times v1 bar.

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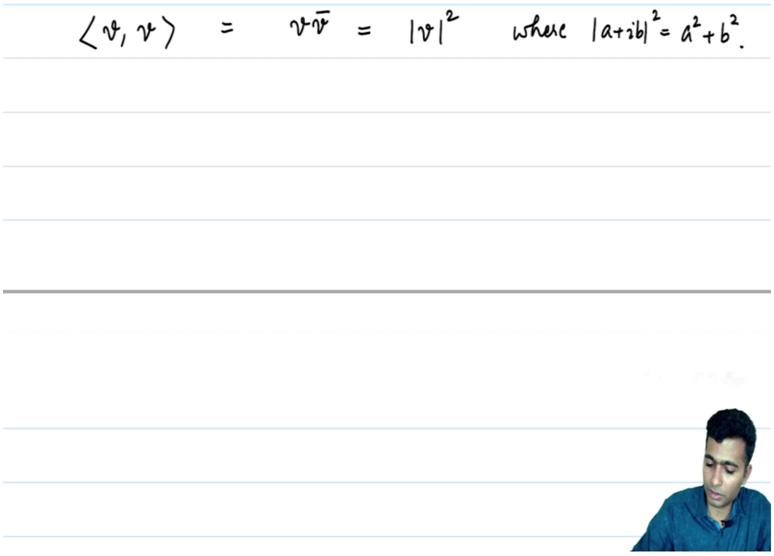

But what is the conjugate of its conjugate? Conjugate of v conjugate of a number of a complex number will give you back the complex number itself. So, maybe I should write it carefully. This is going to be v2 bar v1 the whole bar. If you carefully observe that is precisely what the conjugate means, right? And what is this? This is nothing but the inner product of v1. So, this is okay, let me write 1 more step.

This is v1 times v2 bar the whole bar, which is equal to the inner product of v1 and v2 bar. And therefore, v2, v1 is the conjugate of v1, v2 bar. So yes, that is getting satisfied. And how about inner product of v with itself, that is just going to be equal to v, v bar, which you should go back and check is just going to be mod of v square the absolute value, where mod of a plus ib square is equal to a square plus b square, which is a real number, which is a positive real number.

If both a and if one of a and b are not 0, both are 0 then this is 0 vector and this will be just be 0. So, yes so, this is satisfying all the properties that we need. So, let me show you the definition once more, our definition which is being boxed right now in green will be the definition of an inner product on C. And taking clue from this definition.

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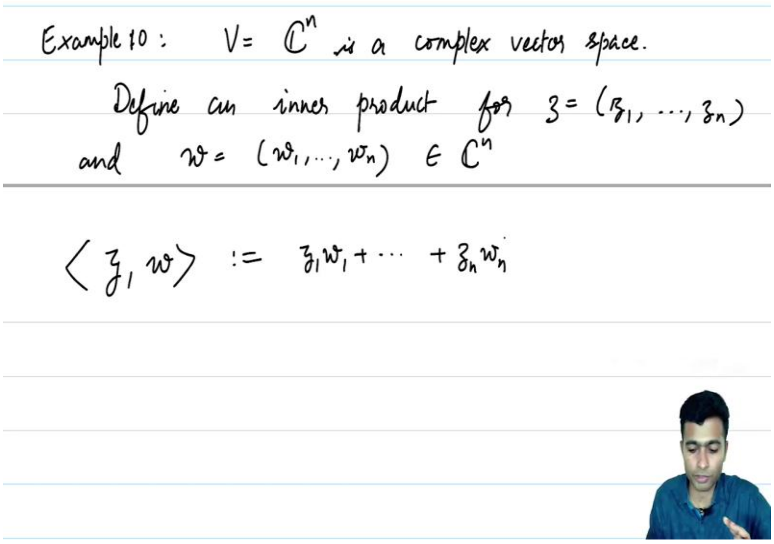

Let us look at the next example number 10. So, what is an example, next example let V be equal to C n be a complex vector, okay. So, this is the, so, V be equal to Cn is a complex vector space. So, as we have already discussed this is a vector space over C, the vector addition is component wise, scalar multiplication is also component wise. We would like to define an inner product on this. Define an inner product.

So, let us imitate, what we did in Rn. So, let for z equal to say z1 to zn and w be equal to w1 to wn complex numbers both in Cn. Inner product of z with w is being defined to be, how about we imitate the definition in Rn, this will just turn out to be z1, w1 plus dot, dot, dot, zn, wn. If n is equal to 1, however, we will just fall back to the example 9, which we just did, and we saw that that is going to fall apart. There is something seriously wrong which will happen if we define it like this. However, from what we did in 9, we can take inspiration and define it as z1, w1 bar, z plus z2, w2 bar up to zn, wn bar.

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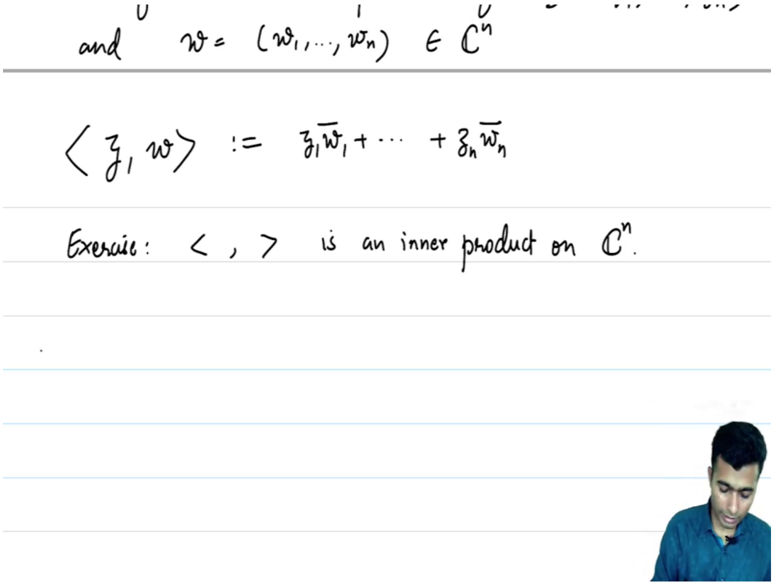

And I will however, leave this as an exercise for you to check that this is an inner product, is an inner product on V complex vector space Cn. Okay.

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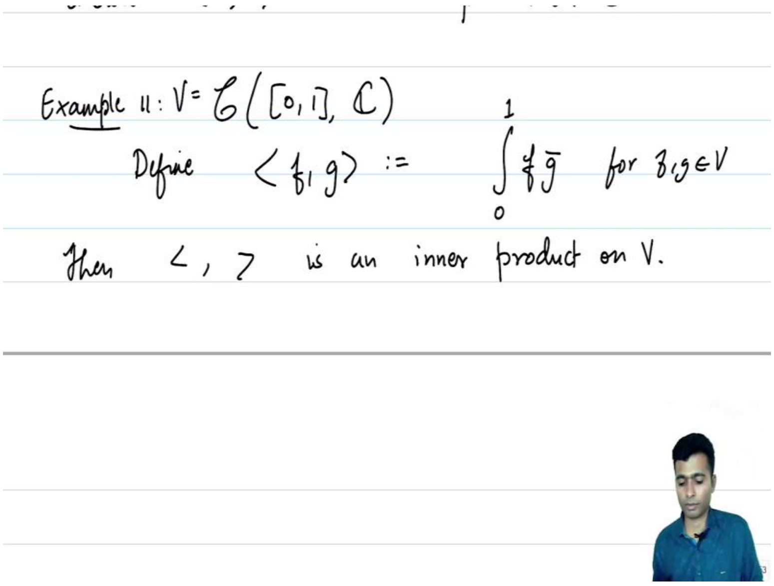

So, let us next consider another example,11. Let us look at complex valued functions, continuous functions from 0, 1. Define the inner product of 2 such functions, so we have already noted that this is a vector space over C. And let us define it to be the inner product like we have defined earlier, but let me just use, just simple notation to write it as fg bar. So, just like in example 10, if it is to be an inner product, the conjugate should get involved and this for f comma g in V, so, V is this. Then this is an inner product on V, okay.

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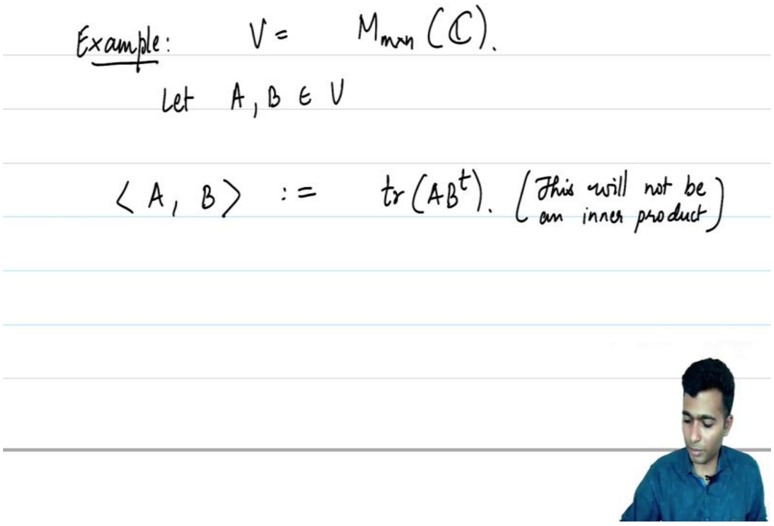

So, one final example. So, let us consider V to be M m cross n of C. So, given a matrix, so, we are defined recall that. So, let A comma B be matrices in V and we had defined A comma B to be equal to trace of AB transpose. But the same problem is again going to come up here. The fact that the conjugate is not being involved is going to be a problem. So, this will not be, you should check that this will not be an inner product, because it is a complex vector space. The conjugate linear property will break down, even the positivity will break down. This will not be an inner product.

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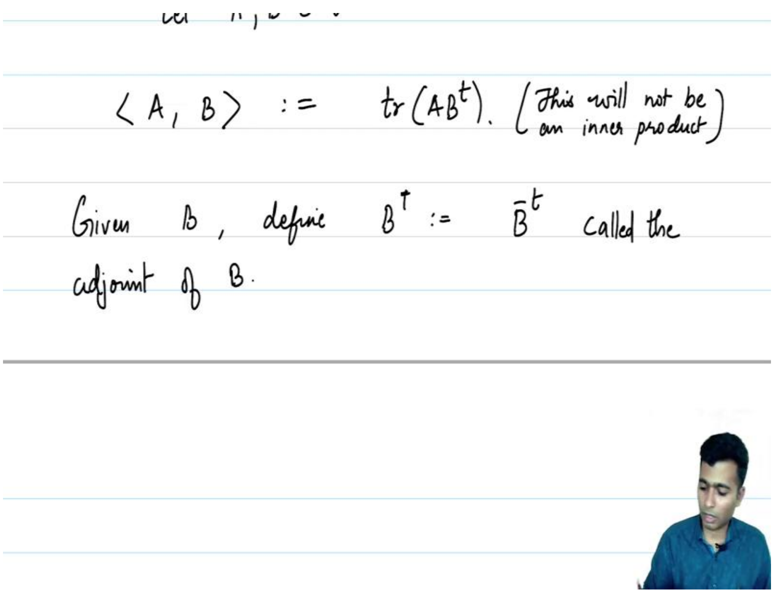

So, what do we do? So, we define given a matrix B. Let us define the adjoint. So, given B define the adjoint to be equal to B bar transpose, which is the conjugate of every entry and then transpose is taken. You transpose or which are, it’s a commutative operation. So, you transpose the given matrix and look at the conjugate of each of the entry that is called the adjoint of B, called the adjoint of B.

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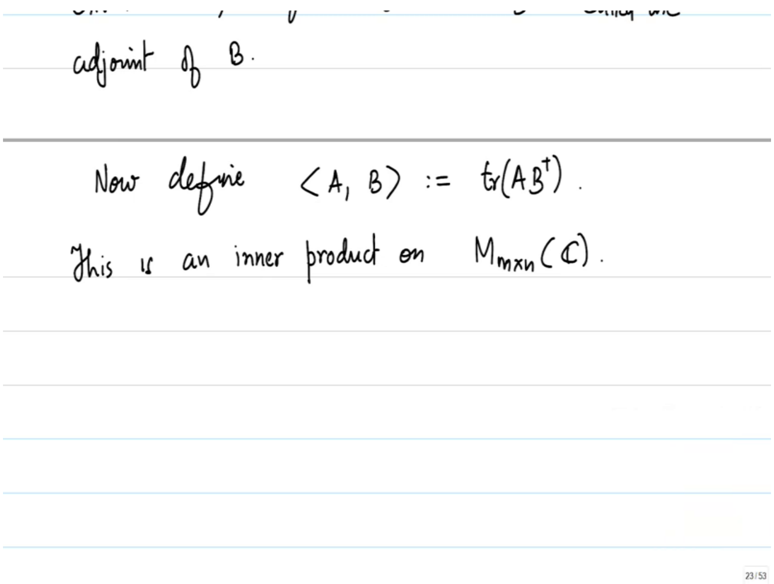

Now define the inner product of A with B will be equal to trace of AB adjoint. And then this will turn out to be, this is an inner product on V. This is an inner product on the complex vector space of m cross n matrices over C. So, we have seen that most of the vector spaces that we have seen in the examples can be made into an inner product space. Most of the vector spaces can be given an inner product, converting them into an inner product space.

So, the question could have already, the question can be asked, why not just start off with the notion of an inner product space? The right answer to that is that the more structure you add, it is more rigidity that you bring into the vector space. So, the minimum amount of operations, namely vector addition and scalar multiplication itself gave us all the results that we wanted. For example, in the last 7, 8 weeks, all the results we could consider by just considering these 2 operations. And we do not need an inner product to conclude any of this.

So, it will be good to identify what are those minimum set of assumptions which are needed to prove certain results. Moreover, if we want, if we put more structure on a space, and if we want to look at linear transformation which preserves this structure as well, namely the inner product as well, they are what are called as isometrics and the number of such maps is start coming down.

So, the study becomes more difficult, the rigidity that comes along with more structure is not very desirable. So, that is the reason why we will not always look at inner product spaces to begin with, only when there is a need, will be imposed an inner product and, and study more about it. Of course, it is a very powerful tool and a lot is do able with an extra inner product which can be imposed. We will see what can be done over the next few days.

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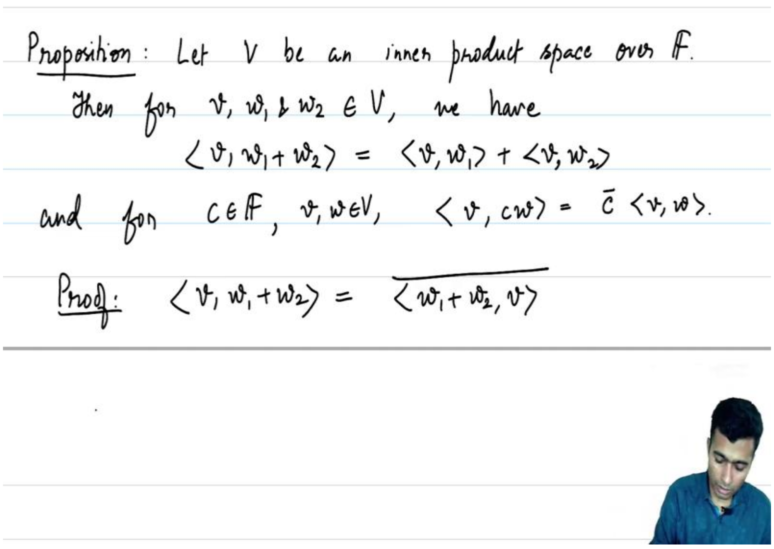

All right, so, let us look at a couple of propositions, before we conclude. So, as you can see the definition, if you go back. There are the properties, the properties are here. So, we are only demanding linearity in the first variable, why have we not demanded linearity in the second variable. So, the second variable, the linearity in the second variable, it is actually not linear, it is conjugate linear.

So, let us see. So, let V be an inner product space that means V is a vector space over F. So, let me just put the possibility of the vector space being a real vector space or a complex vector space, and that it is an inner, inner product space tells, tells us that there is an inner product. Then so denote with inner product, usual inner product with the notation being the usual one. Then for v, w1, and w2 in capital V, we have v, w1 plus w2 is equal to v, w1 plus v, w2 and not just this for a C in F and v comma w in capital V inner product of v with cw is equal to c bar times v comma w.

So, let us quickly prove this. So, what we have is conjugate linearity in the second variable, it is not just linear if you notice c is not coming out, it is c bar, it is c bar that is coming out. So, let us first check for the first identity namely what is this? This is nothing but by the conjugate linearity of our inner product. This is just bar.

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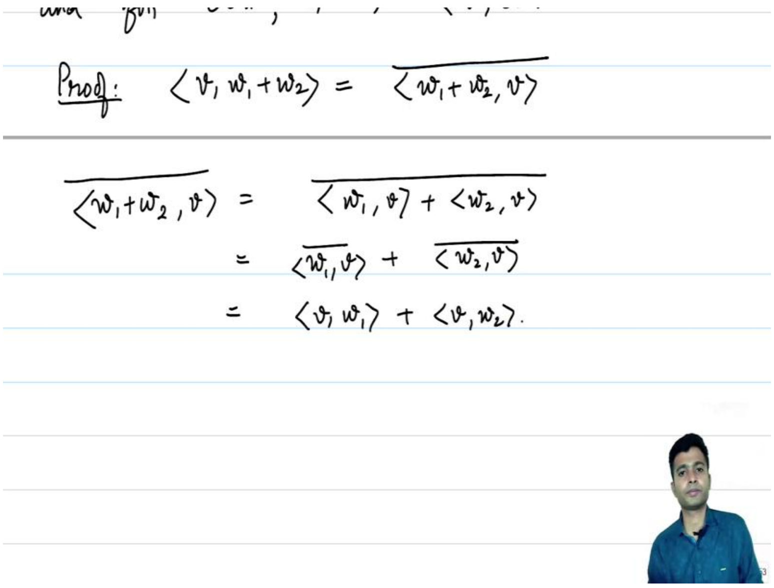

And let us analyze what is w1 plus w2 comma v, this is nothing but w1, v plus w2, v. So, what is the conjugate, the conjugate will be the conjugate of the sum, but you should check that the sum of the conjugate of 2 complex numbers is equal to the conjugate, the conjugates additives, this is just going to be equal to this plus w2 comma v. Now we use the conjugate linearity of the inner product again to conclude that this is v comma w1 plus v comma w2 and we are done.

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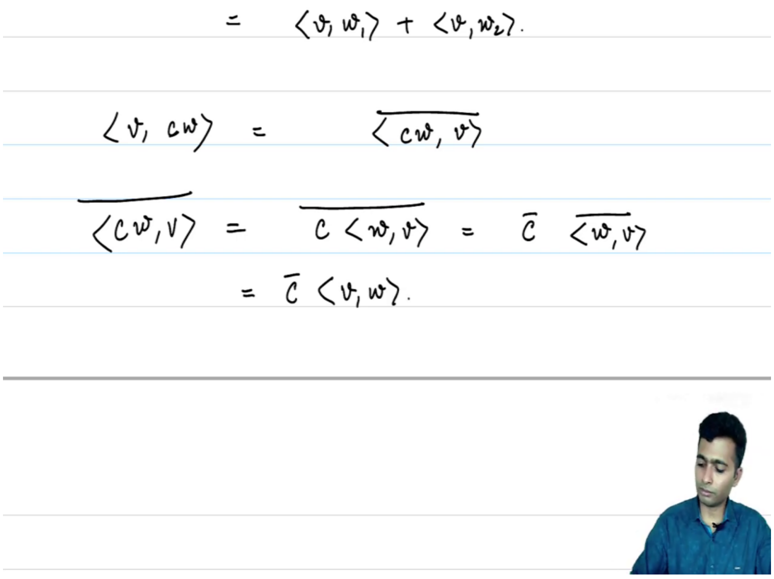

How about the second one? It is very similar the proof, the techniques are very similar. Let us look at c times w. This is just the conjugate of the inner product of cv, cw and v, but what is the inner product of cw and v? This is nothing but c times the inner product of w comma v. And let us now take the conjugate, this conjugate is equal to this. Okay, this line is not straight let me try it again. And the product of, the conjugate of product of 2 complex numbers is the product of the conjugate.

So, this is going to be c bar times w comma v bar. But what is the value, what is w comma v inner product of w comma v bar? It is nothing but the inner product of v comma w and that is what gets us to.

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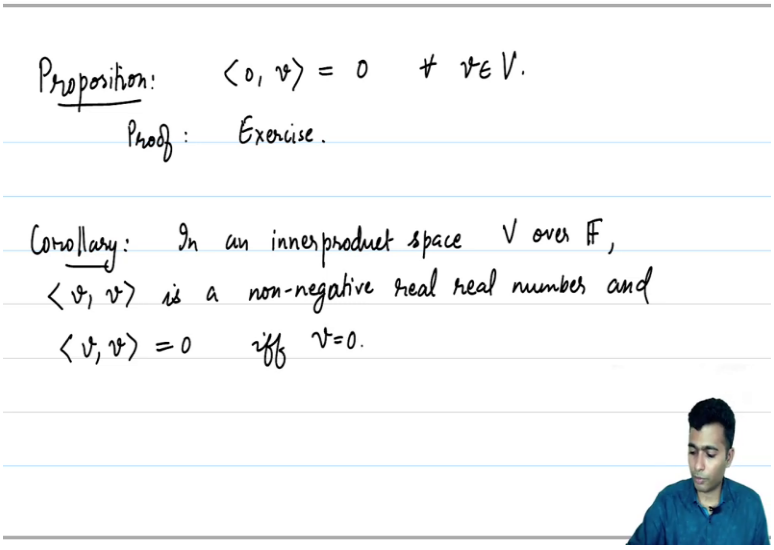

Okay, one more observation. The inner product of 0 with any vector is 0. So, I will just leave that as an exercise. Let me just write it down is equal to 0. So, this 0 inside the inner product is the vector and in the RHS, it is the scalar 0. So, depending on what vector space it is, it will be either the 0 in the complex numbers or it will be the 0 in the real numbers.

So, for all v in capital V. So, have been sufficiently big to imply that this is true in a real vector space or a complex vector space where inner products are being defined. Okay. So, the proof is an exercise. So, one thing to note is that if v is a non zero vector v comma v is a positive real number and that for v comma v is always going to be a non negative real number and it will be 0 if and only if V equal to zero. So, let me just note it down.

So, corollary in an inner product space V over F, inner product of v with itself is non negative real number. So, if you notice, only for non zero vectors that we demanded positivity and we know that for 0 vector it is going to give you back the scalar 0. So, non negative real number. So, in every case it will be a non-negative real number and v comma v is equal to 0 if and only if v is equal to 0, okay. So, next we will be discussing various properties of inner products.