Linear Algebra
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Lecture 1.3
Vector Subspaces

We have seen that a vector space is a set, which is closed under vector addition and scalar multiplication and such that these operations satisfy a list of properties. We also saw a plenty of examples of vector spaces. Next, let us define what a vector subspace is and let us look at a few examples of vector subspaces as well.

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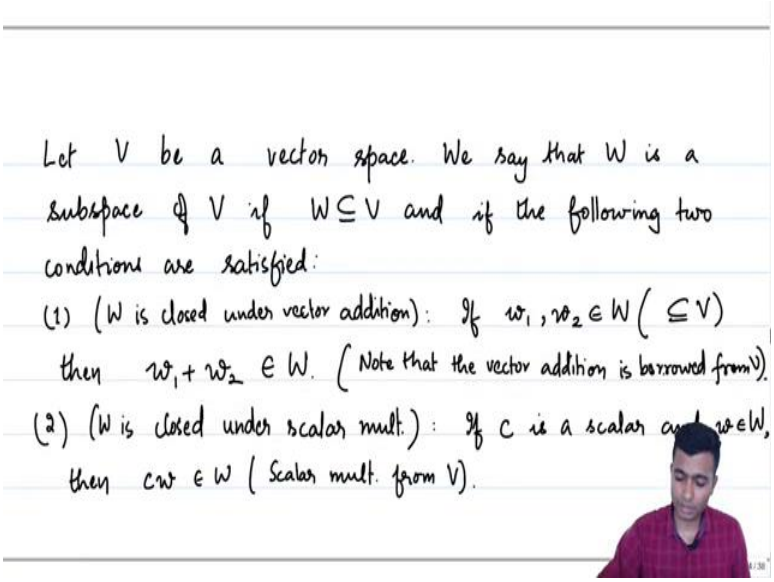

So to begin, let us start with a vector space. We say that set w is a subspace of V, if w is a subset of V to begin with, and if you take any elements any 2 elements in W, look at the sum of the 2 vectors, say w 1 and w 2 are the vectors in capital W, look at w 1 plus w 2. This makes sense because w is a subset of V and V there is a vector addition already defined. So now look at whether w 1 plus w 2 is in our subset W. So, in other words is, check whether w is closed under vector addition, the vector addition which is borrowed from V. And similarly check whether w is closed under the scalar multiplication, which is again borrowed from capital V.

If both a tradition and scalar with respect to vector addition and scalar multiplication, our w is closed, then we say that w is a subspace of V. So let us note it that V be a vector space. We say that w is a subspace of V if w is contained in V and if the following 2 conditions are satisfied as already noted. The first 1 is w is closed under vector addition. If w 1 and w 2 vectors in capital W are elements in capital W as of now w is just some set right, so if w 1 and w 2 are in w remember that this is contained in V, this is a subset of V. Then w 1 and w 2 when considered as a as vectors in capital V, we can add them and then see whether this is in capital W. So, note that the vector addition is borrowed from V.

Not just vector addition, we also demand that w is closed under scalar multiplication just write mult in short for multiplication, and if c is scalar or a real number, and w is an element of capital W, small w is an element of capital W, then c times w is in capital W, just like in the previous case scalar multiplication from capital V. So if these 2 conditions are getting satisfied, then we say that W is a subspace and we say that W is a subspace of V, sometimes it is also traditional to call it a vector subspace of V.

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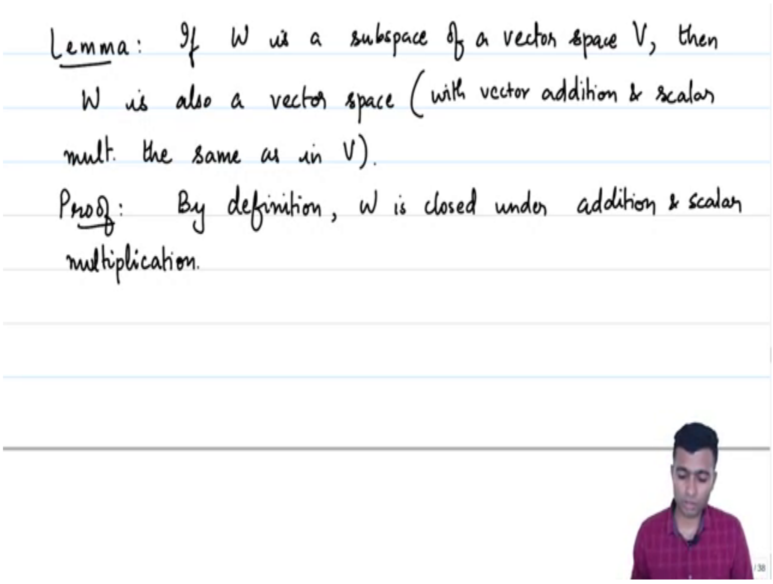

All right, let us look at some of the consequences. First consequence is that if W is a subspace of V then W is itself a vector space with the vector addition and scalar multiplication being the once borrowed from capital W. So let me just state it as a Lemma, if W is a vector subspace, I will slowly drop writing the word vector subspace and just write as subspace, it should be in vector subspace. If w is a subspace of a vector space V then W is also a vector space. I will just leave it, maybe I should not leave it ambiguously with vector addition and scalar multiplication the same as in V.

So, the definition itself checks for the requirement of whether the vector addition and scalar multiplication is closed, or rather whether w is closed under the vector addition and scalar multiplication. So, proof will immediately indicate that by definition, recall what the definition was, you will see that W is closed under not…stop using the word vector addition and just call it addition and scalar multiplication. But we have a long way to go, we need to check for properties 1, 2, 3 up to 8. So let us just revisit the properties.

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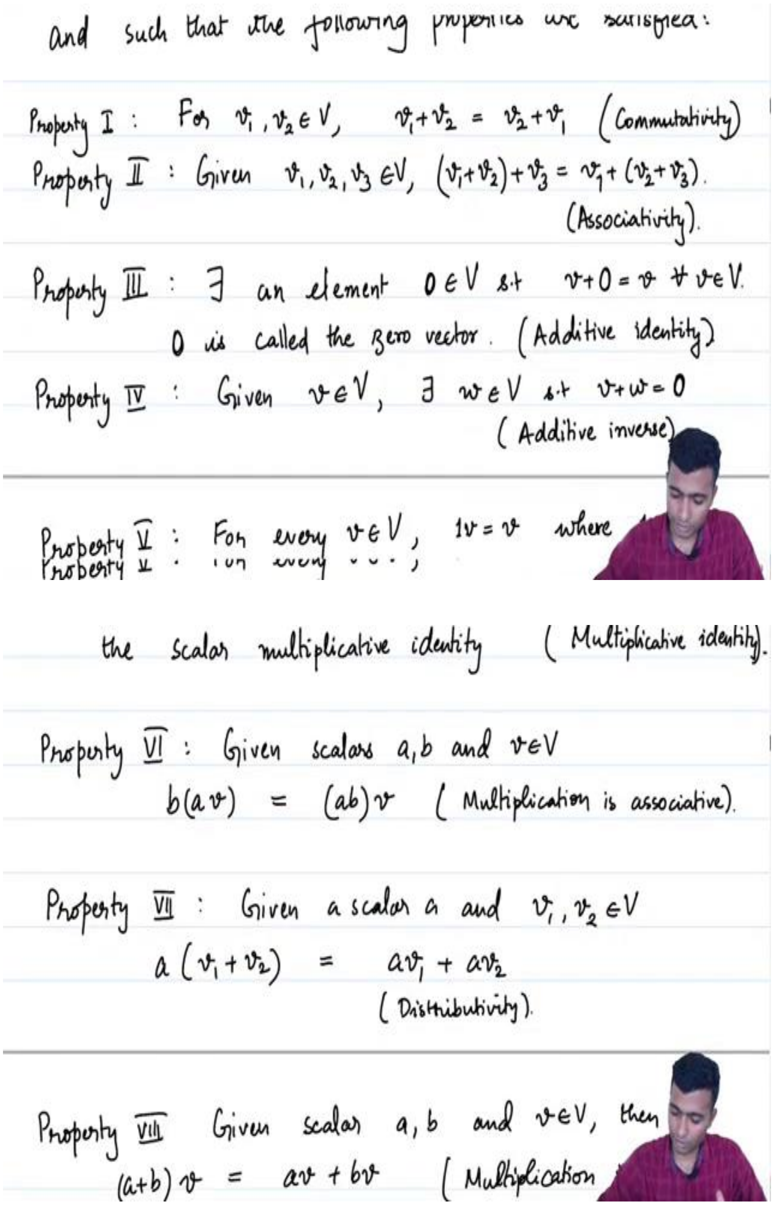

It is written above, let me just point out that property 1, property 2, property 1, property 2, property 5, property 6 and property 7 and even property 8. So, except properties 3 and 4, every other property is automatically satisfied because it is satisfied for V. So let me just mention that, I will let you think about it as well.

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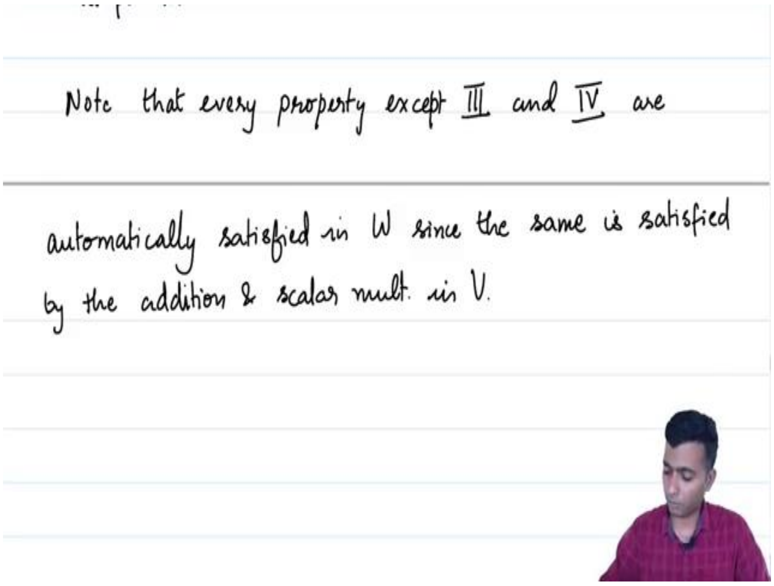

Note that, every every property except when we say 3 and 4 or 4 and 5, we should not make a mistake with that, so let me again notice it carefully property is 3 and 4, every property except 3 and 4 are automatically satisfied in W, since the same is satisfied in V, since the same is satisfied by the addition and scalar multiplication in V. So we just need to worry about properties 3 and property 4, what was property 3? If you go back and check property 3 stated that every the vector space should necessarily have a 0 vector, a vector such that if you add any other vector to it, you get back that same vector. So if V is added to 0 we get back V.

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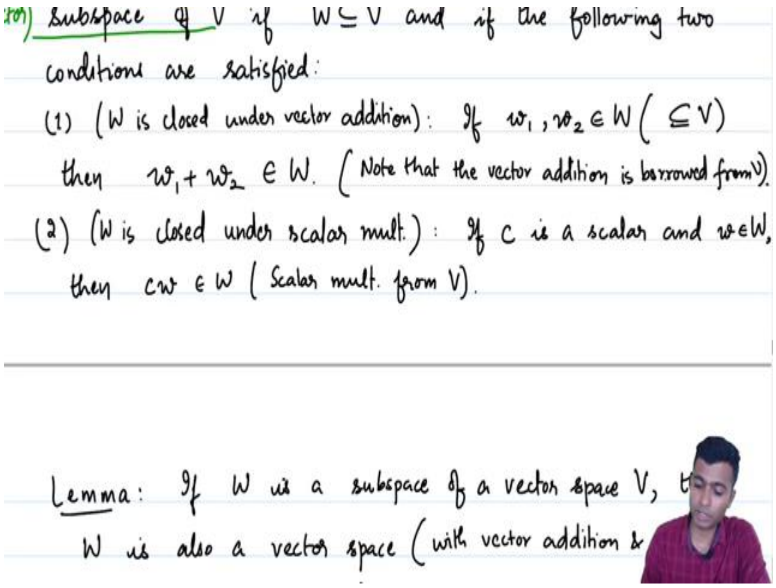

Okay, so we do not have much information about W except the fact that W is closed under vector addition and in this case, very crucially W is closed under scalar multiplication.

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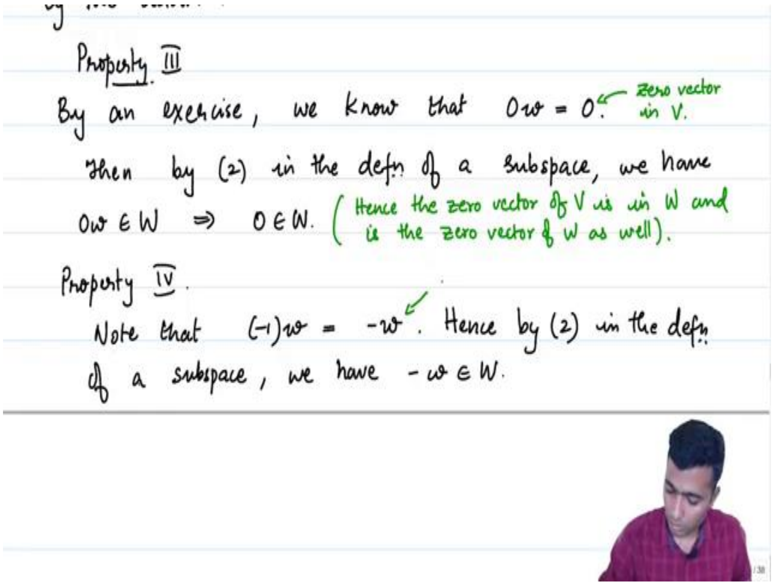

Now by an exercise that we did in the previous session, by an exercise we know that if you consider the scalar 0 and look at the scalar multiplication of 0 with some vector w and capital W, this should end up with the 0 vector. So, if you have not done that exercise, please sit and do the exercise and convince yourself that if you multiply the scalar 0 to a vector, you get the 0 vector. Then if 0 times w is equal to the 0 vector, by property 2, then by property 2 in the definition subspace of a vector subspace, we have 0 times w is in capital W because w is closed under scalar multiplication which implies that 0 belongs to capital W. And therefore, the 0 vector of V, so, let me again use colours to tell that this is the 0 vector in V, and this is also the 0 vector in V.

And what we have essentially shown is that the 0 vector of V is in W, and is the 0 vector of w as well, alright so, that takes care of property 3. How about property 4, property 4 states that given any vector in capital W, that should necessarily be an additive iverse in capital W. Again, let us invoke the same exercise from the previous session again note that, if you have not done this please do it, minus 1 times w is equal to minus of w.

So, this implies that again by 2, by let write it down in words hence by 2 in the definition of subspace, which is that w is closed under scalar multiplication, we have minus of W is an element of capital W. So again, keep this in mind when I say that her this is the additive inverse w of in capital V. No, it is not very surprising that the additive inverse is in capital W as well, because if you remember, one of our other exercises was to say that given any vector, the additive inverse is unique, so we already have an additive inverse in capital V. So it better be in capital W as well, right. Otherwise there is a potential problem of having more than 1 additive inverses in capital V because the 0 is the same, right.

Okay, so we have essentially proved almost all properties of properties in the definition of a vector space. And therefore, we have established that capital W is a vector space as well. Let us now look at many examples, examples of subspaces.

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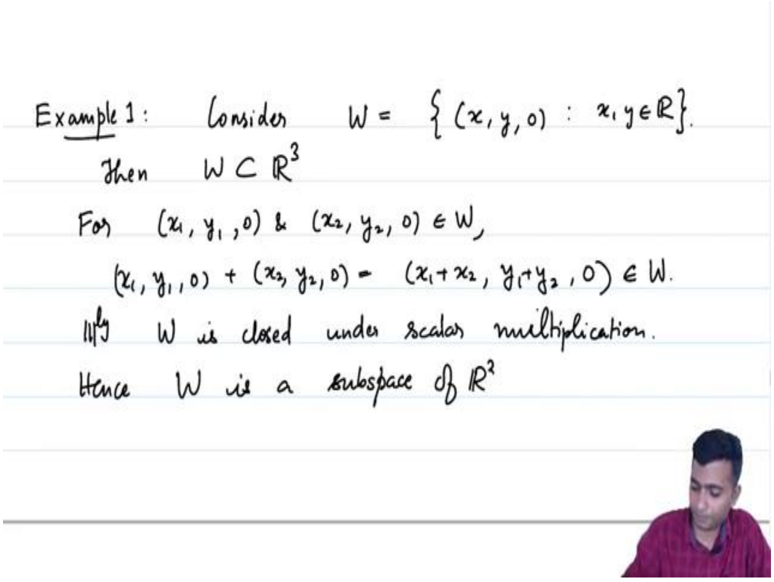

So, the first example so let me write it as example 1. So consider w to be equal to the set of all x, y, 0 such that x and y are in capital R. Then notice that w is contained in R 3, x, y, z, our coordinate is 0, that is what w is capturing is the set of all those points in R 3 whose z coordinate is 0. Then notice for x 1, y 1, 0, and x 2, y 2, 0, in capital W, by borrowing the vector addition from R 3, x 1, y 1, 0 plus x 2, y 2, 0 is equal to x 1 plus x 2, y 1 plus y 2, comma 0, which again has the z coordinate 0 which belongs to capital W. Therefore, our w is closed under the vector addition which is borrowed from R 3. Similarly, let me leave it as an exercise to check that w is closed under scalar multiplication hence, w is a subspace of R 3. Okay.

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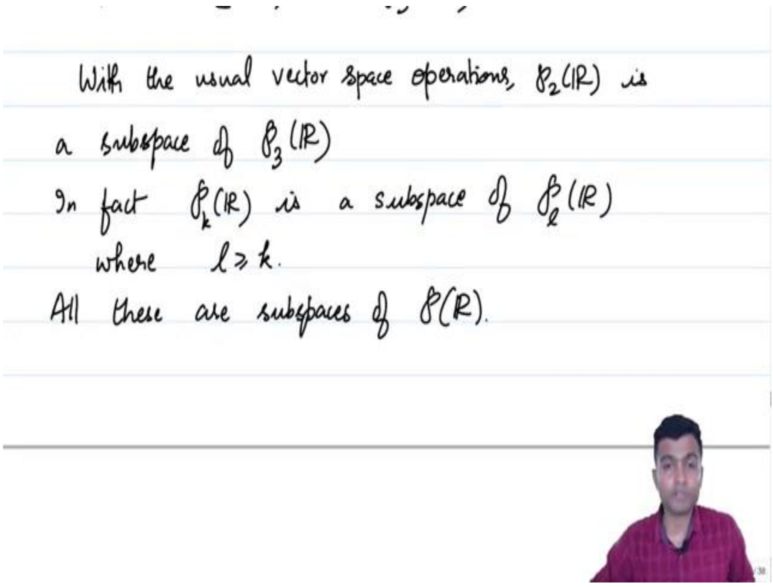

Let us look at another example. So, if you look at P 2 of R then it is a subset of P 3 of R, right any degree to any polynomial of degree less than or equal to 2 naturally be of degree less than or equal to 3. And if you remember, if you recall how the addition operation was defined in P n of R, it is just the usual polynomial addition and scalar multiplication. And therefore I will allow you to check that with the usual vector space operations P 2 of R is a subspace of P 3 of R. In fact, let me write down a chain of subspaces P k of R is a subspace in fact, is a subspace of P L of R where L is greater than or equal to K. All these subspaces of P of R, P 2 of R turns out to be just a subspace of P 3 of R. Right?

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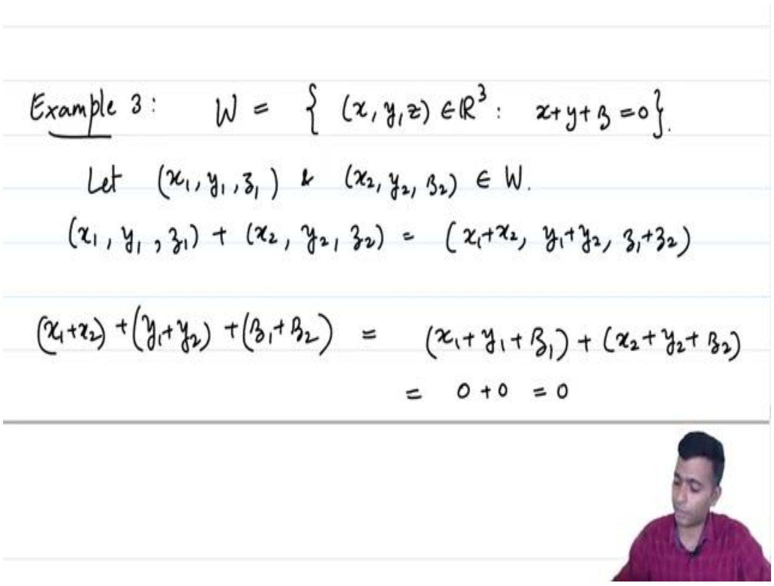

Oh, an important thing to note is that we cannot realize R 2 as a subspace of R 3. R 2 is not even a subset, is not a subset of R 3. It is not hence a subspace of R 3, we might be able to find vector subspaces of R 3 which are identical to R 2 as we have seen here. This is something which is identical to R 2, but it is certainly not equal to R 2. This is an element in R 3, which has three coordinates R 2 is the Cartesian product of R with itself twice just twice, not a subspace not a vector subspace. Okay, let us look at more examples. Your numbers example 3. So, example 3, I would like to consider the set of all x, y, z, again in R 3 such that x plus y plus z is equal to 0. So for example, 1, minus 1 is 0 is an element of W, or 1, 0, minus 1 is an element of W. 2, 3, minus 5 is an element of W and so on.

I would like to claim that W is a subspace of R 3. So we just have to check that W is closed under vector addition, which is borrowed from R 3 and scalar multiplication, which is also borrowed from R 2. So in order to do that let, x 1, y 1, z 1, and x 2, y 2, z 2 be in capital W. Then let us look at x 1 plus y 1 plus z 1 plus x 2 plus y 2 plus oh, this is what I would like to finally have, I am sorry, but what is x 1, y 1, z 1 plus x 2, y 2, z 2, this is equal to x 1 plus x 2, y 1 plus y 2, z 1 plus z 2? Now the question is does this belong to capital W, we will check whether this belongs to capital W by checking for whether the coordinates are up to 0.

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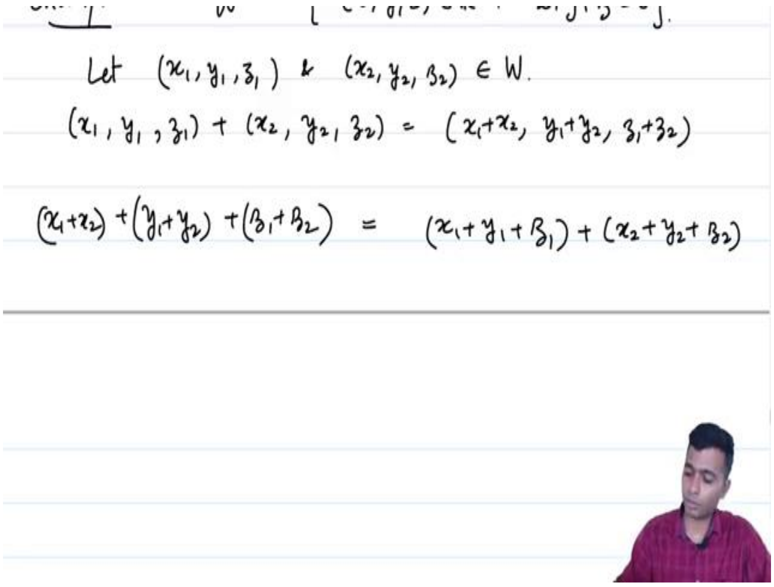

So, consider then x 1 plus x 2 plus, y 1 plus y 2 plus z 1 1 plus z 2, this is the sum of the coordinates, but these are all real numbers which are being added up and we know that the order the addition is commutative and also the order in which we add up these numbers do not matter right. So this I will leave you to check that this is nothing but after all the commutating of elements and re-bracketing terminologies so re-bracketing the variables we get this is equal to x 2 plus y 2 plus z 2, I am sorry, these are not variables, the coordinates. But what do we know about x 1, y 1, z 1; x 1 y 1, z 1 belongs to capital W. And any vector which belongs to capital W should have its coordinates, summing up to 0 that is how our w is being described right? And therefore this is equal to the scalar 0 and the next 1 is also equal to the scalar 0 which is equal to 0. And we are done and we have shown that w is closed under scalar multiplication.

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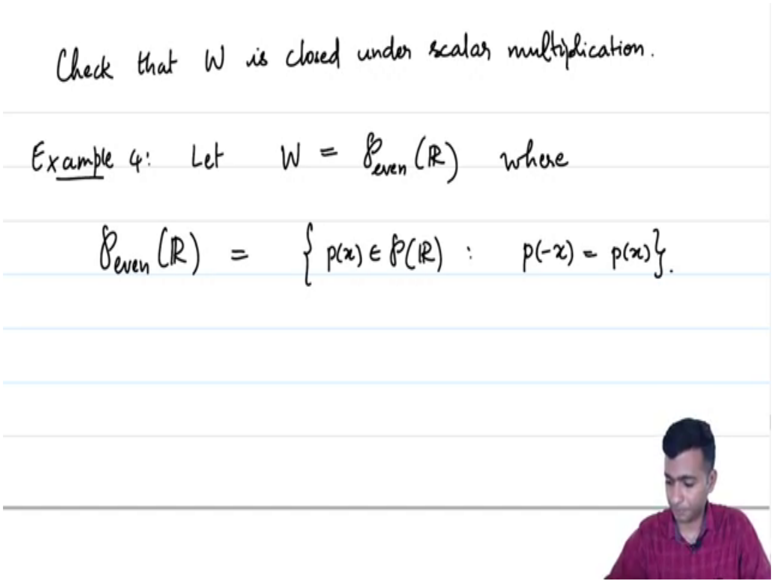

It is quite similar, if you want to show that the, oh we have just shown that the vector addition is closed, it is quite similar to show that the scalar multiplication is also closed. Check that I do not even want to call it an exercise check that scalar multiplication is also closed, oh scalar multiplication cannot be closed, the subspace w is closed, w is closed under scalar multiplication, so it is important to write it down precisely.

Next example, example 4 consider the subspace w so, let w be equal to P even of R where P even of R is the even polynomials, the set of all polynomials in P of R, which satisfies the following property, P have minus x is also equal to p of x then I would like to claim that w is a vector space sorry vector subspace, of course, it is a vector space as well, more precisely, it is a vector subspace of P of R. So then, let me just write it down as a claim formal claim.

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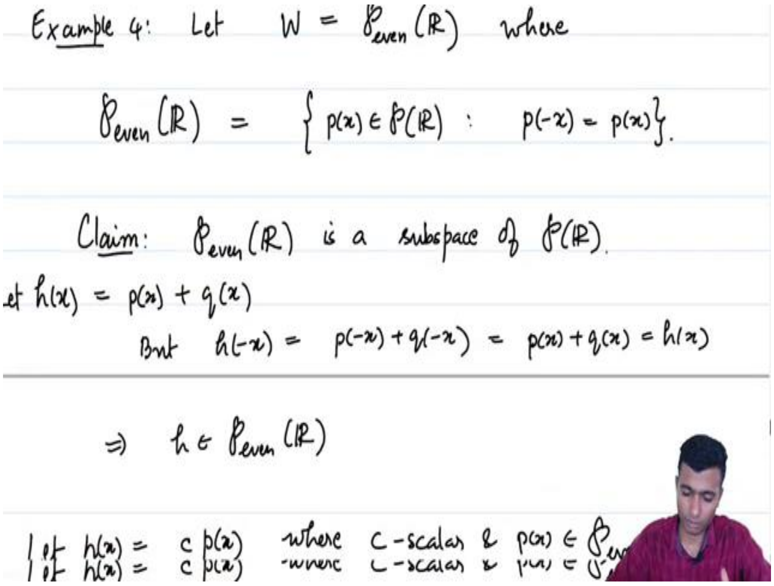

P even of R is a vector subspace of P of R. So let us take 2 such vectors and try to see whether this belongs to p even of R. So let R of x or let me call it h of x be equal to p of x plus Q of x. We want h of x to be in P even of R that will happen when h of minus x is equal to h of x, but h of minus x is equal to P of minus x plus q of minus x, but both p and q belong to P even of R, which is equal to hence p of x plus q of x. But what is this, this is exactly equal to h of x and this implies that h also belongs to P even of R.

And how about scalar multiplication that is P even of R closed under scalar multiplication. Let C be some scalar, let h of x be equal to c times some polynomial p of x, where c is a scalar and p of x is in P even of R. Let us check whether h is now in P even of R so to do that what is h of minus x, this is nothing but c times P of minus x, but c is in P even of R. So this is equal to c times p of x, by the very definition of P even of R, which is equal to nothing but h of x, this implies that h is in P even of R. So we are done, this proves that we have established the claim, hence the claim is established because that is precisely what the definition of subspace says that it should be closed, it should be a subspace of P of R which is true and that it is closed under the vector addition and scalar multiplication.

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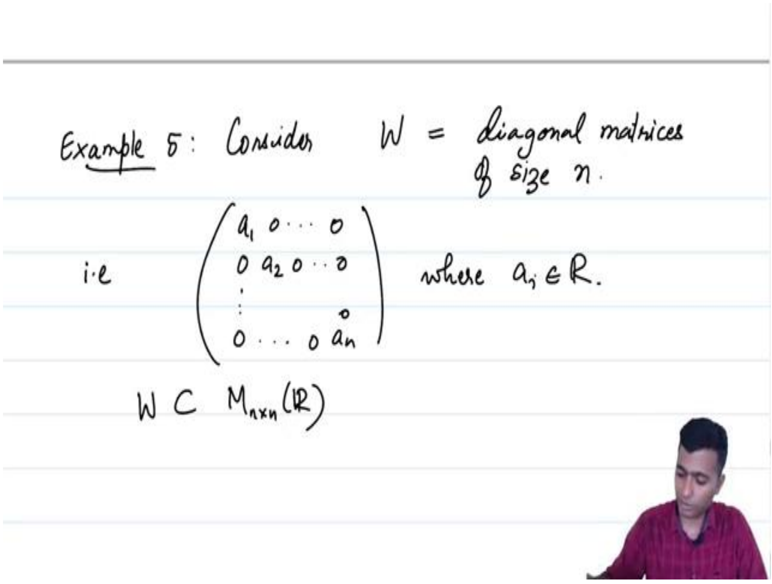

So let us look at 1 more example, so consider all the diagonal matrices, w to be all the diagonal matrices of size n, so what is W, W will be ie matrices of the type a 1 000 0 a2 0 00 0 dot dot dot 0, a n right, this is exactly what the diagonal matrices are. So, where is the sitting inside, notice that w is sitting inside M m cross n of R. Sometimes M m cross n of R is written as m n of R, so m cross n is just abbreviated to just m if m is equal to n. Right. So I will leave it as an exercise for you to check that W is a subspace, maybe I should call it something else, D for diagonal matrices , it does not matter. W is a subspace of M m cross n of R so diagonal matrices are subspaces of M n of R. So in particular, if you just consider diagonal matrices of size n that is a vector space.

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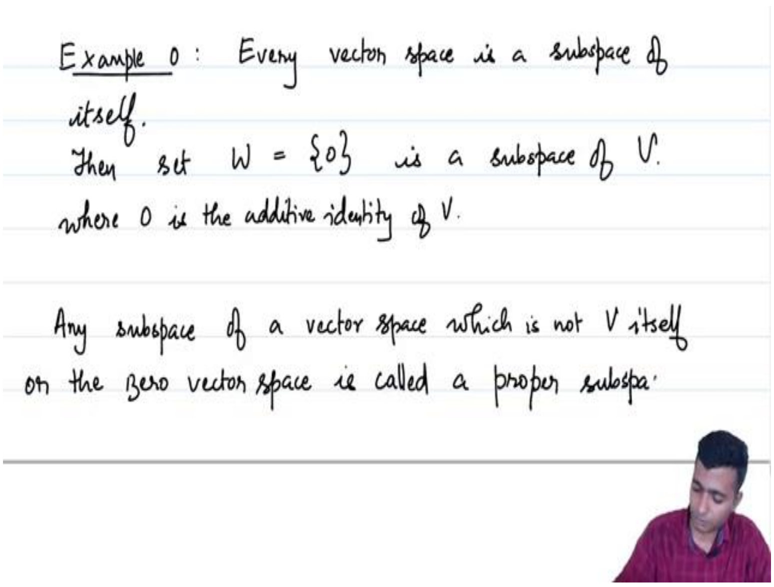

Actually, we should be looking at the most, the first example should have been the following. So this may be should be example 0 which I am putting at the end. Every vector space is a subspace of itself, so it quite straightforward because V itself is a subset of V for sure and it is certainly closed under vector addition and scalar multiplication. So it satisfies all the conditions for a subset being something being a vector subspace. The set W, the 0 vector space is a subspace of V, where the 0 here is the 0 vector in the additive identity in capital V, where 0 is the additive identity.

So the second one needs a capital observation because the 0 is being added to itself here and because here is the additive identity, you get back 0 and hence this is the 0 vector space we had defined. But yes, it is also realized as the subset of W here and yes, it is a it is a vector subspace of capital V. So these are the first examples of subspaces I would have liked to give but that is okay. So any subspace of vector space which is not the 0 vector space or the entire vector space is called a proper space. Any subspace of a vector space V which is not v itself or the 0 vector space is called a proper subspace.

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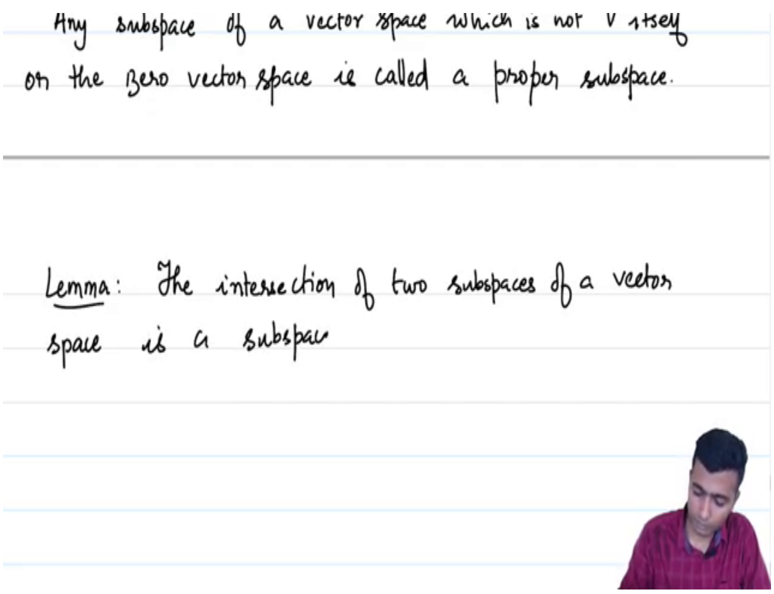

Okay, let me conclude this video by noting that let me just write it as a Lemma, the intersection of 2 subspaces will again be a subspace, 2 subspaces of a vector space is again a subspace.

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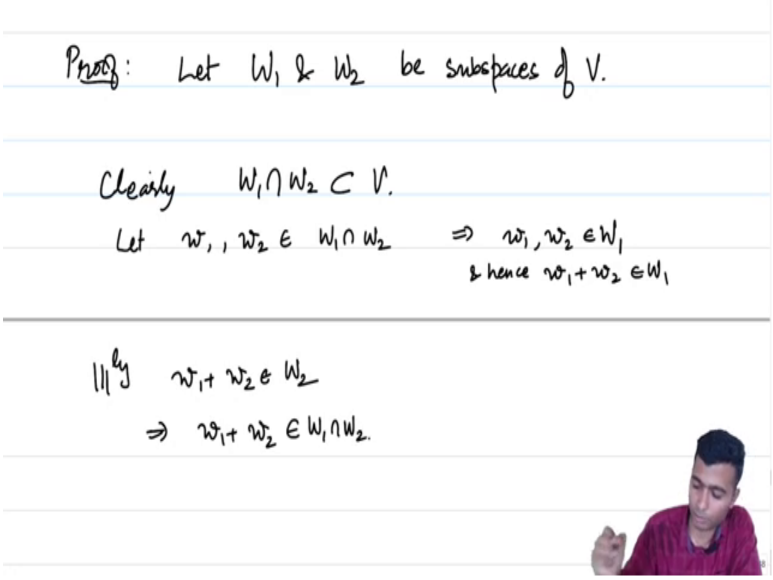

Let me just give a quick proof of this, let W 1 and W 2 be subspaces of V, or consider the intersection W 1 and W 1 intersected with W 2. If you consider W 1 intersected with W 2 then clearly, W 1 intersected with W 2 is a subspace of V after all W 1 is a subset of V, sorry it is a subset of V, W 1 is a subset of V, W 2 is a subset of V, therefore, the intersection is also a subset of V, we only need to check whether it is closed. So, let W 1,W 2 be in W 1 intersected with W 2, but both W 1 and W 2 are subspaces. This gives, W 1, W 2 is in capital W 1 and hence, W 1 plus W 2 is in capital W 1. Similarly, W 1 plus W 2 is also in capital W 2, which implies W 1 plus 2 is in capital W 1 intersected with capital W 2.

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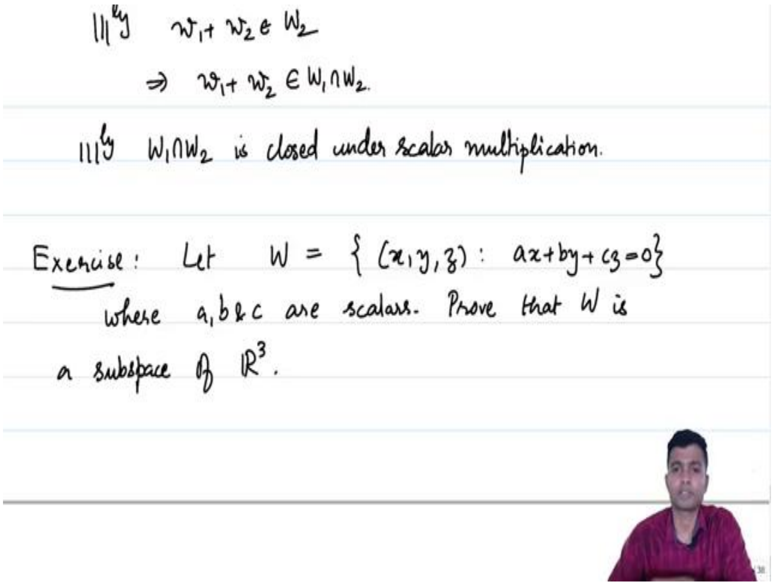

It is very similar to check that W 1 intersected with W 2 is closed under scalar multiplication. So, now we have plenty of examples of vector spaces and after developing the notion of a vector subspace, we now have a family of, so I will just conclude with an exercise for you which will give you plenty more examples of subspaces. So let W be the set of all x, y, z such that a x plus b y plus c z is equal to 0 where a, b, and c are scalars. Prove that W is a subspace of R 3. And if you prove this exercise for every scalar every collection of scalars a, b, c, look at W corresponding to the equation a x plus b y plus c z is equal to 0 that is going to be a subspace. We have many many examples of subspaces, all these are vector spaces. So, yes now we have plenty of examples of vector spaces. So in the next video, we will go forward by developing the notion of what is called as a linear combination.