Linear Algebra
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Lecture 8.1
Characteristic Polynomial

In the last week, we introduced the notion of eigenvalues and eigenvectors of a linear operator on a vector space V. We also defined the analogous notions for an N cross N matrix. We thereafter defined what is the characteristic polynomial of an N cross N matrix A and we observed that the roots of the characteristic polynomial turned out to be precisely the eigenvalues of the matrix A. We also use this to study the particular example of the Fibonacci rabbits. In this week, we extend further the study of eigenvalues and eigenvectors, we begin by recalling the definition of a characteristic polynomial.

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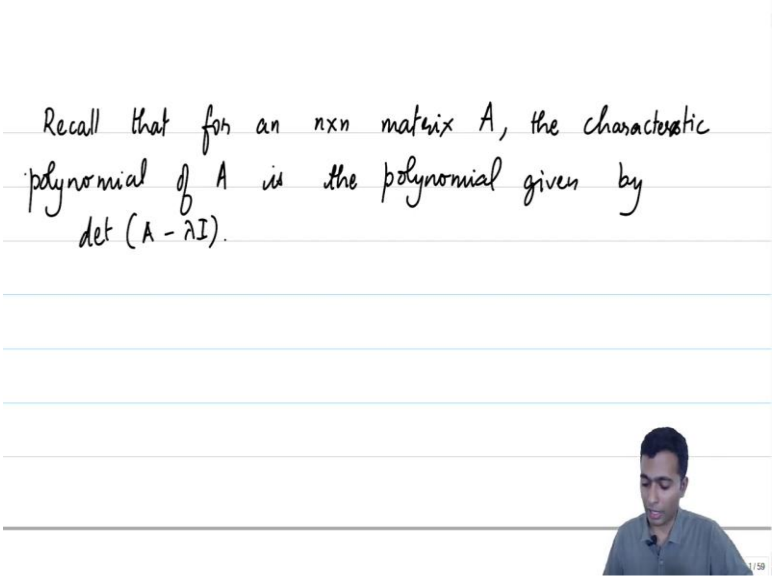

So, recall that for an N cross N matrix A, the characteristic polynomial of A is polynomial given by determinant of A minus lambda I. So, notice that this is a polynomial in lambda, so yeah, we will come to that in A minute.

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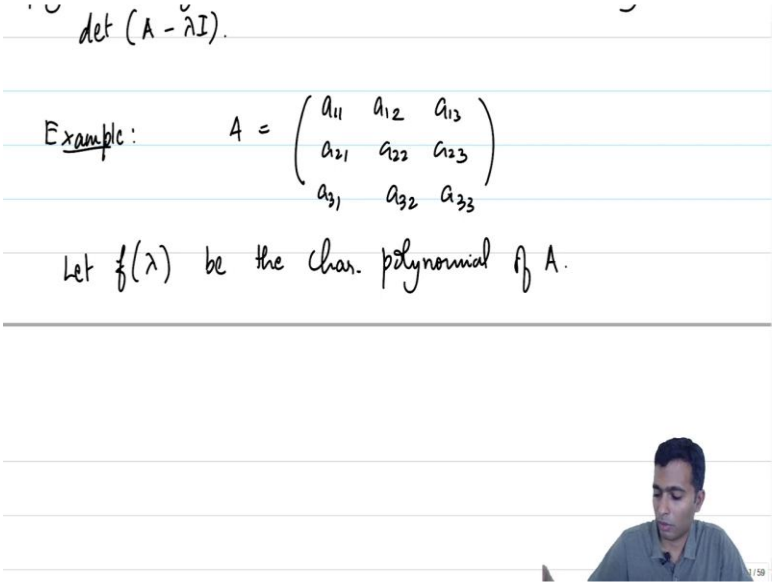

Let us look at an example, what is it, if it is some arbitrary 3 cross 3 matrix A 1 1, A 1 2, A 1 3, A 2 1, A 2 2, A 2 3, A 3 1, A 3 2, A 3 3. Suppose, some arbitrary 3 cross 3 matrix is given, then F of lambda, so let me just write it like this. So, let F of lambda be the characteristic polynomial of A, let us short, in short, let me write it as c h A r, polynomial of A.

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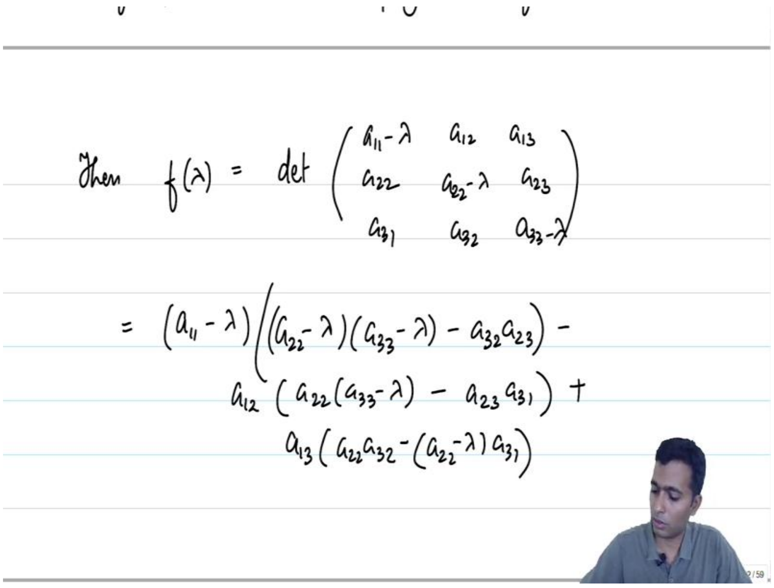

What do we know about A in of F, what do we know about F? Then F of lambda is just the determinant of this matrix A 1 1 minus lambda A 1 2, A 1 3, A 2 2, A 2 2 minus lambda A 2 3, A 3 1, A 3 2, A 3 3 minus lambda. So, if I am to write down the expression for the determinant using the cofactor expansion, this is, this will just turn out to be A 1 1 minus lambda into the first cofactors A 2 2, will be like this.

Maybe I rub this and write it again. A 2 2 minus lambda into A 3 3 minus lambda minus A 3 2 into A 2 3 minus A 1 2 times A 2 2 into A 3 3 minus lambda minus A 2 3, A 3 1 plus A 1 3 times A 2 2, A 3 2, minus A 2 2, minus lambda into A 3 1. So, this is what the expression for the characteristic polynomial A will turn out to be.

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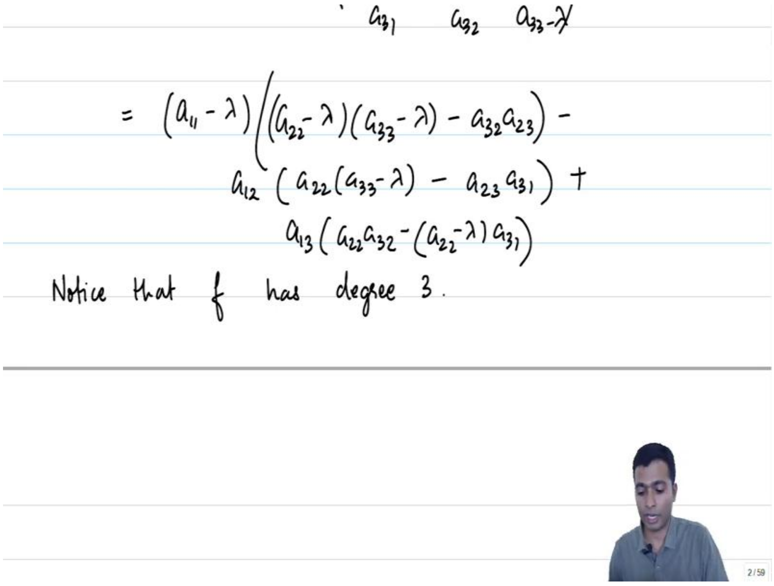

If you notice carefully I will not expand it out any further, if you notice carefully, the polynomial of this particular polynomial is a polynomial of degree 3. Notice that F has degree 3, so observe that I am now using lambda instead of X for the indeterminate, which is just the same no confusion should come up because of that.

All right, so, maybe 1 more example would be to. So, so, as you can see, the characteristic polynomial has a tendency to be quite messy. For example, if you look at higher or rather N greater than 3, the corresponding expression will be far more complex and, it has a tendency to be messy. However, in the particular case, when our polynomial, sorry when a matrix is a diagonal matrix, the characteristic polynomial turns out to be particularly nice.

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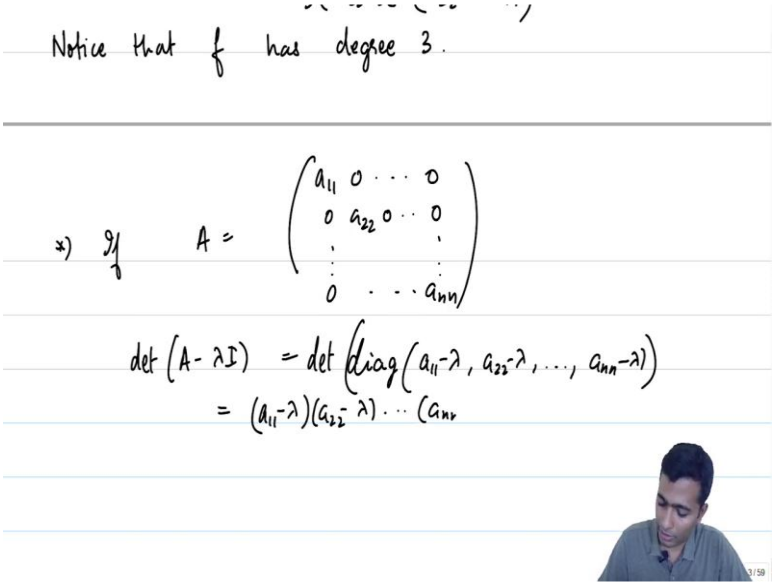

So, if so, this is maybe another example, if capital A is something like say, A 1 1, 0, 0, A 2 2, 0, 0 to A n n, suppose this is our basically diag of A 1 1, A 2 2, up to A n n, then what will be the determinant of A minus lambda I. Let me write it in the compact form, this is going to be determinant of the diagonal matrix consisting of A 1 1 minus lambda A 2 2 minus lambda A n n minus lambda, this is just going to be the product of these entries which will be A 1 1 minus lambda into A 2 2 minus lambda A n n minus lambda. So, yes, so diagonal matrix we can very easily compute the characteristic polynomial. Now, let us explore some more properties of the characteristic polynomial.

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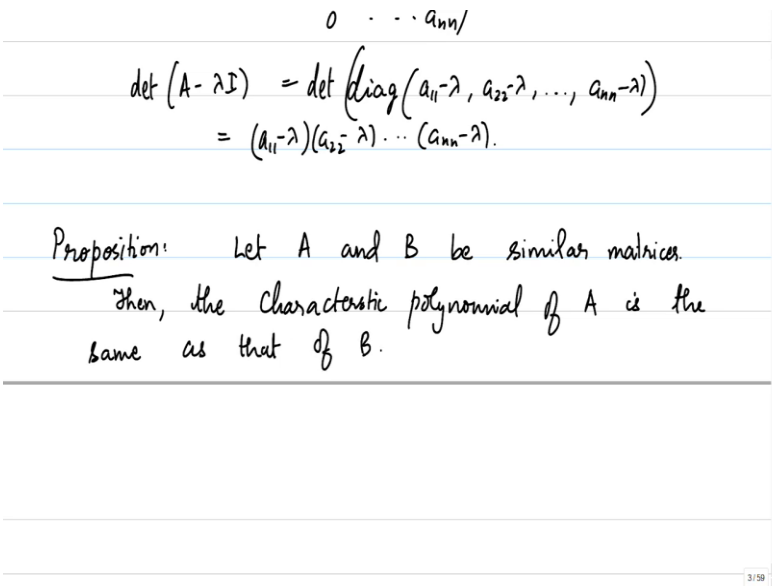

So, the characteristic polynomial has this particularly nice property that it is invariant under similarity. So, let me write it down as a proposition and elaborate on it. So, let A and B be similar matrices. So, recall that A and B are similar, if there exists a polynomial Q such that B is Q A, Q inverse, so let them be similar matrices.

Then the characteristic polynomial of A is the same as the characteristic polynomial of B. The characteristic polynomial of A is the same as that of B. So, let us give a proof of this and then discuss the implications of the proposition. So, notice once more, the proposition tells us that the characteristic polynomial is invariant under the property of similarity, in other words, if you have 2 similar matrices, the characteristic polynomial is the same.

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Proposition: Let A and B be similar matrices. Then, the characteristic polynomial of A is the same as that of B.

Proof: Let $f(\lambda) = \det(A - \lambda I)$

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B is similar to A $\Rightarrow \exists\, Q$ s.t.

$$B = QAQ^{-1}$$

Then

$$\det(B - \lambda I) = \det(QAQ^{-1} - \lambda I)$$
$$= \det(QAQ^{-1} - (\lambda I)QQ^{-1})$$
$$= \det(QAQ^{-1} - Q(\lambda I)Q^{-1})$$

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$$= \det(QAQ^{-1} - (\lambda I)QQ^{-1})$$

$$= \det(QAQ^{-1} - Q(\lambda I)Q^{-1})$$

$$= \det(Q(A-\lambda I)Q^{-1})$$

$$= \det(Q)\det(A-\lambda I)\det(Q^{-1})$$

$$= \det(Q)\det(Q^{-1})\det(A-\lambda I)$$

$$= \det(QQ^{-1}(A-\lambda I))$$

$$= \det(A-\lambda I)$$

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So, look at a proof of this. So, what is the characteristic polynomial of A, the characteristic. Let, F of lambda be the characteristic polynomial of A, this is equal to determinant of A minus lambda I, this is what the characteristic polynomial of A is. Let us observe what it means for B to B similar. B is similar to A, this implies that there exists some Q, such that B is equal to Q A Q inverse, some invertible matrix Q such that B is equal to Q A Q inverse. And what is the characteristic polynomial of B? It is the determinant of B minus lambda I.

So, let us explore what is this. Then determinant of B minus lambda I in particular is the determinant of the matrix Q A Q inverse minus lambda I. But notice that lambda I is just a scalar multiple of the identity in particular you should sit down and check that it commutes with every matrix. So, this is nothing but determinant of Q A Q inverse minus lambda I into Q Q inverse, Q Q inverse is just the identity and because it commutes this will just turn out to be Q A Q inverse minus Q lambda I Q inverse.

That is good, because now we can write this as determinant of Q times A minus lambda I times Q inverse. And what do we now know about the determinant? Determinant is that special function, which when you apply to a product of matrices will be the product of the determinant. So, this is determinant of Q, determinant of A minus lambda I, determinant of Q inverse. But then these are real numbers, this is just determinant of Q times determinant of Q inverse times the determinant of A minus lambda I.

And again, this is just now, this is just the product of all these terms, this is the determinant of Q Q inverse times A minus lambda I, which is the same as determinant of A minus lambda I. So, one question you might be wondering about is, why did we not write this is equal to this directly, because we do not know how A minus lambda I looks like, based on the complexity of how A is, A minus lambda I need not commute with Q or Q inverse. So, we can we cannot directly write the green, sorry red equality which has been just pointed out.

So, we have go through this roundabout way to finally get to this and this is the final thing that we wrote is nothing but F of lambda, which is the determinant of A minus lambda I, which is the determinant, which is the characteristic polynomial of A.

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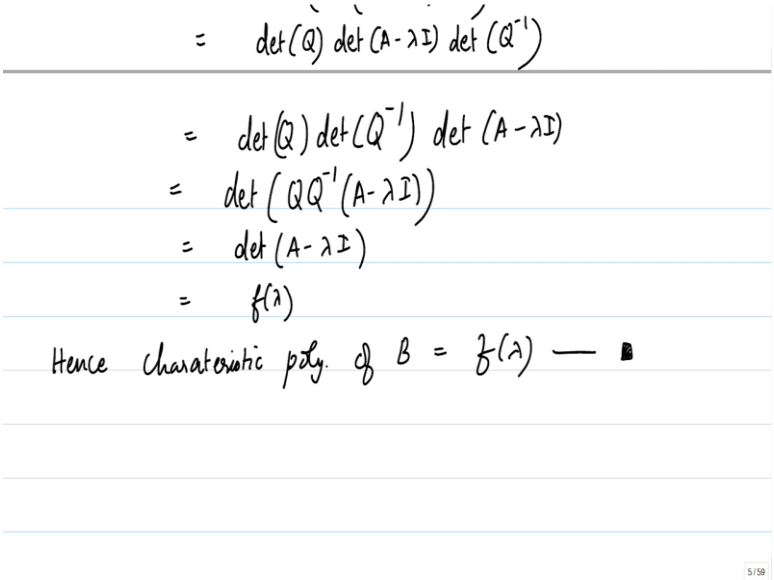

So, yes, so the characteristic polynomial, hence characteristic polynomial of B is equal to F of lambda, which is the characteristic polynomial of A and we have proved the result.

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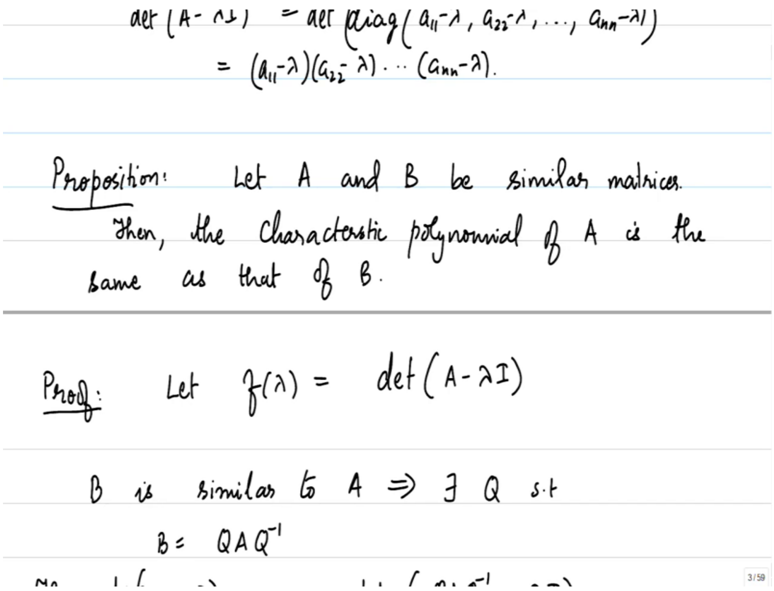

So, let us go back to the statement once more and try to see the implications of this. What this tell is, is that if two different matrix are similar, the characteristic polynomial is the same. Now, if you take a linear operator T, on a vector space V and you look at basis beta, then the matrix of T with respect to beta, so T beta, beta, this is an N cross N matrix, let us call it A and let us pick another basis beta prime.

And you look at T beta prime beta prime, let us call it B from one of the theorems, which we proved in the last week. We know that A and B are similar there exists some matrix Q such that B is equal to Q A Q inverse. And what this proposition tells us is that the characteristic polynomial of A and B is equal is the same. So, in other words, this enables us to talk about the characteristic polynomial of the inner operator T.

So, for all practical purposes, studying characteristic polynomials of N cross N matrices, we uses all the information and necessary tools to talk about the characteristic polynomial of a linear operator which is now well-defined. Alright, let me not go into that and spend any more time, let us continue with the analysis and study of the characteristic polynomial of a given matrix. So, before we jump into any more properties, let us notice what we are doing, when we are computing the characteristic polynomial.

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$$f(\lambda) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix}$$

So, notice that F of lambda is just the determinant of A 1 1 minus lambda A 1 2 up to A 1 n, A n 1, A n 2, A n n minus lambda. So, this you can think of now as the determinant of a matrix, which has polynomial entries, polynomials in lambda indeterminate being lambda. So, the next proposition tells us that if we look at the determinant of such a polynomial with entries in P 1 of R. So, notice all of these A 1 1 minus lambda up to A n n minus lambda, all these are polynomial entries with a degree less than or equal to 1, so the constants have degree 0 and the non-constant polynomials have degree 1. So, this is a polynomial with increase in P 1 of R.

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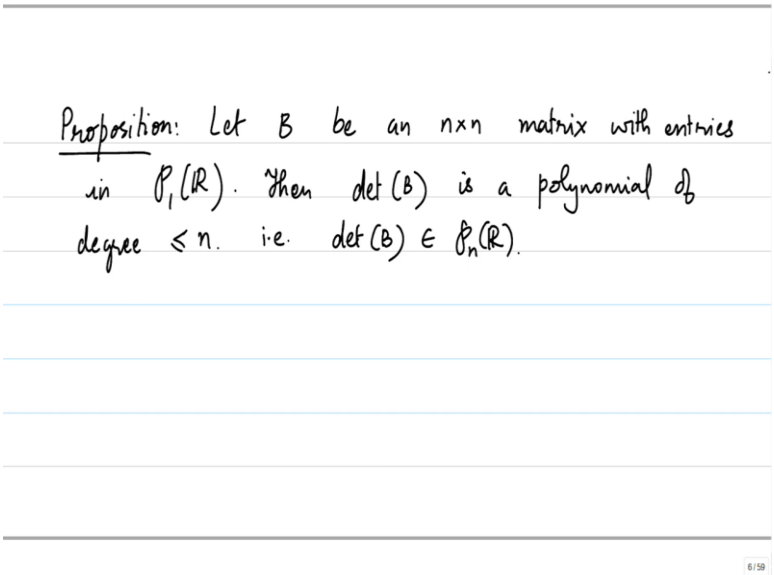

The next proposition tells us that the determinant of such a matrix we will be able to say something about the degree as well. So, let me write down the proposition and then tell you conclusion. So, let A be an N cross N matrix with entries in P 1 of R, so, like in the case when we computed the characteristic polynomial, A minus lambda I, for example is 1 such example, when our A was A was the not this particular A some other A, maybe I should just call it B or let it be A be, may will call it B.

So, let B be an N cross N matrix with entries in P 1 of R. Now, the situation fits, so look at 1 of the A's which we considered earlier and we look at A minus lambda I. A minus lambda I equal to B is a perfect example for a matrix N cross N matrix with entries in P 1 of R. The conclusion of the proposition is that, then determinant of B is a polynomial of degree less than or equal to n, i.e. det B is in P n of R that is quite nice, because if we managed to prove this proposition, what we have essentially proved is that the characteristic polynomial of a given matrix A is a polynomial of degree less than or equal to n.

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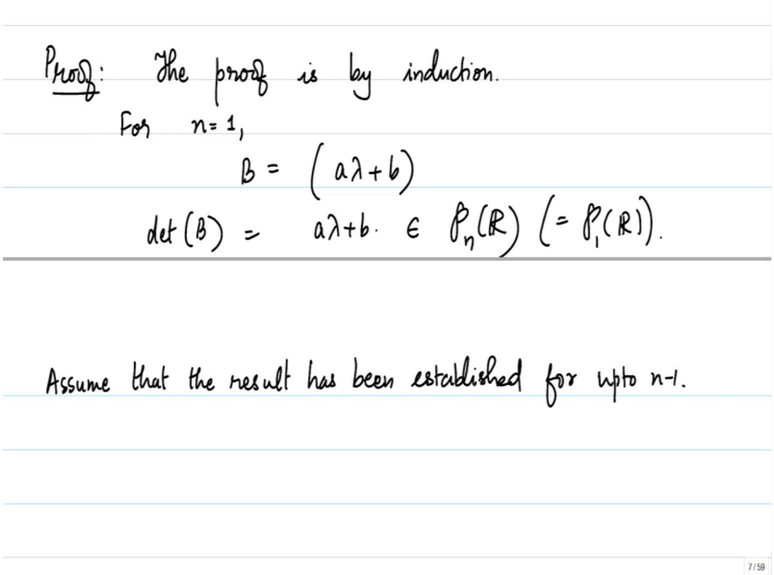

So, let us look at a proof of this proposition, the proof is by induction. So, let us start the induction by n equal, by looking at n equal to 1, so for n equal to 1, what will our B be? B will be just a 1 cross 1 matrix, so it will look like A lambda plus B, small A lambda plus small b. And what is determinant of B? Determinant of B is just going to be the exact same entry which is A lambda plus B, which in particular is in P 1 of R.

So that is what we had P n of R here, because n is equal to 1, which is equal to let me write it in brackets P 1 of R. So, yes, so the n equal to 1 case is easily getting satisfied. Now, let us assume the strong induction hypothesis, assume that the result has been established for up to n minus 1. So, we would like to now take the case when A is an N cross N matrix and then proof the result. So, how about jumping into that.

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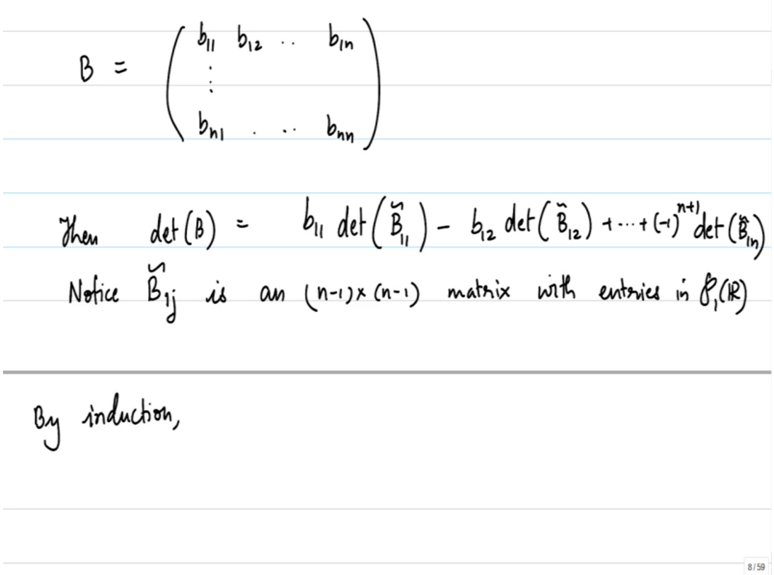

So let, let us look at the, so let us call the matrix B to be B, did I call it B while starting off? Yes. So, let B be something like B 1 1, B 1 2, B 1 n, B n 1 to B n n, notice does that now each of these B (())(18:19) are in P 1 of R, so they are polynomials of degree less than or equal to 1. Now, let us look at the cofactor expansion to calculate the degree of P sorry, the determinant of P.

So, the cofactor expansion of B can be written in the following manner. Then determinant of B will just turn out to be equal to B 1 1 times, let me put it like this, determinant of B 1 1 Tilda minus B 1 2 times determinant of B 1 2 Tilda and so on. Maybe I will write the last term minus 1 to the power n plus 1 times determinant of B 1 n Tilda. Now, notice that B I, B 1 J Tilda, this is an n minus 1 cross n minus 1 matrix.

What does B 1 1 Tilda for example? B 1 1 Tilda was the matrix obtained by deleting the first row and the first column. B 1 2 Tilda was the matrix obtained by deleting the first row and the second column and so on, so they are all B I, B 1 J Tilda is an n minus 1 cross n minus 1 matrix with entries in P of R not far. So, remember that all the entries are degree less than or equal to degree 1 polynomials entries in P 1 of R. And now the induction hypothesis is enforced.

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By induction, $\det(\tilde{B}_{1j})$ is a poly of degree at most $n-1$.

$\Rightarrow$ $b_{1j} \det(\tilde{B}_{1j})$ is a poly of degree at most n.

Hence their sum is a poly of degree $\leq n$

Therefore $\det(B) \in \mathcal{P}_n(\mathbb{R})$ —— $\blacksquare$

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By induction, because this is n minus 1 cross n minus 1 by induction determinant of B 1 J Tilda is an n minus, is a polynomial of degree at most n minus 1, it is in P n minus 1 of R that implies that B 1 J times the determinant of the B 1 J Tilda, this is a polynomial of degree at most n. Why is that the case? Because B 1 J has degree at most 1.

Determinant of B 1 J Tilda that has degree at most n minus 1, product of the two polynomials will have the sum of the degrees of the two polynomials will be at most and P n of R is a vector space if you add polynomials of degree less than or equal to n, we have a polynomial of degree less than or equal to n. Hence, their sum is a polynomial of degree less than or equal to n. Therefore, degree of, sorry determinant of B is in P n of R, so we have established the result.

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Corollary: Let A be an $n \times n$ matrix with real entries
Then the char. poly of A is a polynomial of degree at most n.

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So, as a corollary, corollary immediate corollary is that let A be an N cross N matrix in R, let A be an N cross N matrix with real entries, then the characteristic polynomial of A, this is a polynomial of degree at most n, is a polynomial of degree at most n, so that is some advancement.

However, I would say that we are not good enough yet to study the characteristic polynomials, because it is just saying it is at most n. So, the question naturally comes up is the characteristic polynomial having a degree equal to n or does it have degree less than n, so we will take the study a bit further and we will in fact, say that it has some nice form, which is captured in the next proposition.

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Proposition: Let $n \geq 2$. $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$. Then the characteristic poly of A is given by

$$f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + g(\lambda)$$

where $g(\lambda) \in P_{n-2}(\mathbb{R})$.

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So, let A be the matrix A 1 1 to A 1 n, A n 1 to A n n, matrix N cross N matrix with real entries. Then the characteristic polynomial of A it has a particularly nice form, polynomial of A is given by F of lambda which is equal to A 1 1 minus lambda into A 2 2 minus lambda, the product up to A n n minus lambda plus G of lambda. So, notice that I am again treating it as a polynomial in the indeterminate lambda.

And what is G of lambda? Where G of lambda is a polynomial of degree at most n minus 2. So, the proposition actually is saying that F, the characteristic polynomial of A necessarily has degree n, because G of lambda if you notice has degrees less then n minus less than or equal to n minus 2. And A 1 1 minus lambda into A 2 2 minus lambda up to A n n minus lambda is necessarily a polynomial of degree n. So, this indeed answers the question I just asked is the characteristic polynomial of degree n or is it less than n, it is equal to n.

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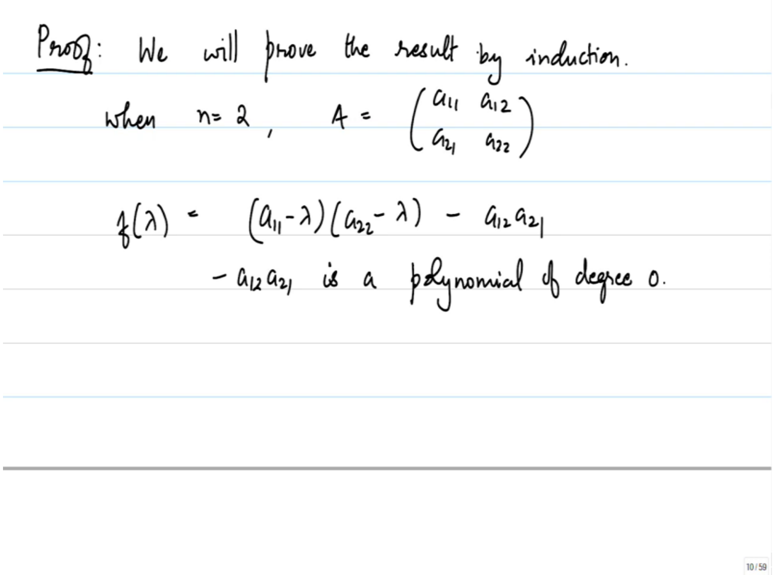

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$$f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + g(\lambda)$$

where $g(\lambda) \in P_{n-2}(\mathbb{R})$.

Let us give a proof of this, the proof will again be by induction. We will prove the result. Oh yes, so I think I should put 1 condition here, let n be greater than or equal to 2, we need at least n equal to 2 to say something like that, otherwise, the notion for G of lambda does not make sense. So, n greater than or equal to 2, this particular statement is true. So, we will prove the result by induction.

So, prove start off the kick start the induction. Let us consider the case when n is equal to 2. So, when n is equal to 2, what is our A? A is just going to be A 1 1, A 1 2, A 2 1, A 2 2. And what is our F of lambda? That is just going to be, A 1 1 minus lambda into A 2 2 minus lambda minus A 1 2 times A 2 1, well that is good. This is exactly what we were trying to establish.

Notice that minus of A 1 2, A 2 1, this is a polynomial of degree 0, is a polynomial of degree 0, which is 2 minus 2. So, it is in P n minus 2 of R. So, yes, so the induction hypothesis does start with n is equal to 2, let us assumed that it is satisfied for up to n minus 1 and we will prove that for n the result is true.

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$- a_{12}a_{21}$ is a polynomial of degree 0.

Assume that the result is true upto $n-1$.

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So, assume that the result is true, result is true up to n minus 1.

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Assume that the result is true upto $n-1$.

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

$$f(\lambda) = \det \begin{pmatrix} a_{11}-\lambda & \cdots & & a_{1n} \\ \vdots & a_{22}-\lambda & & \vdots \\ & & \ddots & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{pmatrix}$$

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Now, let us look at the matrix A, which is an N cross N matrix, where the entries are given by say A 1 1 up to A 1 n, A n 1 up to A n n. What will be the characteristic polynomial F of lambda that is just going to be determinant of the matrix A 1 1 minus lambda A 2 2 minus lambda along the diagonal I am writing the entries first, the remaining terms are as usual. So, this is exactly what our F of lambda is.

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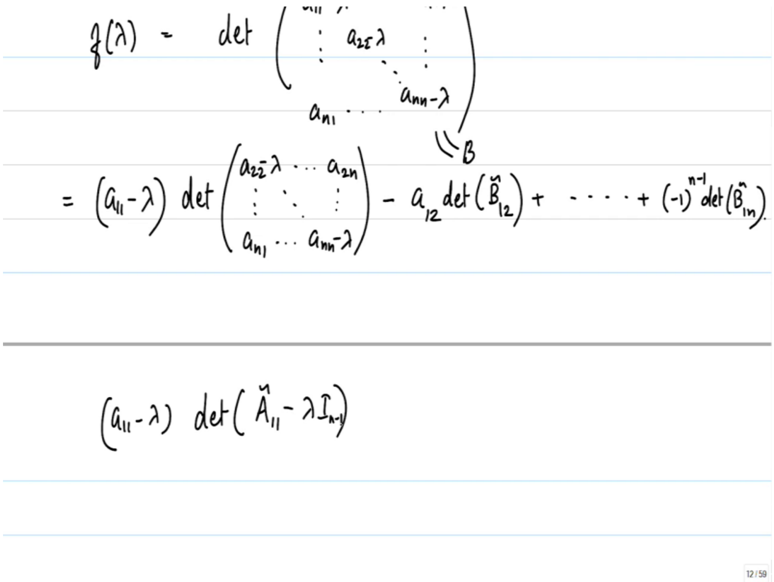

Now, let us look at the cofactor expansion from the first row, what will the cofactor expansion will be? It will be A 1 1 minus lambda times, so this is the cofactor expansion of the determinant times the determinant of the matrix A 2 2 minus lambda A n 1, A n n minus lambda A 2 n that will be the first term, what will be the second term? The second term will be minus of A 2 1, sorry A 1 2 times, so let me just write it as B 1 2 Tilda determinant of B or determinant of A.

Yeah, let me write it as B 1 2 Tilda, where B 1 2 Tilda is if you call this matrix equal to B, this is the relevant minor and so on. Let me write down the final term, this is just minus 1 to the power n plus 1 determinant of B 1 n tilda, this is the cofactor expansion of the determinant of B, where B is the matrix A minus lambda I.

So, what is the first let us go one after the other. Let us focus on the first the first term if you notice, this is A 1 1 minus lambda into the determinant of A, let us put it this way, A 1 1 Tilda minus lambda I, where A 1 1 Tilda is the matrix obtained by removing the first row and the first column of A. So, A 1 1 Tilda minus lambda I n minus 1 in this case. So, now, this is just the characteristic polynomial of the matrix A 1 1 Tilda.

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$$(a_{11}-\lambda)\det(\tilde{A}_{11}-\lambda I_{n-1})$$
$$= (a_{11}-\lambda)\left((a_{22}-\lambda)(a_{33}-\lambda)\cdots(a_{nn}-\lambda)+\tilde{g}_{11}\right) \quad \text{where } \tilde{g}_{11}\in P_{n-3}(\mathbb{R})$$

$$= \underline{(a_{11}-\lambda)(a_{22}-\lambda)+\cdots(a_{nn}-\lambda)+g_{11}} \quad \text{where } g_{11}\in P_{n-2}(\mathbb{R})$$

characteristic poly of A is given by

$$\underline{f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + g(\lambda)}$$

where $g(\lambda)\in P_{n-2}(\mathbb{R})$.

Proof: We will prove the result by induction.

when $n=2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$

$$f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$f(\lambda) = \det\underbrace{\begin{pmatrix} a_{11}-\lambda & \cdots & a_{1n} \\ \vdots & a_{22}-\lambda & \vdots \\ & \ddots & a_{nn}-\lambda \\ a_{n1} & \cdots & \end{pmatrix}}_{B}$$

$$= \underline{(a_{11}-\lambda)\det\begin{pmatrix} a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}-\lambda\end{pmatrix}} - \underline{a_{12}\det(\tilde{B}_{12})} + \cdots + (-1)^{n-1}\det(\tilde{B}_{1n}).$$

$$(a_{11}-\lambda)\det(\tilde{A}_{11}-\lambda I_{n-1})$$
$$= (a_{11}-\lambda)\left((a_{22}-\lambda)(a_{33}-\lambda)\cdots(a_{nn}-\lambda)+\tilde{g}_{11}\right) \quad \text{where } \tilde{g}_{11}\in P_{n-3}(\mathbb{R})$$

And the induction hypothesis kicks in, we will be able to conclude that this is equal to A 1 1 minus lambda into, let us call it F, let it will be F A 1 1 Tilda of lambda plus G 1 1 of lambda, by the induction hypothesis at this is of degree n minus 1, oh no, no, in fact, we can say a lot more, this will, by induction hypothesis this will turn out to be A 2 2 minus lambda into A 3 3 minus lambda up to A n n minus lambda plus G, well let us put at z 1 1.

So, notice what was our statement? The statement was let I am underlining it in green, the characteristic polynomial has this particular type, this is the form of the characteristic polynomial of A. In our case, our A now is A 1 1 Tilda, which is the matrix with diagonal A 2 2, A 3 3, up to A n n, this is exactly what it will turn out to be. Well, this is equal to A 1 1 minus lambda A 2 2 minus lambda plus up to A n n minus lambda plus A 1 1 minus lambda into G 1 1 Tilda.

Okay, what was G 1 1 Tilda? Where G 1 1 Tilda was some polynomial in P n minus 1 minus 2, which is P n minus 3 of R. So, it is a polynomial of degree less than or equal to n minus 3, a polynomial in lambda. If you multiply it to a degree 1 polynomial, so, this will just turn out to be equal to let me just write it as G 1 1, where G 1 1 is A 1 1 minus lambda into G 1 1 Tilda, which is now in P n minus 2 of R. So, the first the first expression here which I am underlining now in Let me see where it is, which am underlining now in green, this is going to look like this. Let us focus on the next term which is being underlined in red.

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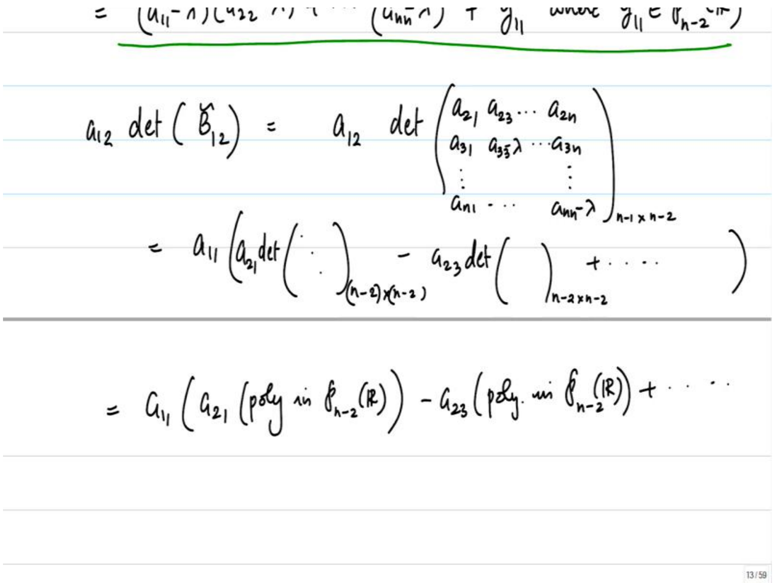

Now, if you notice carefully, what is this? This A 1 2 times determinant of B 1 2 Tilda, but this is equal to A 1 2 times the determinant of, what is B 1 2 Tilda? It is the matrix obtained by removing the first row and the second column. So, this is just going to be equal to A 2 1, A to the second column is removed, first row and the second column, so A 2 3, up to A 2 n, A 3 1, the second column is removed, A 3 3 minus lambda A 3 n, A n 1 up to A n n minus lambda, this is exactly what B 1 2 Tilda will be like.

Now, look at the determinant of this matrix by looking at the cofactor expansion along the first column here, first column of this matrix, the matrix, which I just put in the box, what will it be? This will be some A 1 1 into A 2 1 times some matrix, which is an n minus, let us see, this is an n minus 1 cross n minus 1 matrix. So, this is going to be an n minus 2 cross n minus 2 matrix, then there will be a minus A 2 3 times the determinant of something precise this is A 2 1 determinant of a matrix.

Similarly, this will be A 2 3 times determinant of another matrix n minus 2 cross n minus 2, the terms I will not write, because they are not important. The thing to notice is that by the previous proposition that we proved this determinant will again turn out to be a polynomial of degree less than or equal to n minus 2. So, this is going to be A 1 1 into A 2 1 times some polynomial in P n minus 2 of R plus or rather minus A 2 3 times a polynomial in determine in P n minus 2 of R that is what matters and so on, everything will be like that.

And notice A 1 1, A 2 1 all these are constants or rather scalars, real numbers, degree 0 polynomials.

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$(n-2)\times(n-2)$ $\quad$ $n-2\times n-2$

$$= a_{11}\left(a_{21}\left(\text{poly in } \mathcal{P}_{n-2}(\mathbb{R})\right) - a_{23}\left(\text{poly. in } \mathcal{P}_{n-2}(\mathbb{R})\right) + \cdots\right.$$

$$= g_{22} \in \mathcal{P}_{n-2}(\mathbb{R})$$

$$\underbrace{\begin{pmatrix} \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}-\lambda \end{pmatrix}}_{B}$$

$$= (a_{11}-\lambda)\det\begin{pmatrix} a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}-\lambda \end{pmatrix} - a_{12}\det(\tilde{B}_{12}) + \cdots + (-1)^{n-1}\det(\tilde{B}_{1n})$$

$$(a_{11}-\lambda)\det(\tilde{A}_{11}-\lambda I_{n-1})$$

$$= (a_{11}-\lambda)\big((a_{22}-\lambda)(a_{33}-\lambda)\cdots(a_{nn}-\lambda)+\tilde{g}_{11}\big) \quad \text{where } \tilde{g}_{11}\in P_{n-3}(\mathbb{R})$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)+\cdots(a_{nn}-\lambda) + g_{11} \quad \text{where } g_{11}\in P_{n-2}(\mathbb{R})$$

So, this will just turn out to be some G 2 2, which is in P n minus 2 of R. So, this red, rather this red will just turn out to be this red and this is exactly what this red turns out to be. Now, I will not go about with the remaining terms, the remaining terms are proved exactly similarly, to be in P n minus 2 of R.

The only thing to note is that when you are now considering the term corresponding to A 1 P, you should expand the determinant of the matrix corresponding to whatever is being noted here, you should expand that long the third row. So, nth, the nth term should be expanded, the cofactor expansion should be along the nth term. And at every stage, we will end up with a polynomial of degree less than or equal to n minus 2. So, finally, what do we have the green thing that I have just written down plus main polynomials of degree less than or equal to n minus 2.

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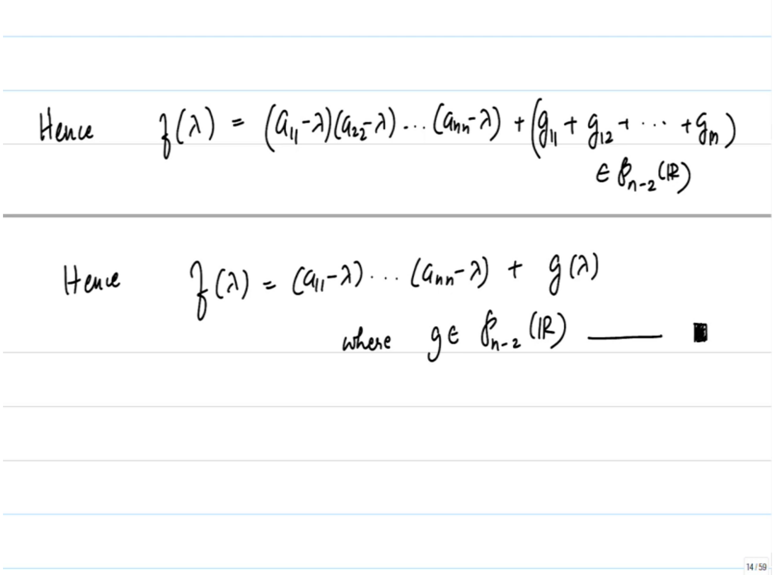

So, hence, our F of lambda is just going to be A 1 1 minus lambda into A 2 2 minus lambda up to A n n minus lambda plus there was a G 1 1 plus G 1 2 plus up to G 1 n, where each of these G I J case are polynomials of degree less than or equal to n minus 2. Hence, F of lambda is equal to A 1 1 minus lambda, the sum of such polynomials G 1 1, G 1 2 up to G 1 n will all turn out to be again a polynomial of degree less than n minus 2, where G is a polynomial in P n minus 2 of R, with this we have established our result. What is that? That is precisely what we wanted to prove, where is it?

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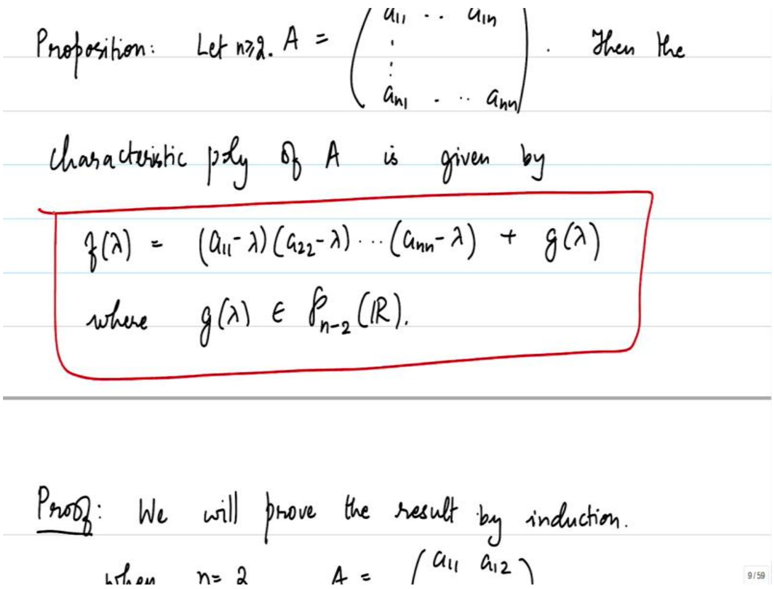

Here F of lambda, putting it in box, this is exactly what we wanted to establish, we have done that.

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$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \underbrace{\left(a_{11} + \cdots + a_{nn}\right)}_{tr(A)} \lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0$$

$$f(0) = a_0$$

$$f(0) = \det(A)$$

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In fact, we can say a little more If you notice our F So, let us keep it here for immediate reference F of lambda is what is it if you expand out A 1 1 minus lambda plus up to A n n minus lambda, this is just going to be equal to minus of 1 to the power n times lambda to the power n plus minus of 1 to the power n minus 1 times A 1 1 plus up to A n n times lambda to the power n minus 1 plus a degree 2 polynomial.

What is A 1 1 plus A 2 2 up to A n n? This is nothing but the trace of the matrix A that is good. So, because in fact the remaining terms if I have to write it out something as A 2 lambda to the power A n minus 2. Let me start from that A n minus 2 lambda to the power n minus 2 plus up to A 0.

Suppose, this is our F of lambda. So, what is F of 0? F of 0 is just going to be equal to A 0 so that means we know what A 0 is if we can get hold of what F of 0 is. But what is F 0? F of 0 is the determinant of A minus lambda is 0, so lambda 0 times I is just going to be determinant of A. So, we know what A 0 is, A 0 is the determinant of A.

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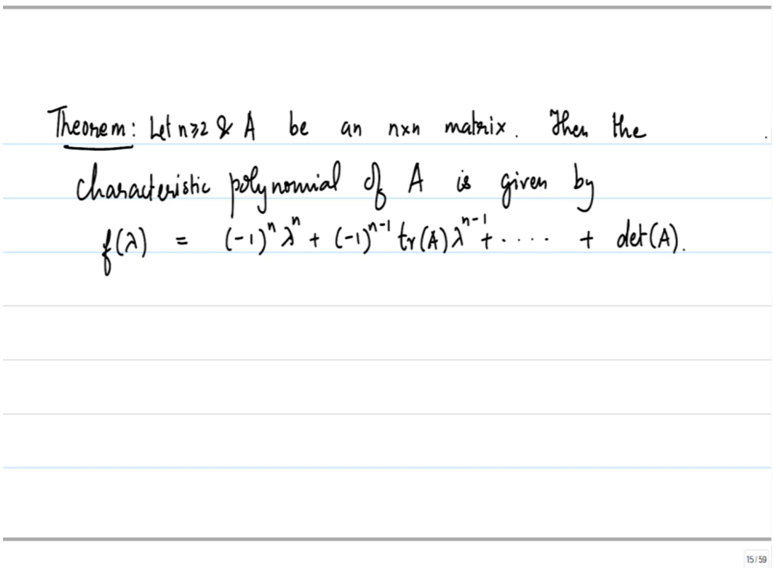

So, let me just write it down as A theorem. So, all the effort we have just spent can be captured in the following theorem. So, let A be an N cross N matrix, so let n be greater than or equal to 2 and A be an N cross N matrix given by, well, let us not bother about what the form is, then the characteristic polynomial, the characteristic polynomial of A is given by F of lambda is equal to minus 1 to the power n lambda to the power n plus minus 1 to the power n minus 1 times the trace of A into lambda to the power n minus 1 plus dot dot dot plus finally, the determinant of A.

So, the characteristic polynomial encodes in its coefficients the trace and the determinant of argument matrix A. So, we have concluded something substantial by studying the characteristic polynomial of a given metrics, and we now know how well the characteristic polynomial looks.

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characteristic polynomial of A is given by

$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} tr(A) \lambda^{n-1} + \cdots + det(A).$$

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$
$$= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$
$$= \lambda^2 - tr(A$$

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Example: $A = (a_{21} \quad a_{22})$

$$f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$
$$= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$
$$= \lambda^2 - tr(A)\lambda + det(A).$$

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So, this is actually quite nice if you look at 2 cross 2 matrix, this is quite straightforward. For example, if A is say, A 1 1, A 1 2, A 2 2, sorry A 2 1 and A 2 2, what is going to be F of lambda? F of lambda will, let us compute it and see. A 1 1 minus lambda into A 2 2 minus lambda minus A 1 2 times A 2 1, which is equal to minus lambda square minus A 1 1 plus A 2 2 times lambda plus A 1 1, A 2 2 minus A 1 2, A 2 1, which if you notice this is equal to lambda square minus the trace of A times lambda plus the determinant of A, which is nice.

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$$= \lambda^2 - tr(A)\lambda + det(A).$$

$$*) \quad A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & a_{nn} \end{pmatrix}$$

$$f(\lambda) = (a_{11}-\lambda)\cdots(a_{nn}-\lambda)$$
$$= (-1)^n \lambda^n + (-1)^{n-1}(a_{11}+\cdots+a_{nn})\lambda^{n-1} + \cdots + a_{11}a_{22}\cdots a_{nn}$$

Well, let us look at what happens when A is diagonal matrix let be equal to A 1 1, 0, 0. another case which can be easily check, then F of lambda, this is just A 1 1 minus lambda up to A n n minus lambda, which we have already noticed is minus 1 to the power n times lambda to the power n plus minus 1 to the power n times A 1 1 plus up to A n n times lambda to the power n minus 1, so sorry n minus 1 plus the constant term here is very clear, it will be A 1 1 A 2 2 up to A n n, which is exactly the determinant of the matrix A.

So, let us now analyze this F of lambda a bit further. So, notice that F of lambda is a polynomial of degree n. So, since F of lambda is a polynomial of degree n, it is possible that we can write it as a product of n factors, so it could possibly factor into n linear terms, let me note that down.

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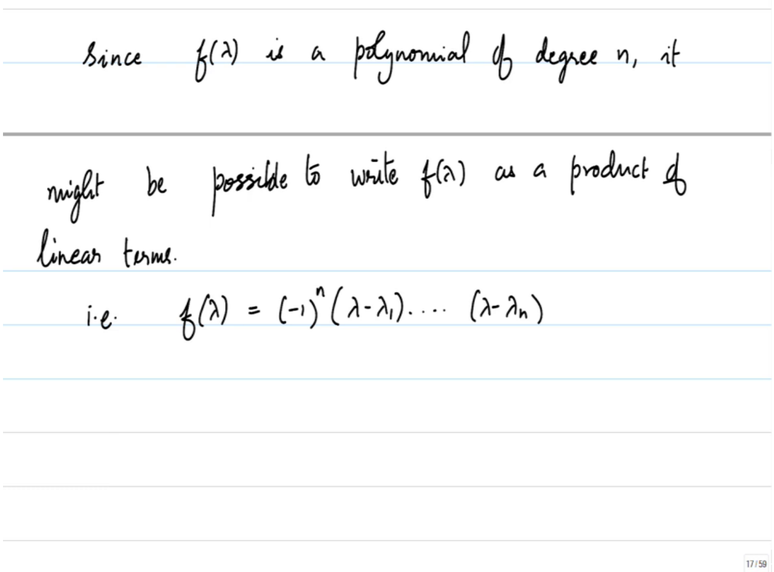

Since, since F of lambda is a polynomial of degree n, it might be possible to write F of lambda as a product of linear factors, not necessarily always possible, when we are considering matrices with entries in real numbers, but maybe sometimes it is possible. So, i.e. we might be able to write F of lambda as minus 1 to the power n term we will be say coming out and it will, it could possibly look like minus 1 to the power n times lambda minus lambda 1 into up to lambda minus lambda n. Not necessary that these lambda I's are distinct, restriction is not being imposed, but it might be possible to write it like this.

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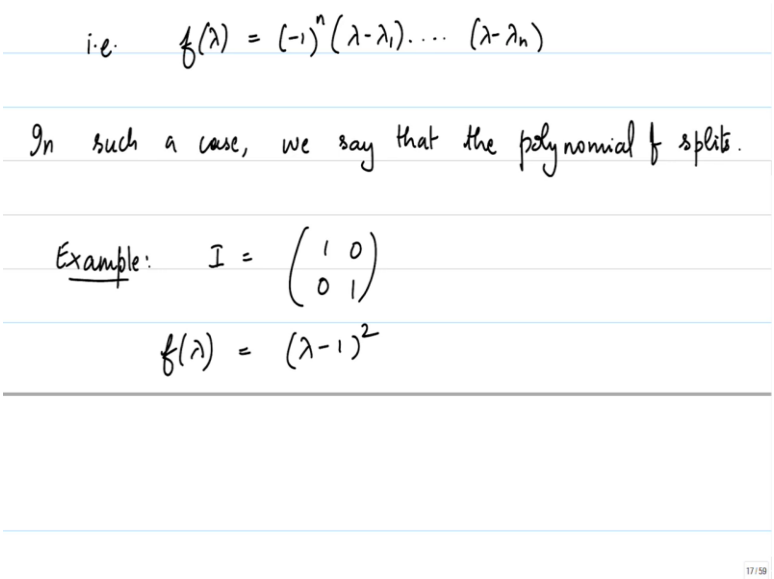

In such a case in such a case, we say that the polynomial F splits. So, let us look at a couple of examples, a couple of examples where it does split and when it does not split. So, let us look at the simplest examples, for example, let A be equal to 1, 0, 0, 1, what is going to be F of lambda? F lambda just will be lambda, 1 minus lambda minus 1, the whole square the identity. So, let me put it like this, I 1, 0, 0, 1, it is just going to be lambda minus 1 the whole square this lambda minus 1 into lambda minus 1 in this case, it does split.

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* $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$f(\lambda) = \lambda^2 - 1 = (\lambda - (-1))(\lambda - 1)$$

* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$f(\lambda) = \lambda^2 + 1$$

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How about when A is say 0, 1, 1, 0, then F of lambda will be lambda square minus 1, which will be lambda plus, maybe I will put like this lambda minus 1 times lambda minus 1. Yet again it splits, it may not give you the impression that it always splits, just tweak the above example 0, 1 minus 1, 0, then F of lambda will just turn out to be equal to lambda square plus 1. We know that this is actually lambda plus I into lambda minus I, where I is the square root of minus 1, but over the field of scalars being real numbers, this particular polynomial does not split.

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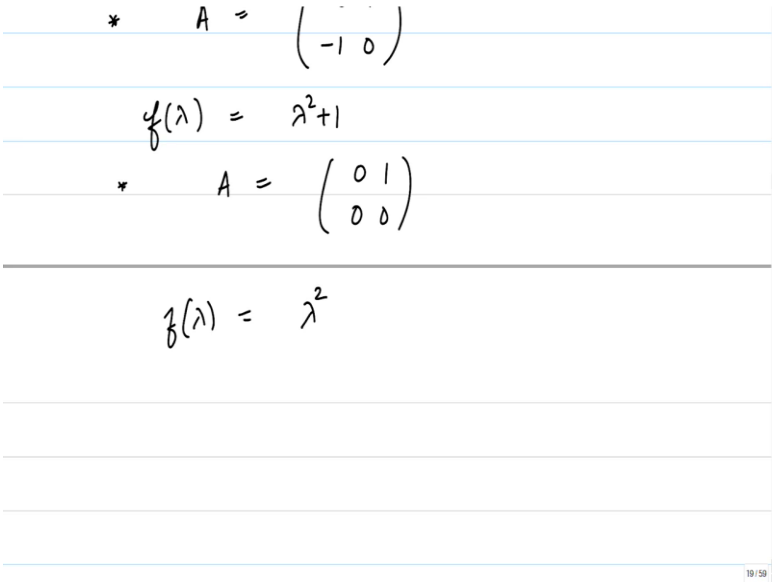

We will also, let us look at 1 another simple example play with zero's and one's, what will F of lambda be here? F of lambda is lambda square. And notice that this is lambda minus 0 into lambda minus 0. So, again, it splits. So, there are cases when it splits and there are cases when it does not split. So, what can we say about when our given matrix is diagonal matrix, that is the case when it is quite interesting.

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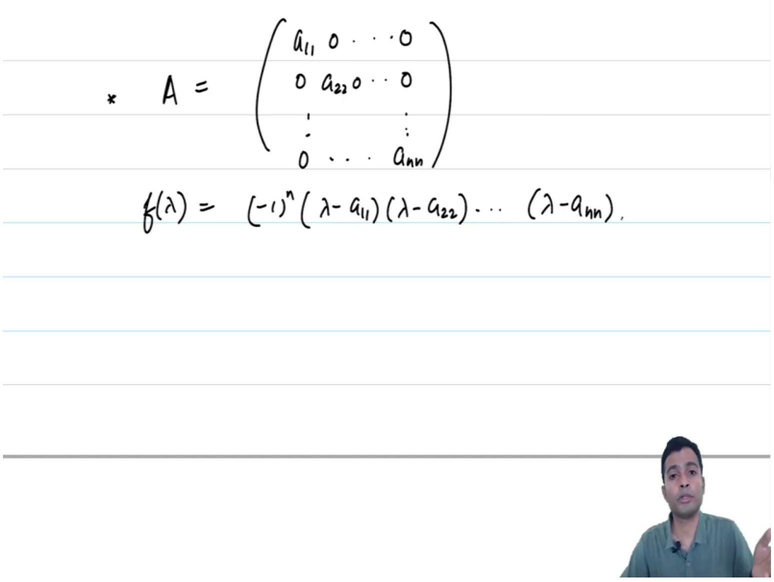

So, if A is something like say A 1 1, 0, 0, A 2, 2, 0 and so on, we know exactly what the characteristic polynomial of A is. Here, F of lambda is just equal to I will write it like this minus 1 to the power n times lambda minus A 1 1 into lambda minus A 2 2 up to lambda minus A n n.

So, in the case of a diagonal matrix, we know that the characteristic polynomial necessarily splits and we also know that if a matrix is diagonalizable, that means that it is similar to a diagonal matrix. And we just proved in the beginning of this video that similar matrices have the same diagonal and therefore the sorry, the similar matrices have the same characteristic polynomial, and therefore the characteristic polynomial of a diagonalizable matrix necessarily split.

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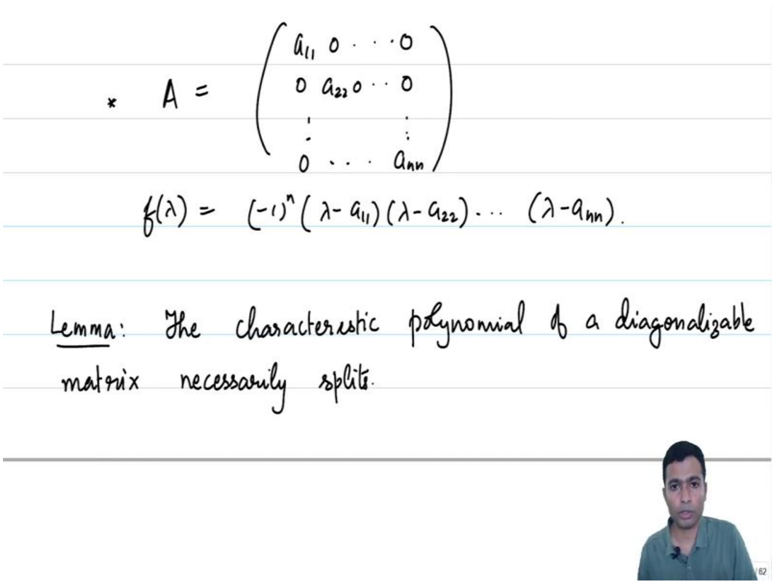

So, notice so I will write down as a lemma, which I will not prove, I just said it in words, the characteristic polynomial, characteristic polynomial of a diagonalizable matrix necessarily splits. So, in other words, if we get hold of a matrix, whose characteristic polynomial does not split, then that matrix is not diagonalizable matrix.

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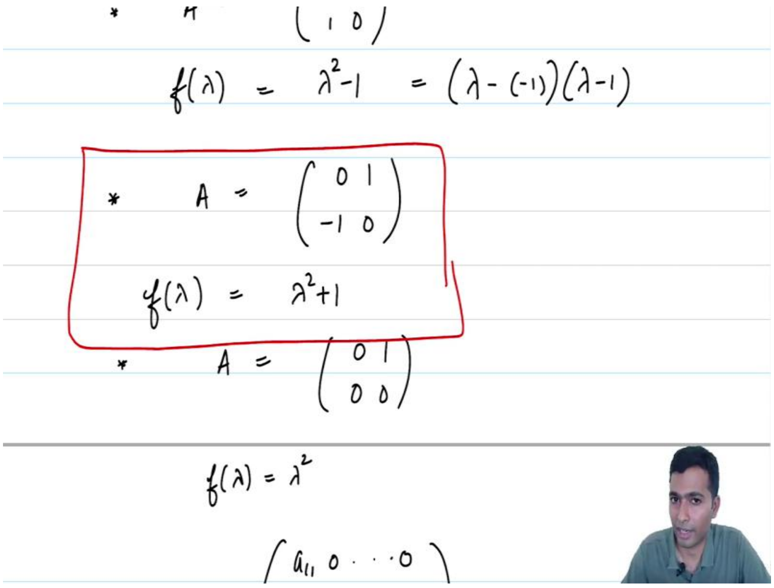

So, in particular, let us come back to this example. In this particular case, which I am now putting in red, the characteristic polynomial does not split. So, we know for a fact that A is not a diagonalizable matrix. So, this is one easy way of checking for whether a given matrix is diagonalizable, but of course, characteristic polynomial is something which we have to compute to do that, but nevertheless once it is computed, if it is easily seen that it does not split, you can conclude that it is not diagonalizable, so this is a necessary condition.

So, the question now pops up is the following if you have a matrix whose characteristic polynomial splits can we conclude that the given matrix is a diagonalizable matrix? So, that, in other words is the condition that the characteristic polynomial splits is this a sufficient condition as well.

So, I will not keep you in suspense and say that it is not necessarily the case that is not a sufficient condition. However, the splitting of the characteristic polynomial can be studied in depth, in order to conclude something about the diagonalizability of the given matrix, which will be the content of lectures to follow.