Linear Algebra
Complex Vector Spaces
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So, in the last week, we were discussing eigenvectors and eigenvalues of given linear transformation. And we also discussed techniques to compute eigenvalues of a given linear transformation by considering its characteristic polynomial, and we studied these characteristic polynomials in great detail to discuss potential diagonalizability of a given linear operator.

So, in this week however, we take off in a slightly different direction. We would like to discuss potential generalization of the notion of length that we are familiar with in say R2 or in R3. So, for example, if a vector 2, 3 comma 4 is given, in3 comma 4 in R2 is given to you. We know that the distance of or the length of this vector to 3 comma 4 is square root of 3 square plus 4 square which is 5.

We would like to ask the following question, can this notion of length which we are quite familiar with in R2, can this notion be generalized to a abstract vector space. So, given an arbitrary vector space, can we say about a notion of length in that vector space. So, that will be the goal of our next immediate goal. However, before we venture into answering any of these questions, we would like to discuss vector spaces over a different field of scalars than real numbers.

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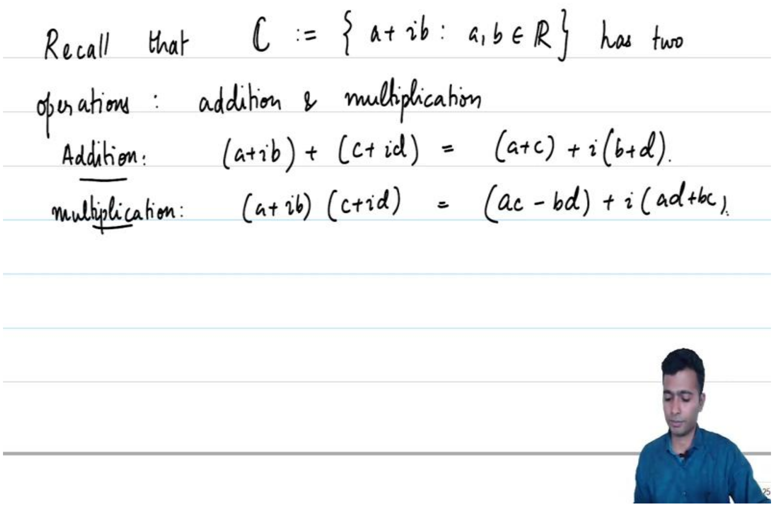

So, let us begin by recalling what our basic properties of complex numbers. So, recall that C which is the set of all elements of the type a plus ib, a and b are real numbers. So, C comes with a natural notion of addition and multiplication. So, it has 2 operations what are the addition of 2 complex numbers and multiplication. And how are they defined? So, addition let us see how addition is defined. If, say a plus id is added to C plus id.

This is just a plus c plus i times, this is not d, I am sorry, this is b. So, i times b plus d and how about multiplication? Multiplication is defined, as a plus ib times c plus id is equal to ac minus bd plus i times ad plus bc. So, just like in the case of real numbers, the complex numbers with these, these or rather the operations that we just defined on the complex numbers also satisfies all those good properties, which we had discussed in the very first week.

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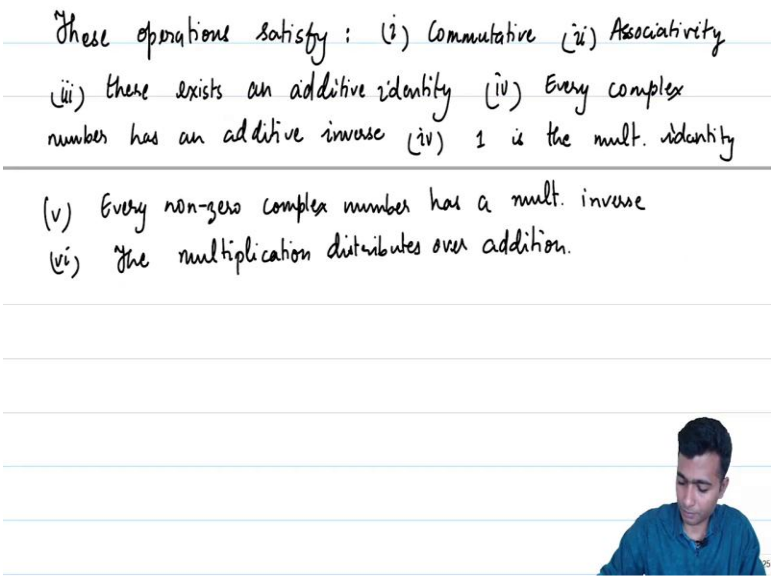

So, complex numbers, these operations rather satisfy all these properties. What are the properties that real numbers satisfied with addition and multiplication? That it was commutative. If you take 2 complex numbers, the order in which we add does not matter the order in which we multiply did not matter associativity, if we add or multiply 3 complex numbers, associativity, the order the, which one we multiplied first and which one was multiplied to it, or which two we added first and the third one which was added to the first, the sum of the first two did not matter.

So, associativity was there and given any complex number, there is an additive inverse, there is an additive inverse. So, if a plus ib is given to you, minus a plus i times minus b is an additive inverse, oh before, additive inverse, I should mention additive identity. So, 0 plus i times 0, there exists an additive identity. If you add any complex number to 0, which is 0 plus i times 0, we get back the complex number. There exists an additive identity, which is 0 plus i times 0.

Every complex number has an additive inverse. What more? There was a multiplicative identity, one in the case of real numbers, here also 1 plus 0 times i, which we will denote as 1 itself again. 1 is the multiplicative identity. What more? Every non-zero complex number, leave that as an exercise for you to explicitly calculate the inverse, every non zero, so am I writing the number wrong, Yes, I am this is 5. Every non zero complex number has an inverse, as a multiplicative inverse.

So, let me write it specifically multiplicative inverse and finally, the multiplication distributes over addition. So, all those properties which real numbers with it is. So, all those properties which real numbers along with its multiplication or addition and multiplication operation satisfied, satisfied by complex numbers as well. So, we may consider complex numbers with the operations of addition and multiplication, which we just defined. We can consider complex numbers also as a potential candidate for being the field of scalars. So, we could, so the complex numbers, we may write it down here.

We could also consider the complex numbers as the field of scalars in order to define a vector space in order to do define a vector space. So, we could define a vector space over complex numbers, sometimes it is called, such vector spaces are called complex vector spaces, then the definition will be the exact same definition as we have given, instead of considering scalars to be real numbers. Now, our scalars will be complex numbers, the same definition can really be taken as a definition for vector spaces over C as well.

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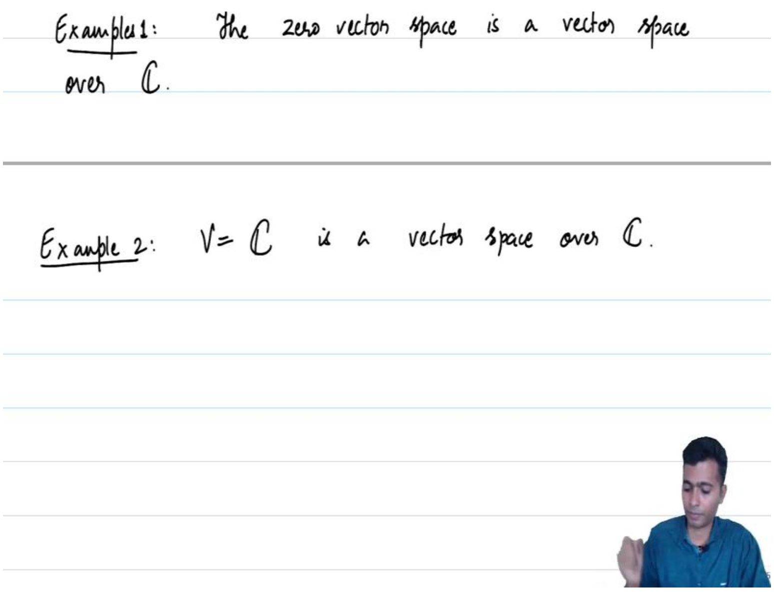

Let us look at a few examples. So, I will not spend too much time discussing the examples again, because we are quite familiar now with the notion of vector spaces. So, are over the field of scalars being real numbers, the notion, the idea of vector space over complex numbers is extremely identical to those notions which we have discussed in detail. We just have to be careful that whenever we consider a scalar now, we are considering complex numbers, if it is a vector space over C.

So, the first example should be the 0-vector space, this is also a vector space over are, over C is vector space, so over C. So, you could take any complex number define the scalar multiplication, addition vector addition could be defined as in the earlier case, take any complex number define the scalar multiplication of that complex number to 0 as 0 itself. And with these operations, the set will turn out to be a vector space over C.

Example 2, so I will not spend any more time discussing the properties 1 to 8. Of course, all those have to be settled, all those have to be satisfied and it is a job for you to do that. This time however, you should be careful, we have to check all those properties 1 to 8 with scalars from complex numbers, all these properties should be satisfied for scalars being complex numbers.

How about the second example, C itself is a vector space. So, V is equal to C is a vector space over C, how is the vector addition, the usual addition and how is the scalar multiplication, take a scalar which is not a complex number and take a vector which is again a complex number, the scalar multiplication will just turn out to be the normal multiplication, this becomes a vector space over C.

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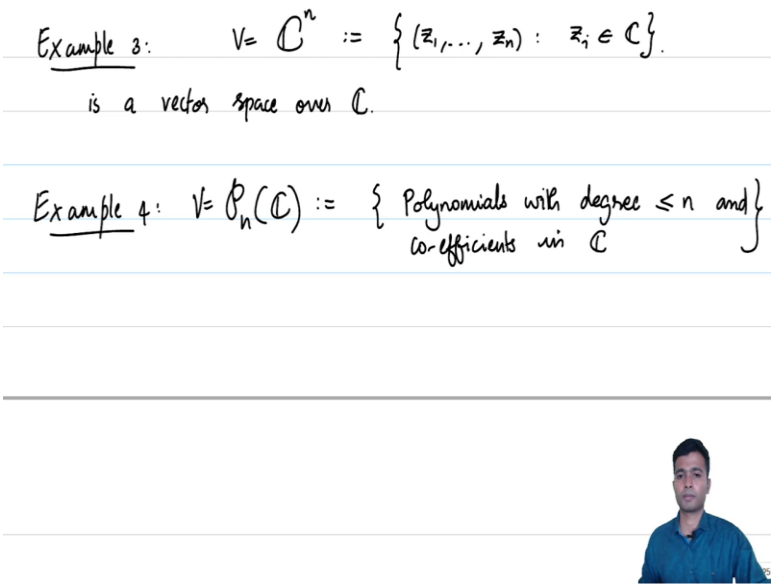

Next example will be the Cartesian product of C with itself n times, you see there is an analogous example we are taking analogous examples. So, over there, Rn was the example of the linear space, here over C, we would like to consider Cn which will be just Z1 to Zn where each of these Zi are complex numbers and how is addition defined? Addition is defined component wise. How is scalar multiplication defined? Again, it is defined component wise. So, let me not venture into that. Let me not spend more time on that this is a vector space over C. How about polynomials? So, that is the next example.

So, let me write Pn of C. So, V is Pn of C in this case. So, notice over there it was Pn of R what would be the, what would be your guess on the definition of Pn of C? This is going to be all polynomials of degree less than or equal to n. However, this time we are going to have coefficients from complex numbers. So, polynomials with coefficients, with degree less than or equal to n, less than or equal to n and coefficients in C. And examples of what is the addition here, right before going to the next example, the addition is just like addition, usual addition in Pn of R just in this case, we are adding complex coefficients rather than real coefficients.

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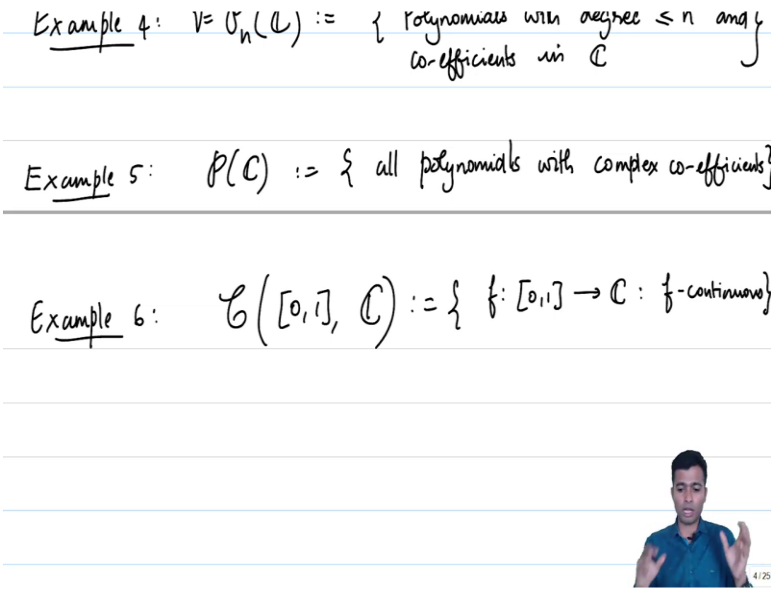

Similarly, scalar multiplication is defined just like in Pn of R. However, now, we are multiplying a scalar which is a complex number to each of the coefficients. Similarly, P of C is the space of all polynomials with complex coefficients. So, let me write more, in a more compact manner. And I write complex coefficients it means coefficients in C. So, the coefficients of polynomials here could be complex numbers. All right, what other examples? Okay next example, Example 6. Let us say we considered, continuous functions, real valued continuous functions.

Now, let us consider, let me put a comma and put a C to denote that now we are considering complex valued continuous functions. So, this is defined to be the set of all f from 0, 1 to C such that f is continuous. How is the addition defined? Again, it will be defined point wise f plus g at a point x will be f of x plus g of x. Now, notice that f of x is a complex number g of f is also a complex number, you can add two complex numbers and get back a complex number.

And therefore, f plus g will now be a function from 0, 1 and into C. And a real analysis course will tell you that some of 2 such complex valued continuous functions will again be a continuous function. Similarly, we define scalar multiplication, we take a complex number, let us call it alpha, alpha times f at a point x will be alpha times f of x, alpha is a complex number f of x is a complex number, a multiplication will give you a complex number, this sends a map from 0, 1 to C. And again, a course in real analysis will tell you that this is also a complex number.

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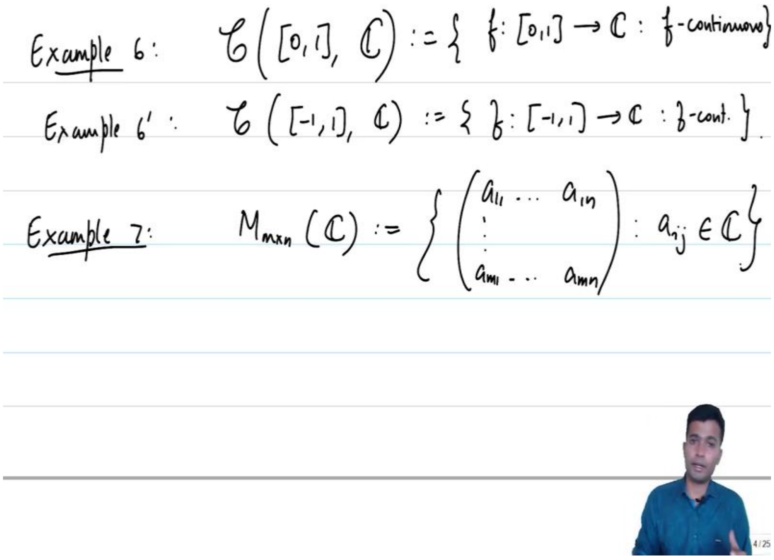

The domain was not special, we could have so maybe 6 prime examples 6 prime, could have been something let us say C minus 1, 1 to C as well. So, this is just going to be f from minus 1. 1 comma minus 1 comma 1 into C and such that f is continuous and defined addition and scalar multiplication similarly. Okay, more examples, a very important example is M m cross n of C. Again, I do not need to really write it down because this is just the matrices a 11 to a 1 n, a m1 to a mn.

But now our aij are not just real numbers, they are potentially complex numbers. And the vector addition is defined component wise, scalar multiplication is also different component wise. So, yes so, we could consider vector spaces over complex numbers and there are so many examples as we can see. Right. So, if you go back to our previous week material and carefully look into it. We were defining a linear combination of vectors in a vector space.

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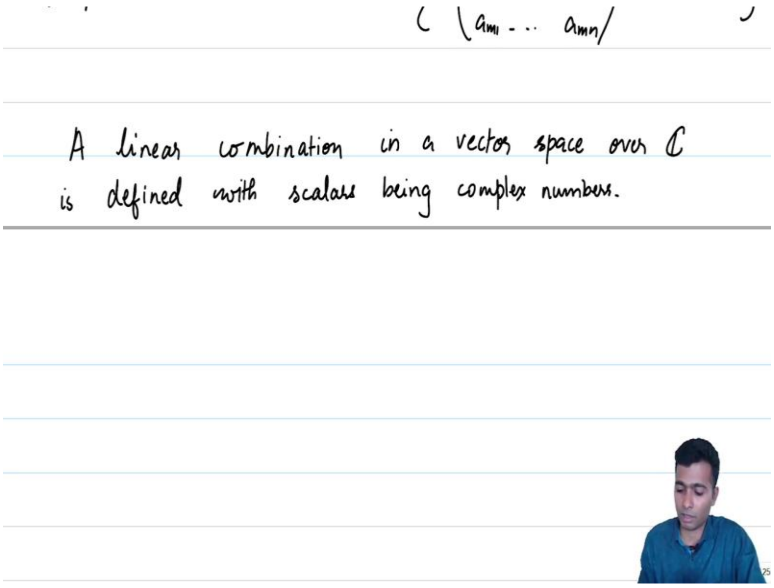

And how was a linear combination defined, the linear combination was defined as a vector linear combination was defined of say V1 to Vn was defined a1 V1 plus a2, V2 to an, Vn. And we were demanding that ai b scalars. Now we could define a linear combination. With, in a complex vector space, in a vector space over C, could be defined, is defined rather analogously, defined with coefficients or scalars being complex numbers. So, in a vector space over C, in one of these examples, which we have just described, a linear combination could be defined analogously by or rather the definition was already given in a very general case, where in we just use the word scalar.

Here our scalar now is complex numbers, we could now define the or rather we do not need to define again, we have already defined the notion of linear independence goes through exactly how we had defined earlier. We could also define span, span now will be all linear combinations with the coefficients or the scalars being complex numbers, so span makes sense, basis makes sense, basis of a complex vector space will be a linearly independent set which spans over a given vector space.

We could talk about the replacement theorem, the dimension theorem, before coming to the dimension theorem, replacement theorem and the fact that a basis, if there is a finite basis, every other basis will have the same size all these results if you go back and check carefully, the proofs, the statement and the proofs will go through for the field of scalars being complex numbers. Main reason being that we have really not used any specific properties of the field of scalars to prove any of these results.

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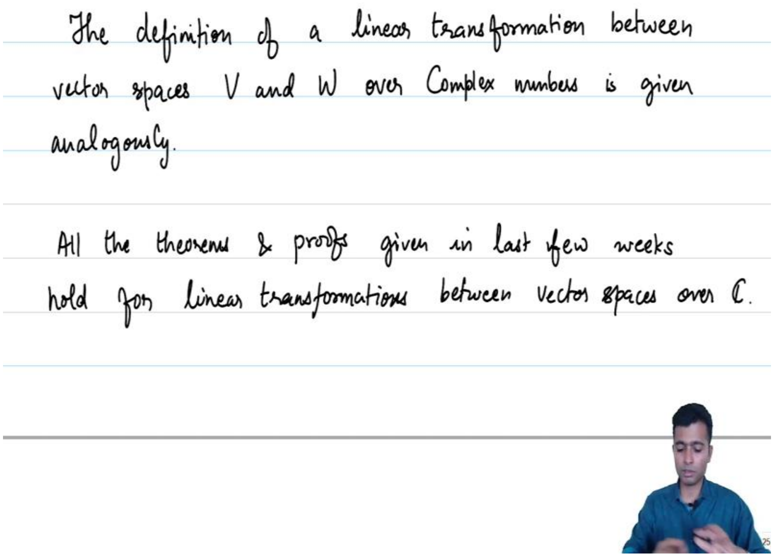

We have just used the properties of the vector space operations for our results to be proved, right. So, we could also define or rather we have already defined what linear transformations are between vector spaces over a given, so, we have, so, the definitions of linear transformation between vector spaces V and W over complex numbers is given analogously. So, the structures are preserved, so vector addition is preserved, there is nothing new there.

However, when scalar multiplication is being preserved now, we demand that the scalar will be a complex number, right. So, a thing to note here is that we cannot talk about linear transformations from a vector space over the complex numbers to a vector space over the real numbers. So, if V is a vector space over C and W is a vector space over r, it does not make sense to demand that there x is a linear transformation from V to W, because the scalar multiplication in V is with respect to complex numbers. And the scalar multiplication in W is with respect to real numbers.

The scalar multiplication of a complex number and a vector in W does not make sense. So, we cannot even demand or we cannot even give sense to a definition of a linear transformation from a vector space over C to a vector space over R or for that matter, a vector space over R to a vector space over C. So, whenever we talk about linear transformations, it will be with, it will be over a, over the same field of scalars. Either it will be over the field of scalars being real numbers, vector spaces over real numbers or it will be between vector spaces over complex numbers.

So, all these statements and theorems, which we have given about linear transformations in the last many weeks, they hold for linear transformation between vector spaces over C as well, should actually go back and carefully look at each of these statements and proves and notice that the proof really works. Even if we are considering complex vector spaces. So, let me just note that all the theorems and proofs given in the last weeks, last few weeks hold for linear transformations between vector spaces over C. So, even for complex vector spaces or vector spaces over the field of scalars being complex numbers, all these theorems hold.

For example, dimension theorem holds, all the consequences of dimension theorem. As for example, in a complex vector space, if the dimension of the vector space is the complex, the dimension is n as a complex vector space then, if you consider a set of n linearly independent vectors in the vector space V then it should necessarily be a basis.

So, not just the results about dimension theorem or its consequences, every definition and every result that followed for example, eigenvalues, eigenvectors etcetera can be defined in the case of complex vector spaces and linear transformation between vector spaces over C. So, in this case now eigenvalue will just turn out to be a complex number instead of a real number. It could be a real number which is also a complex number. But, do keep in mind that when we are considering linear operators from a vector space over C to itself are eigenvalues could be complex numbers.

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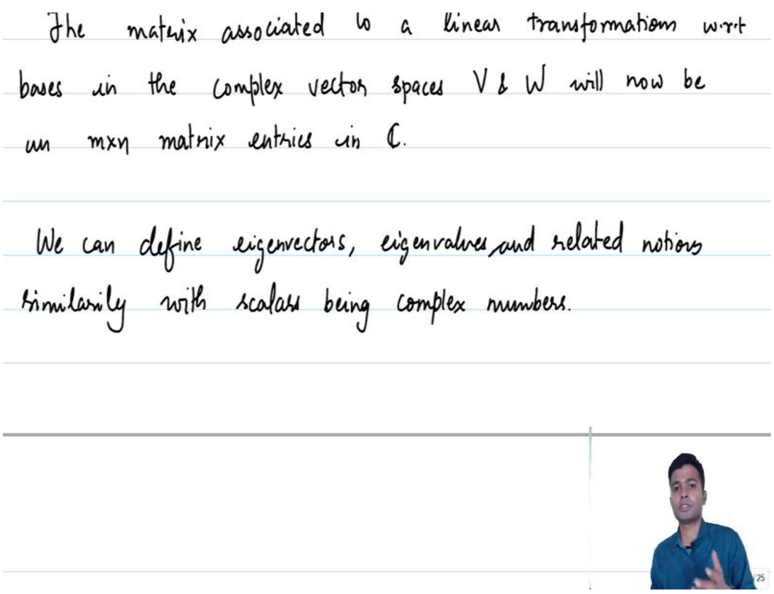

So, eigenvectors will, so, the matrix there could be, so, we can we also have the notion of the matrix associated to a linear transformation will now just turn out to be an m cross n matrix with entries in C, the associated, matrix associated to a linear transformation with respect to basis in the complex vector spaces V and W, finite dimensional complex, complex vector spaces V and W will now be an m cross n matrix, where m and n are respectively dimensions of W and V.

But now the entries will be in C. So, instead of real entries it could have now complex entries, we could also talk about as already noted, we can define eigenvectors, eigenvalues, eigenspaces, the characteristic polynomial and all related notions analogously that notions similarly with scalars being complex numbers. So, in particular eigenvalues could now be complex numbers. Our characteristic polynomial of an m cross n matrix over C or a linear operator on a vector space V over C will now be a polynomial or complex number.

And not just necessarily, not just over real numbers, it could be having coefficients which are complex numbers. Of course, the impact of considering vector spaces over C, the impact of complex numbers and its operations are the various properties of complex numbers and its operations does have some implication on, on the various properties of the linear transformation. However, we will not explore too much in that direction in this course.

So, what I meant is, for example, if our linear operator is on a vector space V over C, then if you consider the characteristic polynomial it will be an n degree polynomial over C. And the complex numbers have an added advantage as compared to the real numbers that you look at any polynomial it should necessarily split into real factors, oh sorry linear factors. That is not necessarily the case in the case of r. For example, when you look at lambda square plus 1, it does not split, but in C every polynomial splits into linear factors.

So, therefore the question of diagonalizability, whether, if our characteristic polynomial splits or not will not be a problem of concern for us. However, we will not venture too much in that direction. Now, let us go ahead and study more ideas and more notions on vector spaces and not restrict in the, in the direction of what will be the implication of various properties of complex numbers.

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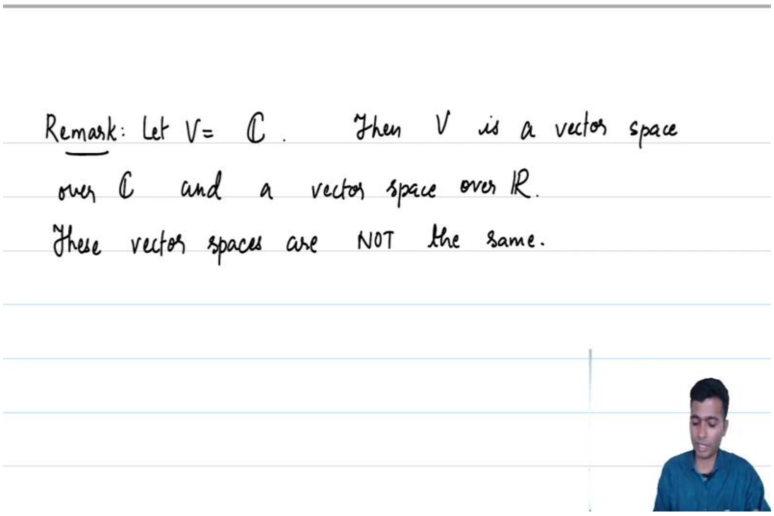

All right. So, let me stop by making a remark here, by making an observation here. So, let me just call it a remark. So, if you consider V to be the complex, set of complex numbers. Then so, let V be the set of complex numbers, then V is both a vector space over C, so a vector space over C as has already been noted. And if you go back to the first week and look at our examples, it is also a vector space over R and a vector space over R.

So, the same set C or V in this case is a vector space over C or the field of scale has been considered as complex numbers and it is also a vector space over R when the field of scalars is being considered over R. However, it should be kept in mind that even though it is the same set which is becoming a real vector space or a complex vector space, both are not the same. These vector spaces are not the same, are not the same. So, remember that vector space is not just the set, it is a set with two more operations, it is the collection of these three objects, the set and the two operations which makes it into a vector space.

So, here are the operations change, when you are considering it as a vector space over R and when we considering it as a vector space over C, the operations change and that is precisely what it means to say that the vector spaces are not the same. Just to give you an idea about why it is not the same, I will allow you to check it, check that the, check a basis for V over R.

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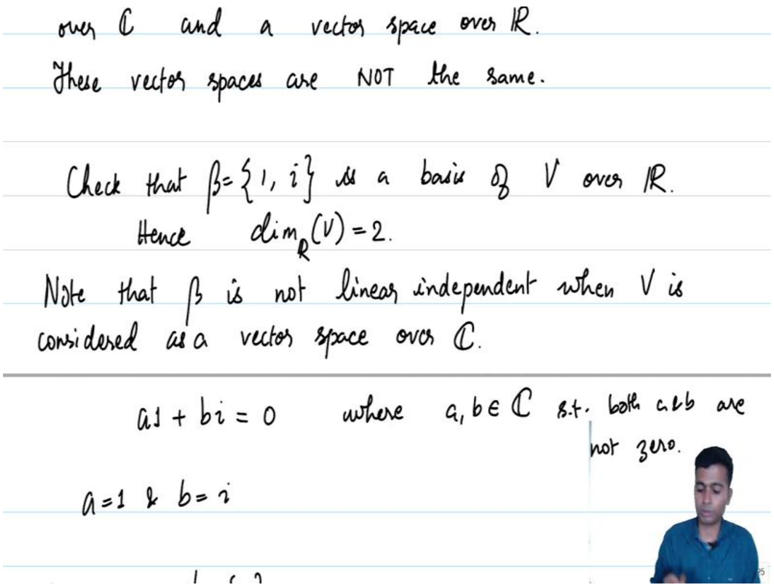

So, check that I just give you an example. 1 comma i is a basis of V or let me just write V in this case it is C over R. However, note that this set is not even. So, let, let me call it B, note that B, beta is not linearly independent. So, these are two vectors in C, right. But these vectors are not linearly independent when you consider V as a vector space over R, when V is considered as a vector space over C. So, when you are considering it as a vector space over C, this is not linearly independent.

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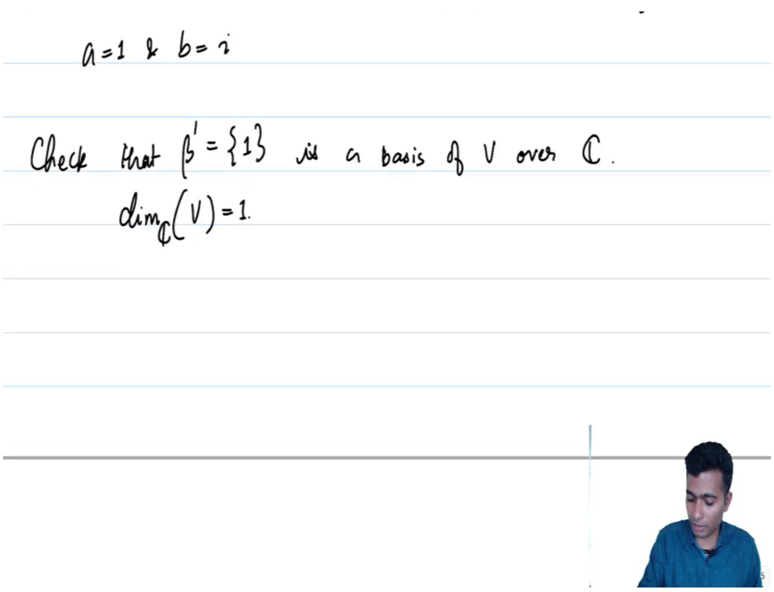

For example, we would like to look at a times 1 plus b times i to be equal to 0. And remember that we now want a comma b to be C not both, and both are not zero such that both a and b are not, are not 0. So, a moment's thought I will reveal that a equal to 1 and b equal to

i. Both are complex numbers. If you put a equal to 1 and b equal to i, what will be the relation that we just wrote? It will be 1 times 1, which is 1 plus i times i, which is minus 1 is equal to 0. So, this is not linearly independent.

So, I would also leave you to check that. 1 is a basis, the set beta prime, which is equal to 1 is a basis of V over C. So, if you consider a vector… the vector space of complex numbers over C it has dimension 1. So, dimension of V over C here is equal to 1, what was the dimension here? Hence, dimension of the vector space V as a vector space over R, this is equal to 2.

So, they are not the same, they are different vector spaces when considered and it should not be confused. So, whenever there is a vector space involved, from now on, we will keep track of whether it is a vector space over real numbers or whether it is a vector space over complex numbers. And we will now jump into the notion of an inner product space, which is the right object to look at in order to talk about links in a vector space.