Linear Algebra
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Lecture 8.3
Multiplicity of eigenvalues

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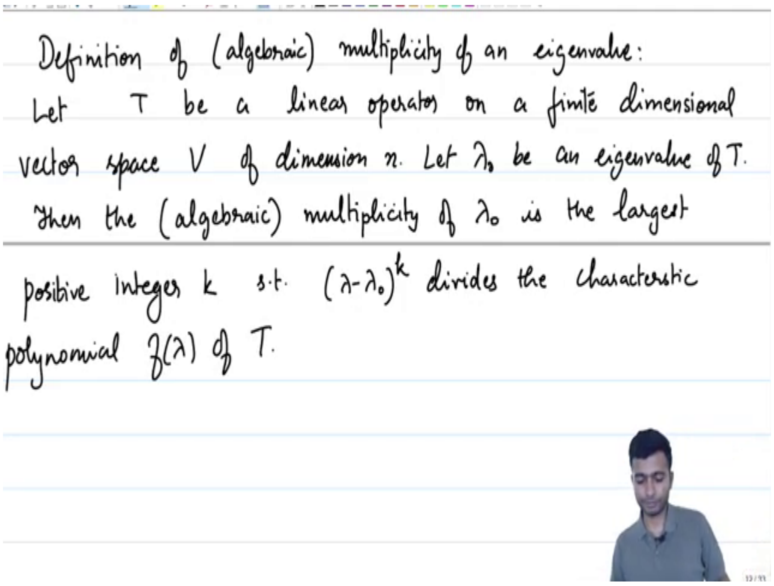

So, let us start with the definition of the Multiplicity of an eigenvalue, definition of, let me put it in a bracket the word algebraic multiplicity of an eigenvalue. So, let T be a linear operator on a finite dimensional vector space V, which has say dimension n and let Lambda naught be a Eigen vector corresponding or Lambda naught be an eigenvalue of our linear operator T.

So, let Lambda naught be an Eigen value of T, then the multiplicity of Lambda naught is the largest positive integer such that Lambda minus Lambda naught to the power k divides the characteristic polynomial of T. So, let me define the multiplicity or maybe let me just write a bracket algebra, this many times also called algebraic multiplicity.

The algebraic multiplicity of Lambda naught is the largest positive integer K such that Lambda minus Lambda naught to the power k divides the characteristic polynomial f of Lambda of T. So, let us go back and look at our examples earlier.

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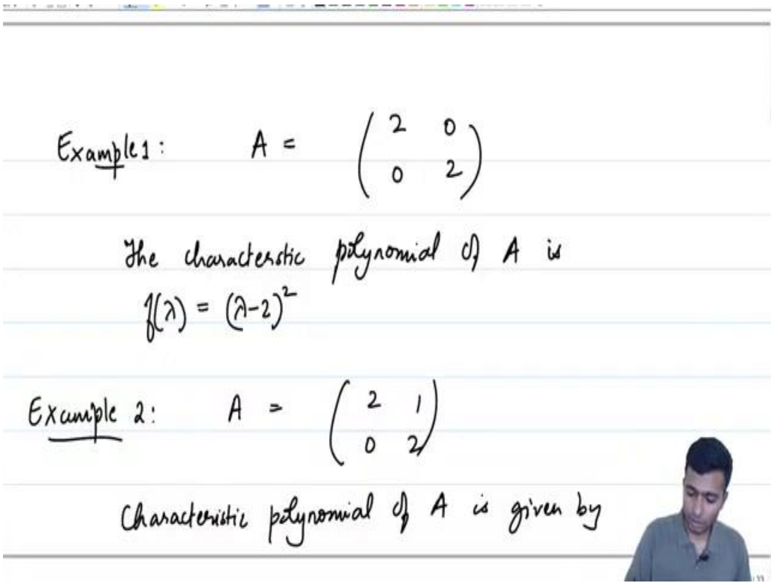

So, the first example the characteristic polynomial is Lambda minus 2 the whole square, the only Eigen value here is 2 and the algebraic multiplicity of the eigenvalue 2 here is equal to 2.

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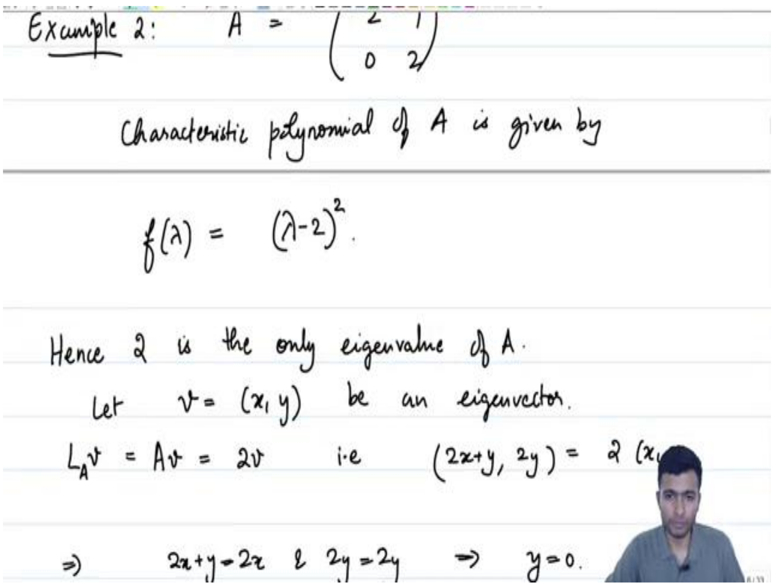

The same is the case with example 2, where the characteristic polynomial is the same.

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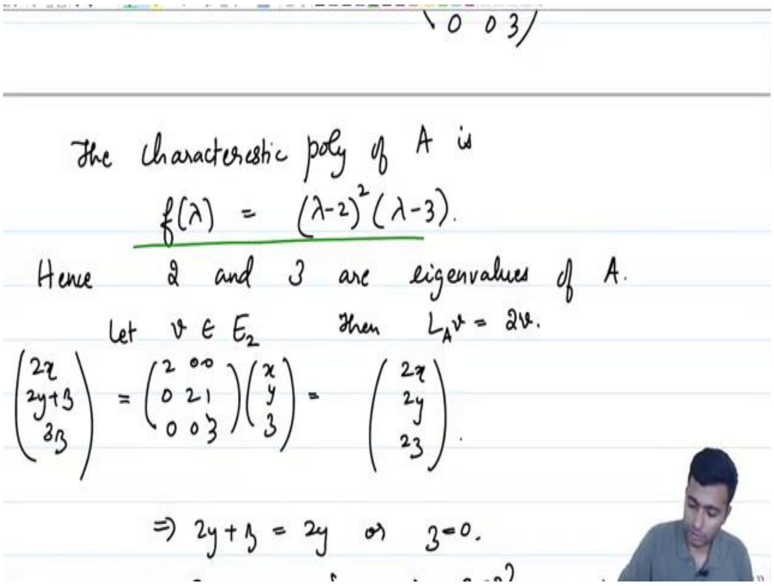

How about example 3? Example 3 has the characteristic polynomial given by Lambda minus to the whole square times Lambda minus 3, there are two Eigen values in the Lambda naught equal to 2 and Lambda naught equal to 3, there are two different Eigen values here, the multiplicity of the eigenvalue 3 is equal to 1 that is the largest number such that Lambda minus 3 to the power k divides our f of Lambda and similarly, the algebraic multiplicity of Eigen value 2 is equal to 2.

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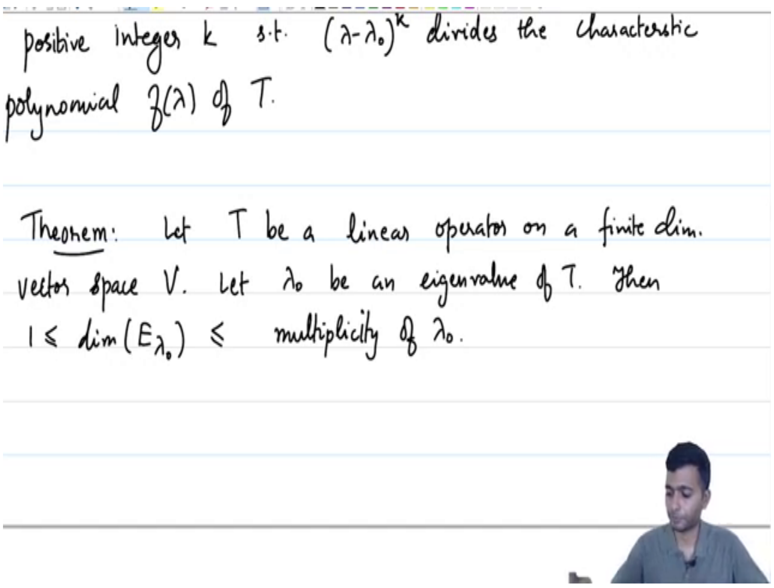

So, why are we concerned about this? The next theorem tells us that the algebraic multiplicity always dominates the dimension of the corresponding Eigen space. So, let me prove that theorem for you. So, let, T be a linear operator on a finite dimensional vector space V and suppose, let Lambda naught be an Eigen value of T.

So, the theorem tells us that, then dimension version of E Lambda naught is less than or equal to the multiplicity, so I will slowly drop the adjective algebraic, so let us just call it multiplicity of Lambda naught. So, notice that if Lambda naught is an Eigen value of T then there exists at least one vector which is non-zero and such that it is the Eigen vector corresponding to Lambda naught. So, therefore the dimension is certainly greater than or equal to 1. This theorem tells us that the dimension has to be less than or equal to the multiplicity of Lambda naught.

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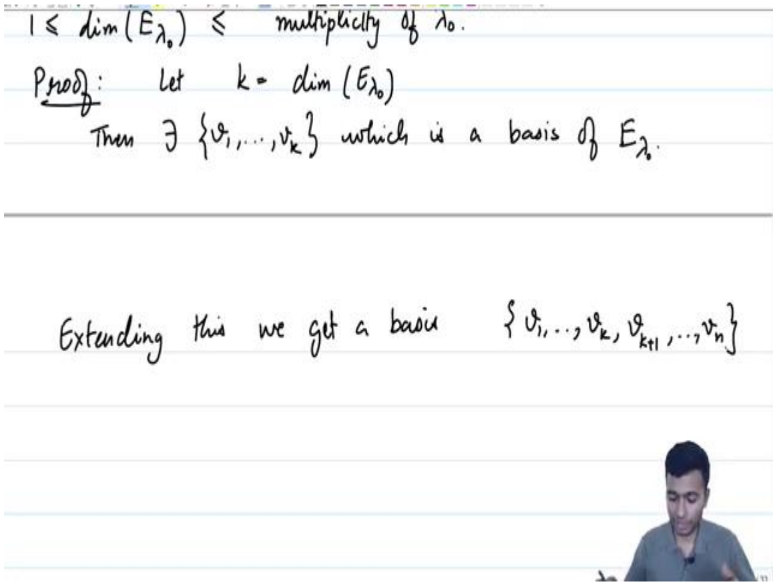

So, let us give a proof of this, so the fact that, so let us give these things names so, let k be equal to dimension of E Lambda naught and therefore, what does that mean? Then there exist v 1 to v k such that v 1 to v k or rather, which is a basis of E Lambda. So, this is a basis of the Eigen space E Lambda naught corresponding to Lambda naught each of the v 1, v 2 up to v k are Eigen vectors with eigenvalue Lambda naught, but any linearly independent set is contained in a basis so we can extend it.

So, extending this, we get a basis which is given by say v 1 to v k which are Eigen vectors and then v k plus 1 to v n, where n is the dimension of our vector space we have slowly stopped putting n as being the dimension of l always assume that n is the dimension of v in the entire lecture.

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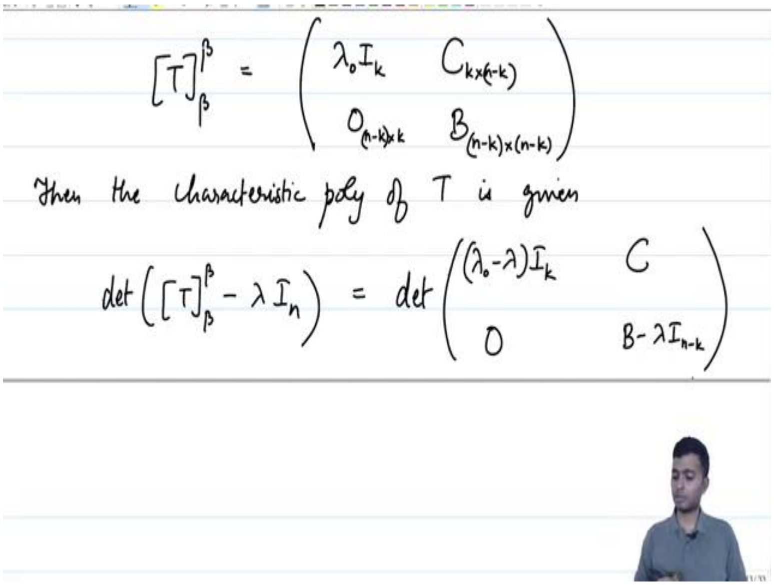

So, let us try to look at how the matrix of T so, let us call this Beta. So, interested in what is T Beta, Beta what is the matrix of T with respect to the basis Beta? What will be the first column? The first column will be T v 1, and what is that? T v 1 is just Lambda naught times v 1, and similarly T v 2 is Lambda naught times v 2, so other bases do not contribute anything.

So, I would say that the first k columns will just have a Lambda naught times I k and here there will be a 0, the 0 matrix which is basically n minus k cross k matrix. And then there will be some matrix here which I would like to split as something into k cross n minus k matrix C and B which is n minus k cross n minus k matrix.

So, what will be the characteristic polynomial of T? So, let us see, then the characteristic polynomial of T is given by, so recall that the characteristic polynomial of a matrix is invariant under similarity and therefore, characteristic polynomial of a linear operator can be defined with respect to any basis.

So, what is this? This is basically a determinant of T Beta Beta minus Lambda times I n, which will just turn out to be equal to the determinant of the matrix Lambda naught minus Lambda times I k, v 0 and the C are untouched, and there will be a B minus Lambda times I n minus k. This is precisely how the characteristic polynomial will behave.

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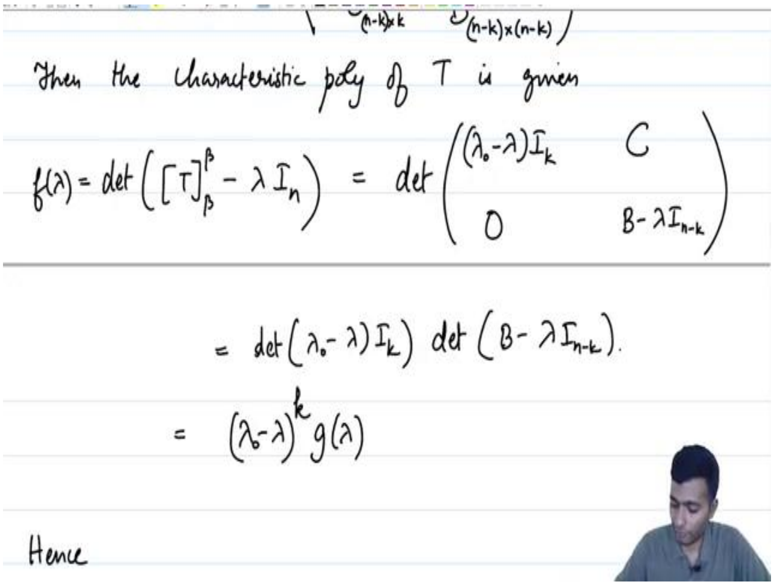

By one of the exercises, which was done in week 6, this is just going to be equal to the determinant of Lambda naught minus Lambda to the power times I k times the determinant of B minus Lambda I n minus k, determinant of block matrices. Recall that this is a block matrix of the correct sizes which was written a bit above here.

So, what is the determinant of Lambda naught minus Lambda times I k? That is just going to be equal to Lambda naught minus Lambda to the power k times some g of Lambda where g of Lambda is the polynomial of degree n minus k, which is basically the characteristic polynomial of B here, so, it is going to be a polynomial of degree at most n minus k.

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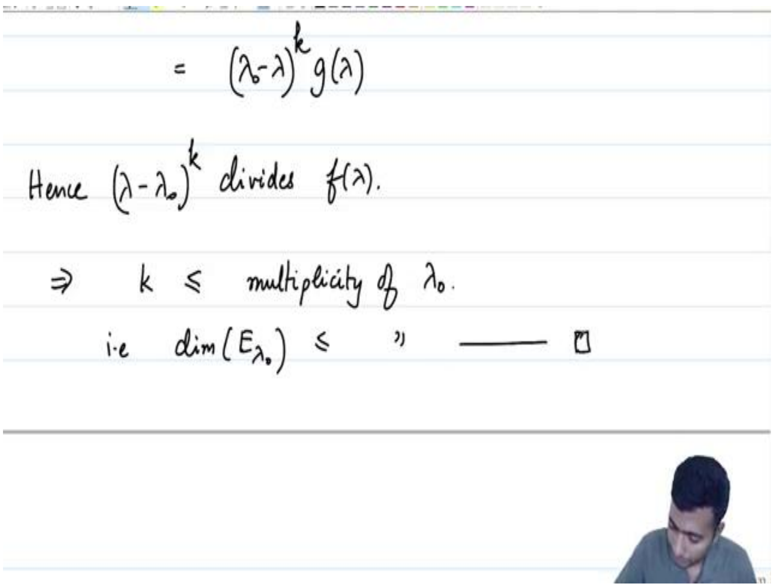

And therefore, this is so what does this mean, this means that k Lambda minus Lambda naught to the power k so, hence, so, what is this? This is f of Lambda and we have that f of Lambda is Lambda naught minus Lambda to the power k times g of Lambda therefore Lambda minus Lambda naught to the power k divides f of Lambda.

But what was the multiplicity of a given eigenvalue, it is the largest positive integer l such that Lambda minus Lambda naught to the power l divides f of Lambda. Therefore, k has to be less than or equal to the multiplicity of Lambda naught, but what was k? K was nothing but the dimension of the Eigen space and this is less than or equal to the multiplicity of Lambda naught and we have completed the proof.

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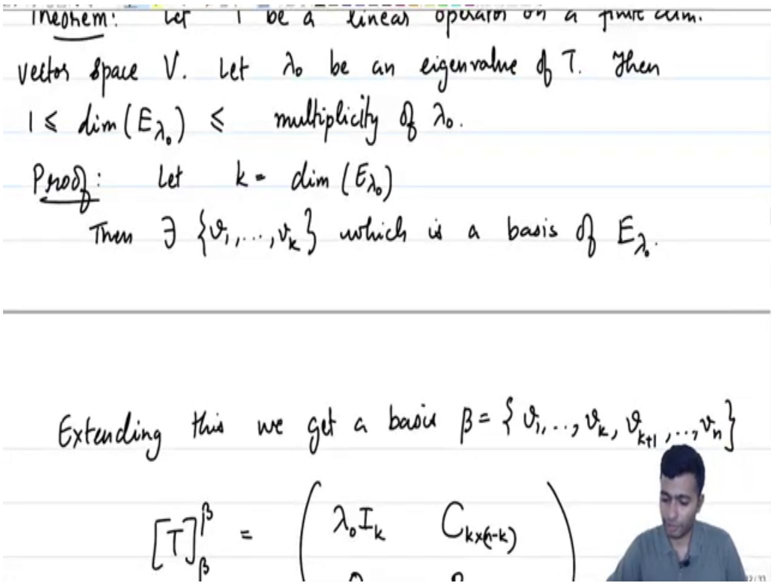

So, let me just show you the theorem we have just proved. We have just shown that if you start off with a linear operator and let Lambda naught be an Eigen value of the given linear operator T, then the multiplicity of the Eigen value always bounds the dimension of the Eigen space.

So, in the previous example, we saw that when the dimension was equal to the multiplicity, our linear operator was diagonalizable. So, if the characteristic polynomial splits and if this inequality which are circled in green, if that turns out to be an equality, then we had that, of course, we already know that it was, we found out that it was diagonalizable by explicitly computing a basis consisting of eigenvectors

So, that yeah this I have already noted that tempts us to conjecture that if in a general n cross n matrix, if we can somehow show that if we are given that the dimension of the Eigen space is indeed equal to the multiplicity, then for all the Eigen values Lambda i, then maybe our linear operator is diagonalizable and that is true and next goal would be to exactly prove that. So, let us gather some of the statements needed to prove what we just mentioned.

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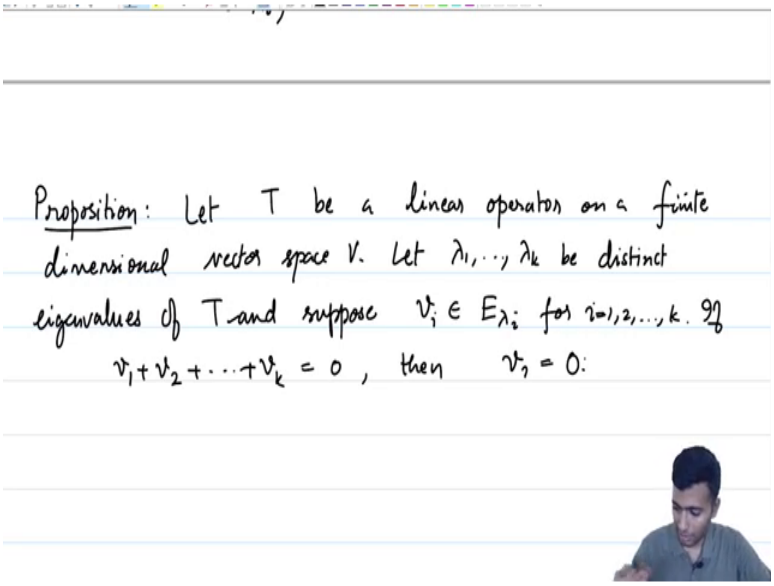

So, the first one is a statement which we have already proved in another case. So, let T be a linear operator on a finite dimensional vector space v. So, let Lambda 1 to Lambda k be distinct eigenvalues of T. Suppose V i belong to E Lambda I, so that means V i are elements in v or vectors in the Eigen space corresponding to Lambda i.

If we have that v 1 plus v 2 plus up to say v k so, this is for i equal to 1 to k. So, for each these we have V i is in E Lambdas. So, suppose v 1, v 2 up to v k all add up to the 0 vector, then V i is equal to necessarily equal to the 0 vector. So, I have slowly stopped bothering about whether it is clear from the context as to the 0 being written as the vector zero or the scalar zero.

I think by now you should be able to quickly catch from the context itself. So, you should carefully look at what the context is and conclude whether our vector is the zero vector or the zero scalar. So, here it is the zero vector as you can see, so I will leave this as an exercise to you.

We have already noted that if Lambda 1, Lambda 2, up to Lambda k are Eigen values and distinct Eigen values and v 1, v 2 up to v k are corresponding Eigen vectors then they are linearly independent, you should use that to conclude that this forces V i x to be equal to be the 0 vector. So, I will leave that as an exercise for you and let me know make next maybe theorem.

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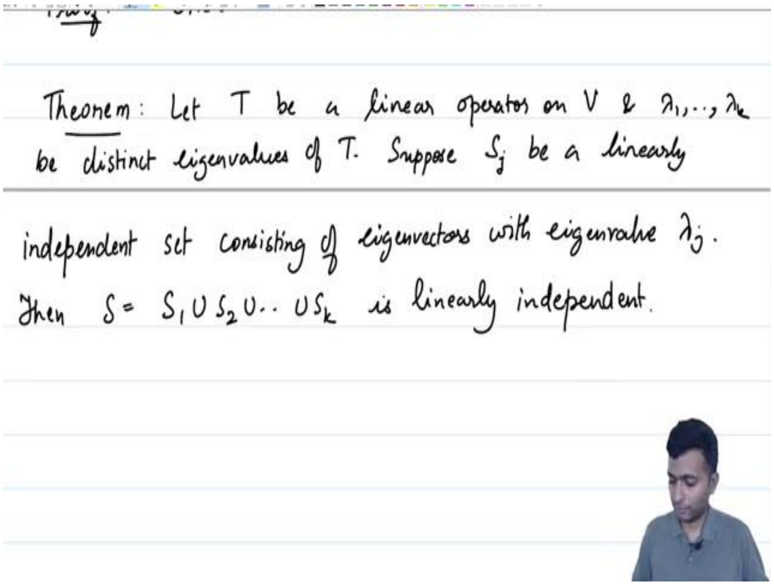

So, let Lambda 1, Lambda 2, up to Lambda k be Eigen values of distinct Eigen values of a linear operator T. And suppose we get hold of set say S 1 which are consisting of eigenvectors with eigenvalue Lambda 1, and suppose S 1 is linearly independent and suppose S 2 is a similar set corresponding to Lambda 2.

So, if you look at Lambda 1, union Lambda 2, the question is, is it again linearly independent? The answer is yes in fact, for each Lambda I, if we can have such an S I consisting of linearly independent vectors of which are Eigen vectors corresponding to Lambda I, then the union will be linearly independent set, let me write it down. So, let T be a linear operator or maybe I should just consider matrices and the same linear operators on V and suppose and Lambda 1 to Lambda k be distinct Eigen values of T.

What is that? I have always carefully written Lambda 1 to Lambda k, k could be strictly less than n that need not be n distinct eigenvalues, if it is having n distinct eigenvalues, then we already know that it is diagonalizable and we do not need to do any of these things. So, in fact, the interesting case comes when case strictly less than n in whatever we are doing right now. So, if suppose, we get hold of S j be a linearly independent set consisting of eigenvectors with eigenvalue Lambda j then S equal to S 1 union S 2 union S k is linearly independent.

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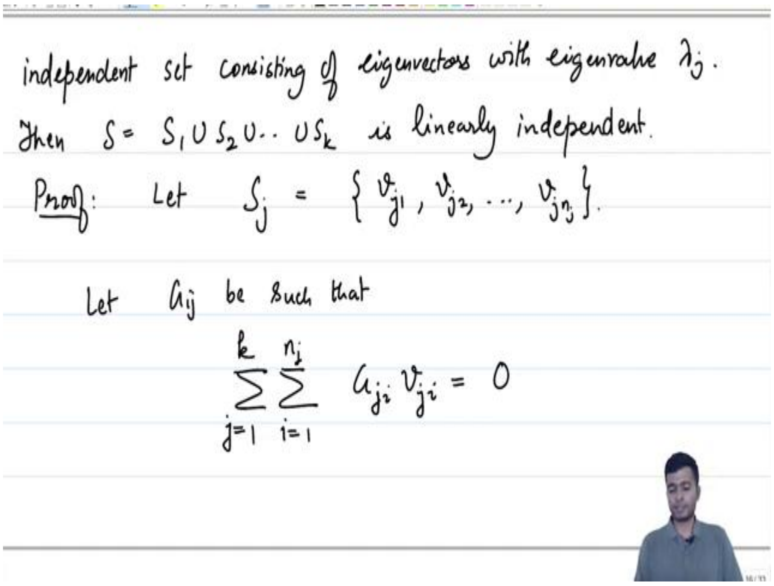

So, let us prove this result so, let each of these S j, just let us give some names to the vectors in each of the S j. So, let S j be given by v j 1, v j 2 up to v j n subscript j, each of the S j consists of such vectors. So, we would like to show that the union is linearly independent. So, let there be a linear combination of these vectors which is equal to the zero vector.

So, let a ij be such that summation okay j might not be a good idea so, let us be a little careful with indices so use, yeah, that is okay. j is going from 1 to k and say i is going from 1 to n subscript j, and suppose a ij v j. I should have just called it S I and it would have been nice but that is okay.

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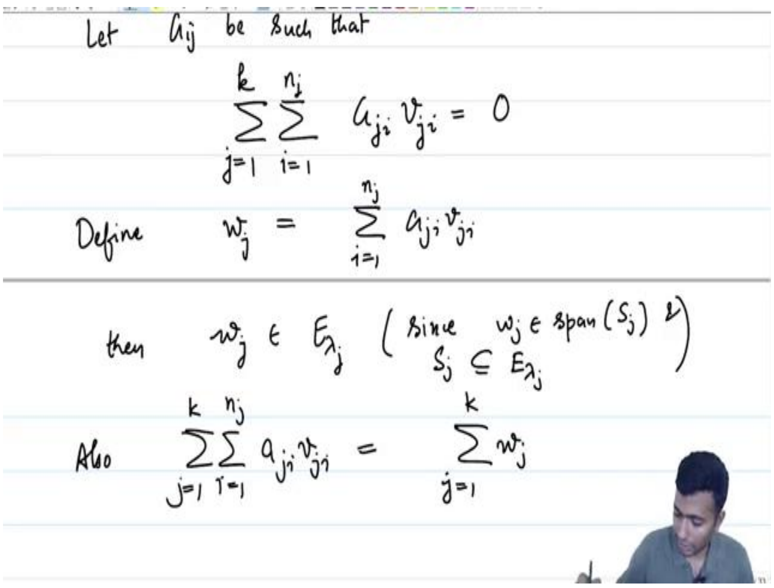

Or let us call it a ji, does not matter, a ji v ji, so i goes from 1 to n and so on. This being equal to the zero vector, so this is a linear combination. A typical linear combination will be like this and we would like to show that each of the coefficients a ji are 0. To do that, what we will do is, we will define w j to be equal to summation i is equal to 1 to n j a ji vji.

So, notice that j is fixed and we are looking at the thing in the bracket here, this is what we are focusing on and we are looking at sum here, what can we say about w j? Each of the v ji belongs to S ji, S j rather this means that w j belongs to span of S j which is contained in E Lambda j.

Why is this? Since S j w j belongs to span of S j, and S j is contained in E Lambda j, because of this each of the W J's belong to E Lambda j. And this expression above also summation j goes from 1 to k, i goes from 1 to n j of a ji v ji is now equal to summation j is equal to 1 to k w j, that is precisely what we have written it as.

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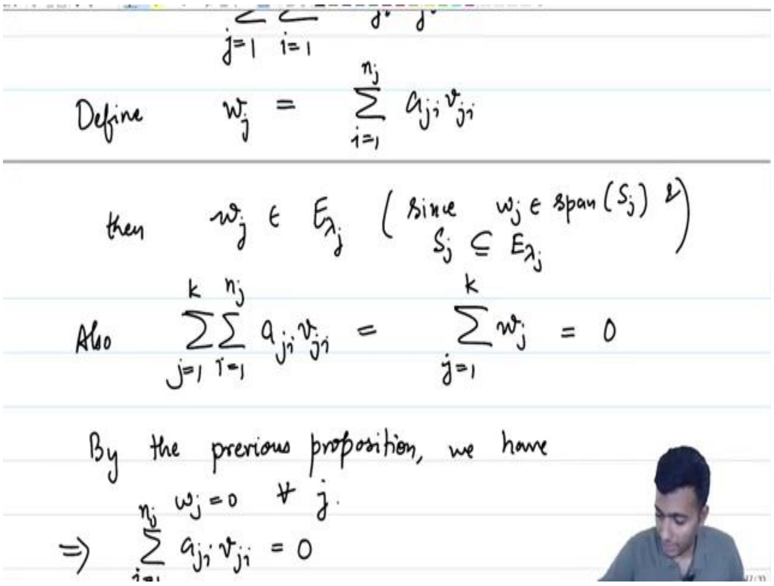

And we are given that this is equal to the 0 vector. But the previous proposition tells us that if we have v 1, v 2 up to v k are Eigen vectors corresponding to or rather elements vectors in the Eigen space corresponding to Lambda 1 up to Lambda k then each of them have to be necessarily 0, by the previous proposition we have w j is equal to 0 for all j, but then what is w j?

Let us get back to what our w j was, w j was the expression which I have put in a box and this implies, so let me write it down this implies summation a ji v ji that i goes from 1 to n j is equal to the 0 vector, but what do we know about S j, we know that S j is a linearly independent set. So, there cannot be a linear combination of vectors in S J which is equal to 0, this implies that so for all j.

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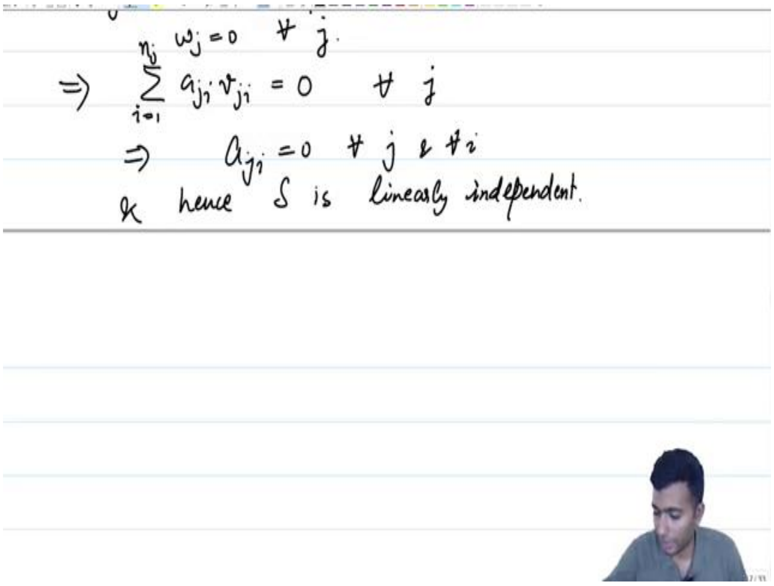

This implies that a j is equal to 0 the scalar 0 for all j and all i, and hence S is linearly independent because we took an arbitrary linear combination to be the 0 vector and we noticed that this forces each of the coefficients to be equal to 0. So, yes this is a linearly independent set.

Now, let us state the main theorem. Main theorem states that if we have a linear operator T and suppose the Eigen values Lambda 1 up to Lambda k, it satisfies the condition that the dimension of the Eigen space. So, suppose the characteristic polynomial of the linear operator splits. So, we are given all this.

So, suppose we are in the situation where the characteristic polynomial of our given linear operators splits, then our linear operator is diagonalizable if and only if the Eigen space, the dimension of the Eigen space of each of the Lambda i or each of the Eigen values is equal to the multiplicity of the Eigen values. So, let me state the theorem.

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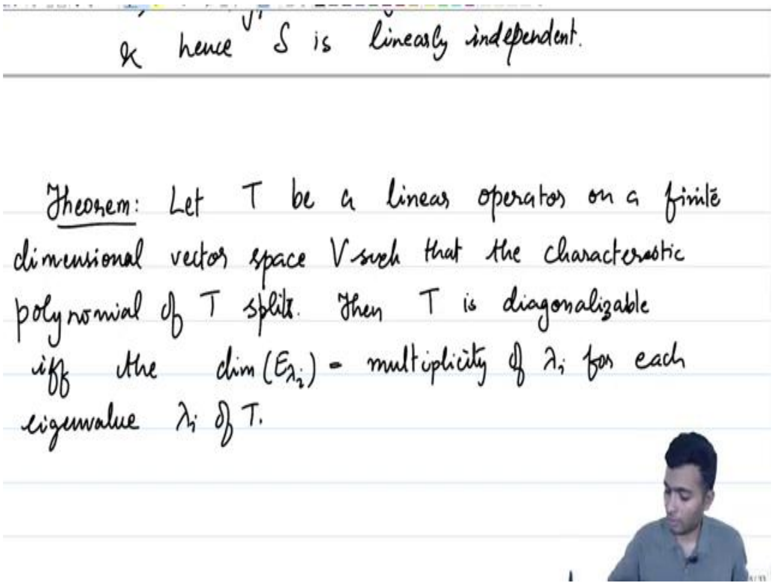

So, let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. So, recall that that was the entire context, we know that if T is diagnosable, the characteristic polynomial splits. We are now studying, given that the characteristic polynomial splits, when can we say that our linear operator T is diagonalizable. So, then T is diagnosable if and only if the multiplicity if the dimension of E Lambda i is equal to the algebraic multiplicity of Lambda i for each Eigen value Lambda I of T.

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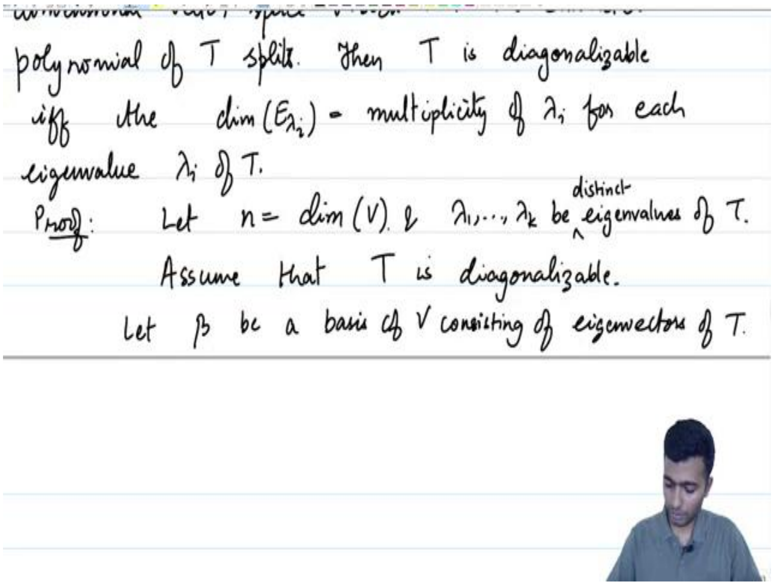

So, notice that it is if and only if statement, it says that if T is diagonalizable, then the dimension is the same, and if the dimensions are equal, then T is diagonalizable. So, let us prove both the directions, so let us first assume that T is diagonalizable. So, we would like to show that the dimension of E Lambda i is equal to the multiplicity for each the eigenvalues Lambda i. So, what does it mean to say that T is diagonalizable? So, let Beta be a basis of V consisting of eigenvectors of T, so let us do one thing.

Let us start the proof one line ahead and let n be equal to the dimension of V, let us call the dimension of V to be equal to m. So, and Lambda 1 to Lambda k be Eigen values of T, so there are k distinct eigenvalues, let me just add the word distinct, there are k distinct Eigen values of T, k is less than or equal to n. Okay that is an exercise for you to show that there cannot be more than n distinct eigenvalues of a linear operator T.

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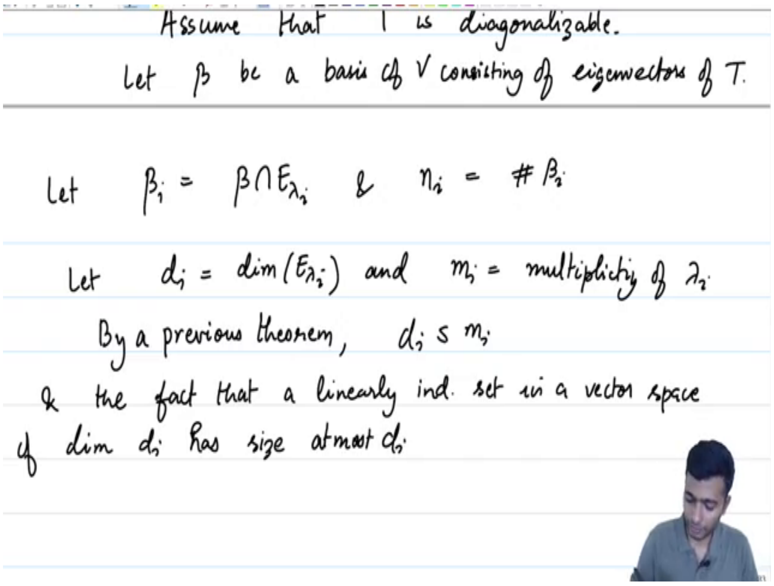

So, we have a basis Beta and let Beta i be equal to Beta intersected with E Lambda i. So, Beta i captures those Eigen vectors in Beta corresponding to Lambda i. So, let and n i be equal to the number of elements in Beta I. Now, n i is the number of linearly independent vectors which are Eigen vectors corresponding to Lambda i. And the first observation is that n i has to be less than or equal to the dimension of E Lambda i. So, let give dimension of E Lambda I some name. So, let d i be equal to the dimension of E Lambda i and m i be equal to multiplicity of Lambda i.

So, we know a few things, we know that d i is by previous theorem or by a previous theorem not the penultimate theorem we have d i is less than or equal to m i. And the fact that Beta i consists of linearly independent Eigen vectors corresponding to Lambda i implies that and the above observation and the fact let me write down the reason, the fact that linearly independent set in a vector space of dimension d i has size less than or at most d i

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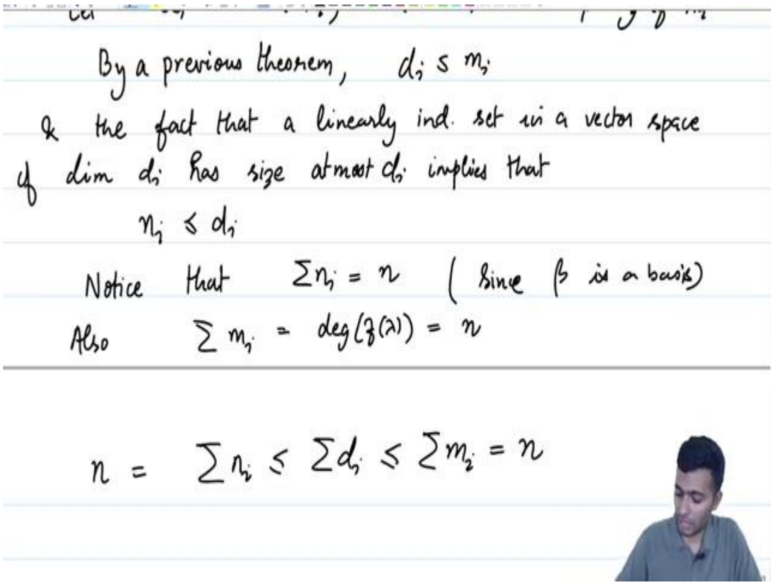

This implies that n i is less than or equal to d i. But we know a few things about n i namely, that summation n i, notice that summation n i is equal to n, because summation n i is the number of vectors in Beta, which is the basis or which is a basis of V and therefore, this is equal to n.

Since, Beta is a basis, also what is summation m i that has to be equal to the degree of f of Lambda, the characteristic polynomial which is equal to n, this is from the explicit form of the characteristic polynomial we have seen in the last week. And what do we hence have, we have that n is equal to summation n i which is less than or equal to summation d i, which is less than or equal to summation m i is again equal to n, so there is a sandwiching that has happened.

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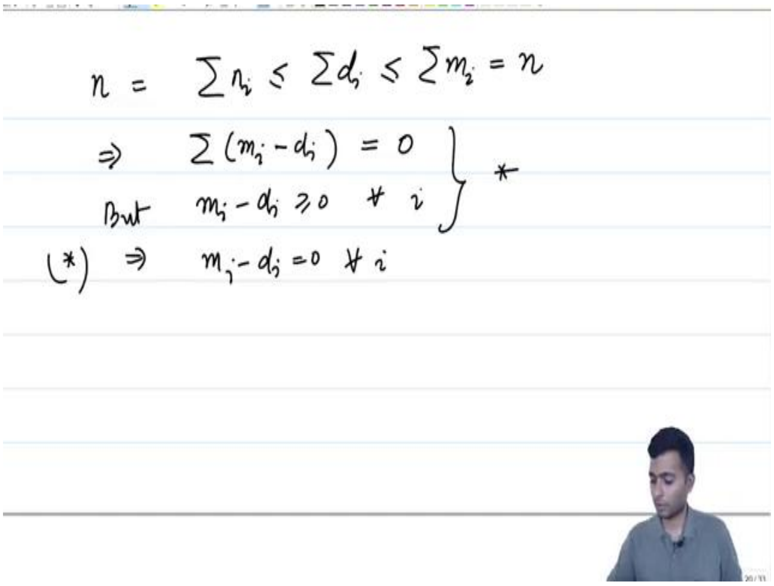

And therefore, summation m i or let me put it this way m i minus d i is equal to 0, but we know that m i minus d i is greater than or equal to 0 by one of the previous theorems. Even if one of them is greater than 0, the sum cannot be equal to 0 because each are non-negative quantities for all I, this implies both star implies m i minus b i is equal to 0 and hence, we have proved one side of the result for all i.

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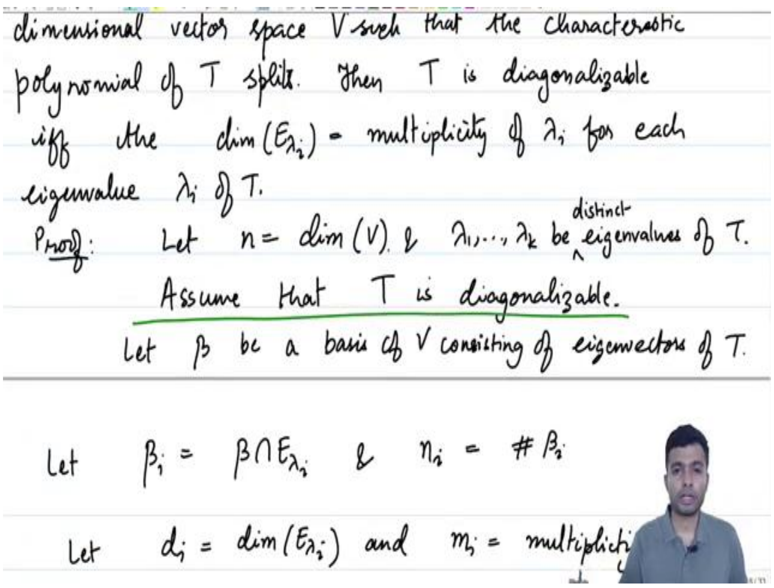

So, what have we proved? We have proved that if we assume that T is diagonalizable then we have shown that the dimension is equal to the multiplicity. Let us now prove that the dimension of E Lambda equal to the multiplicity forces our linear operator to be diagonalizable

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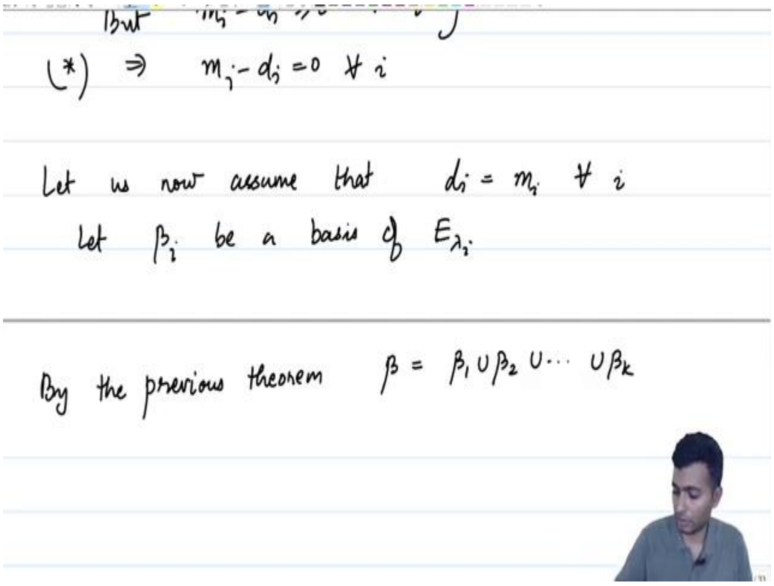

So, let us now assume that m i is or rather di which is less than or equal to m i is in this case equal to m i for all i. So, what does this mean? This means that the dimension of Lambda I, sorry dimension of E Lambda i is equal to the multiplicity that is what it means so, let us do one thing. Let Beta i be a basis of E Lambda i, we know that Beta i has size d i and by the previous theorem, Beta is equal to Beta 1 union Beta 2 union up to Beta k.

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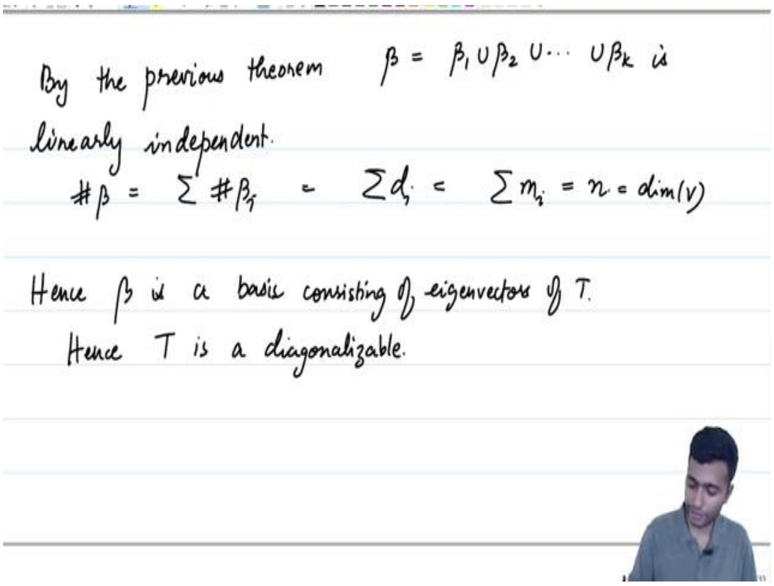

This is linearly independent because they are linearly independent sets in different Eigen spaces and therefore, Beta equal to Beta 1 union Beta 2 up to Beta k are linearly independent. What is the size of Beta? But size of Beta notice that each of the Beta i are mutually disjoint, it is equal to the summation of the size of Beta i, which is equal to the summation of the d i, but this is now equal to the summation of a m i which is equal to n.

Therefore we have a linearly independent set which has size equal to the dimension of V that forces it to be a spanning set and hence a basis, hence Beta is the basis. What is Beta? Beta consists only of Eigen vectors of T therefore, basis consisting of eigenvectors of T, hence T is diagonalizable. So, we have obtained a necessary and sufficient condition on, when T is diagonalizable given that the characteristic polynomial splits.