Linear Algebra
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Lecture 7.4: Computing Eigenvalues

So let us now focus our attention on developing some techniques to evaluate the eigenvalues and the eigenvectors of a given n cross n matrix. So let us begin by key proposition in this direction. So let me write down the proposition.

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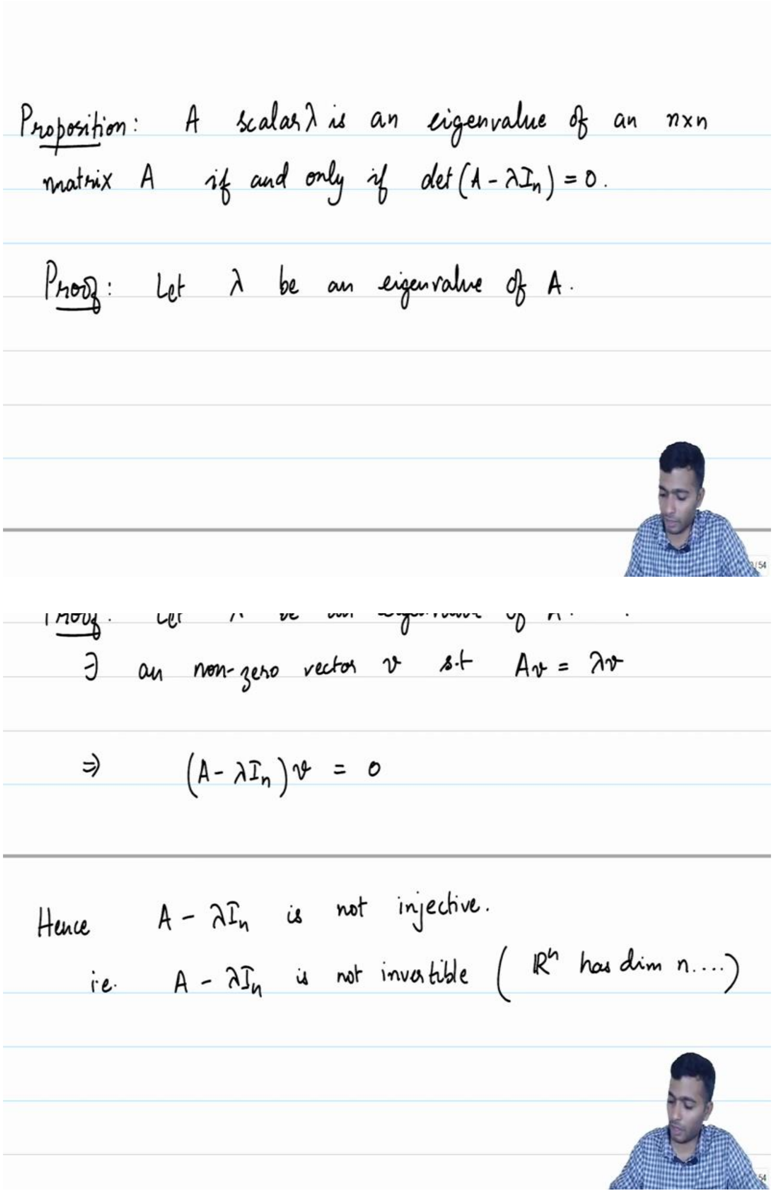

So a scalar lambda is an eigenvalue of an n cross n matrix A, if and only if the determinant of A minus lambda I n is zero. Let us give a proof of this statement. So proof, so let us assume that scalar lambda, let us call it lambda here itself. Let us assume that lambda is an eigenvalue of the matrix A. So let lambda be an eigenvalue of A.

That means that there exists a non-zero vector v, i.e. there exists a non-zero vector v such that A v is equal to lambda v or given lambda. We write it, this implies that A minus lambda I n, where I n is the n cross n identity matrix times v is equal to 0. And v is a non-zero vector. This just implies that A minus lambda n is not an injective linear transformation. Hence A minus lambda I n is or the linear transformation corresponding to this is not injective. Because (that) this means that v is in the null space of A minus lambda I n. Because this means that v is in the null space of A minus lambda I n.

But what can we say about linear transformation from a finite dimensional space to itself, if it is not injective? We can say that a linear transformation from finite dimensional space v to itself is injective if and only if it is surjective and hence invertible. So if it is not injective, A minus lambda I n is not invertible, i.e. A minus lambda I n is not invertible because if it were, so the reason is that this is R n has dimension n, finite dimensional and any invertible linear transformation any injective linear transformation is necessarily invertible. And any invertible linear transformation is obviously necessarily injective. So let me write dot dot dot for you to complete down the reason. So A minus lambda I n is not invertible.

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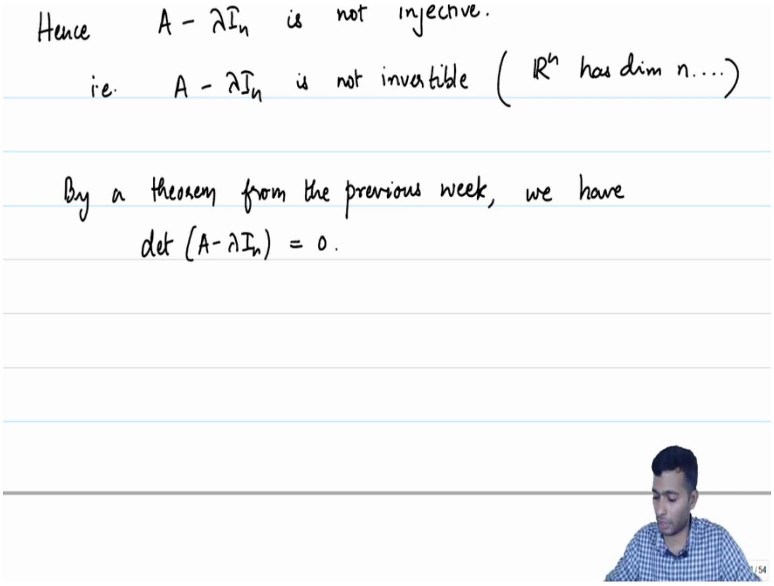

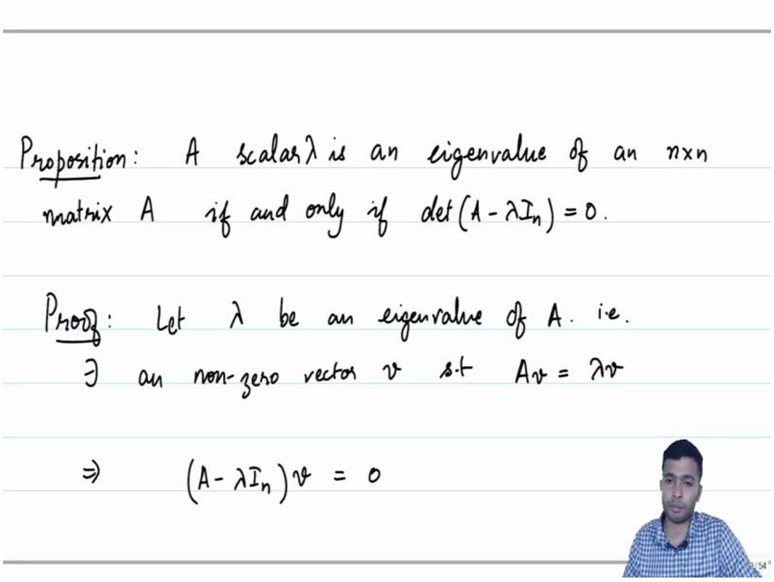

But what had we proved in the last week? We have proved that a vector a matrix is invertible if and only if the determinant is non-zero. Here we have that A minus lambda I n is not invertible. And therefore, that forces by a theorem from the last week, from the previous week, we have determinant of A minus lambda I n is equal to 0. If it were not 0, then A minus lambda I n would have been invertible. And that is precisely what we have set out to prove. If you look at the preposition, it is said that determinant of A minus lambda I n is equal to 0.

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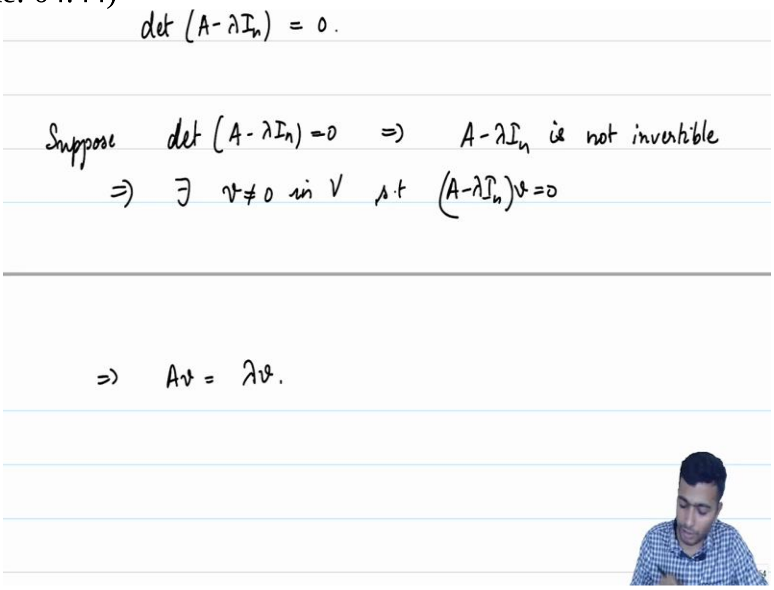

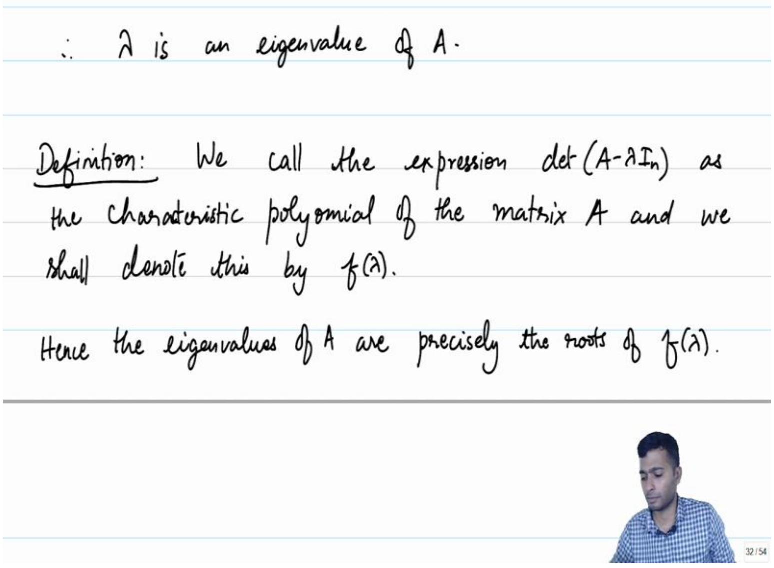

Now let us try to prove the converse. So suppose, it is just the backtracking of the argument. Suppose A minus lambda I n has determinant 0, that implies that A minus lambda I n is not invertible. And if it is not invertible, it is not injective. Because here invertibility and injectivity are the same. It is finite dimensional. This implies that, there exists v not equal to 0 in V such that A minus lambda I n v is equal to 0 v in the null space of A minus lambda I n, which implies that A v is equal to lambda I n v which is equal to lambda v.

Therefore, lambda v is an eigenvector and lambda is an eigenvalue corresponding to v of the matrix A. So we have effectively found out a (character) characterization of when we can say that scalar is a eigenvalue. So this prompts us to give a definition for the number or sorry, the expression determinant of A minus lambda I n. Notice that this is going to be a polynomial expression in lambda, so let me give a definition here.

So we call the expression, call the polynomial in lambda or rather let me call it expression as of now, determinant of A minus lambda I n as the characteristic polynomial of lambda of A. Characteristic polynomial of the matrix A. And we will denote it by f of lambda. And we shall denote in the rest of this lecture this by this polynomial by f of lambda. So what or the above preposition just said is that, lambda is an eigenvalue if and only if it is the root of f. So hence the eigenvalues of A are precisely the roots of f of lambda.

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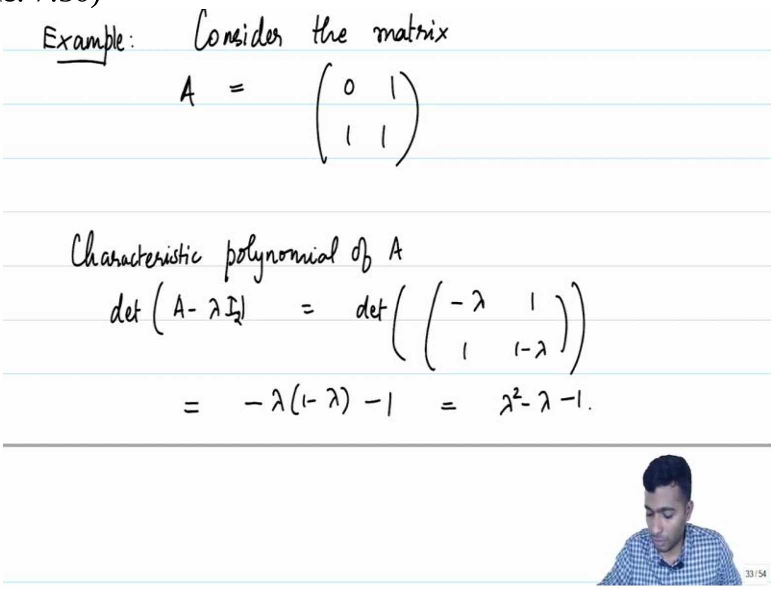

So let us look at a few, let us look at an example, wherein we will do the computing of eigenvalues and eigenvectors and look at some of the implications. So an example, so let us consider the matrix A which is given by 1, 1, 0, 1. Or we denote this 0, 1, so it will be 1, 1. This is it. This is the right expression of example I would like to consider. So let us try to compute the eigenvalues of A. In order to do that, let us compute the characteristic polynomial of lambda. So characteristic polynomial, we have just noticed that the characteristic polynomial, the rules of the characteristic polynomial are precisely all the eigenvalues.

Polynomial of A, this is given by the determinant of A minus lambda times I, here in this case I 2. And that will be the determinant of the matrix minus lambda 1, 1, 1 minus lambda. And this tells us this is lambda times 1 minus lambda minus 1, which is equal to lambda square minus lambda minus 1.

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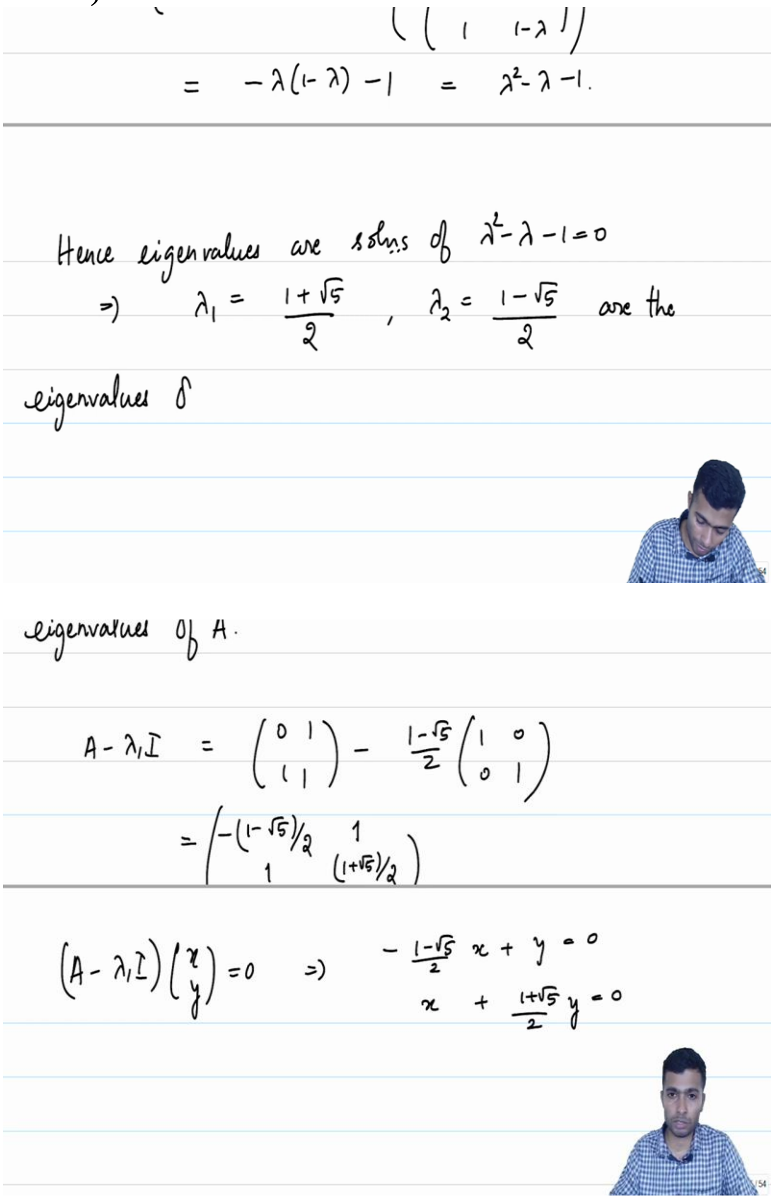

So what are the eigenvalues? Eigenvalues are all the roots of this particular characteristic polynomial. Hence, eigenvalues are solutions of lambda square minus lambda minus 1 is equal to 0. This implies that lambda 1 is equal to 1 plus square root of 5 by 2 and lambda 2 is equal to 1 minus square root of 5 by 2 are the eigenvalues of A. Let us now evaluate the eigenvectors corresponding to lambda 1 and lambda 2.

So to compute the eigenvectors corresponding to each, let us look at the null space corresponding to A minus lambda 1 times identity and A minus lambda 2 times identity. So let us now consider A minus lambda 1 times identity. This is just matrix is 0, 1, 1, 1 minus 1 minus square root of 5 by 2 times 1, 0, 0, 1. And this is 1 (plus) or 1 minus square root of 5 by 2, minus of this, 1 then there is a 1 and there is 1 and there is 1 minus 1 minus square root of 5 by 2, which is 1 plus square root of 5 by 2.

So A minus lambda 1 I something in the null space will be x, y equals to 0, implies minus of 1 minus square root of 5 by 2 times x plus y is equal to 0, x plus 1 plus square root of 5 by 2 times y is equal to 0. So if you observe carefully, if you multiply minus of 1 minus square root of 5 by 2 to the second equation, we get back the first equation. That is not surprising because determinant of a minus lambda 1 I was not non-zero, it was 0. Therefore, it does not have full rank.

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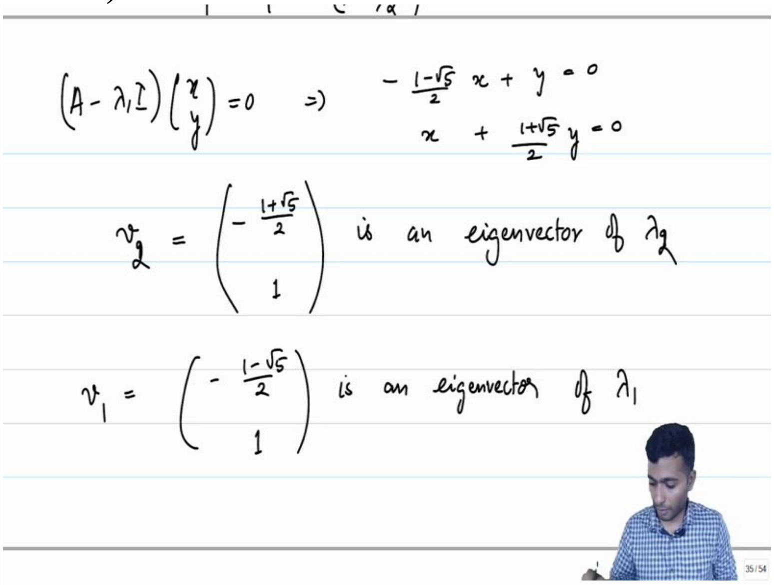

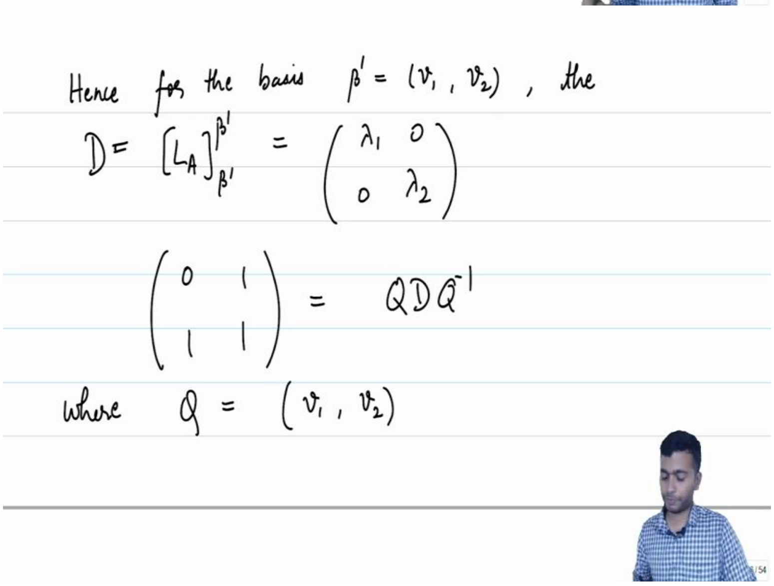

So typical solution will be of this type. Let us write it as, so v 1 equal to 1 and or maybe, 1 plus minus of 1 plus square root of 5 by 2 and 1 is an eigenvector of lambda 1. I leave it you to evaluate that v 2 equal to minus of 1 minus square root of 5 by 2, 1 is an eigenvector. So lambda 1, let us see what was lambda 1, so lambda 1 was 1 plus, so this is let us put lambda 2 here. And this is just going to be v 2 corresponding to lambda 2, and v 1 equal to this will be an eigenvector of lambda 1. Lambda 1 if you recall is 1 plus square root of 5 by 2.

So if you consider the basis, so consisting of the vectors v1 and v2, then our matrix for A will turn out to be a diagonal matrix. That is something which we had proved in the last video. So hence for the basis beta or rather beta prime given by v 1, v 2 are the matrix L A corresponding to beta prime will just turn out to be equal to lambda 1, 0, 0, lambda 2. But if you recall this was nothing but or we had A which is 0, 1, 1, 1 was Q, let us call this D. D Q inverse where D was D is as given above and Q is the matrix given by v 1, v 2.

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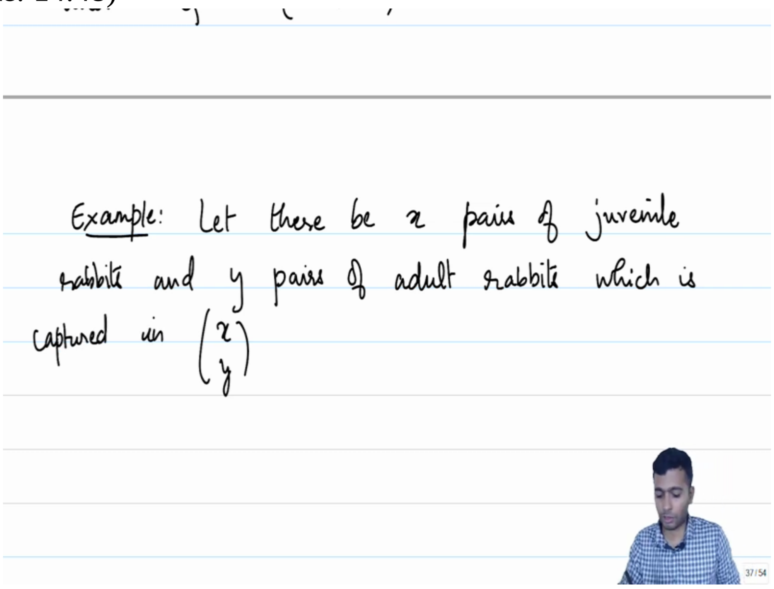

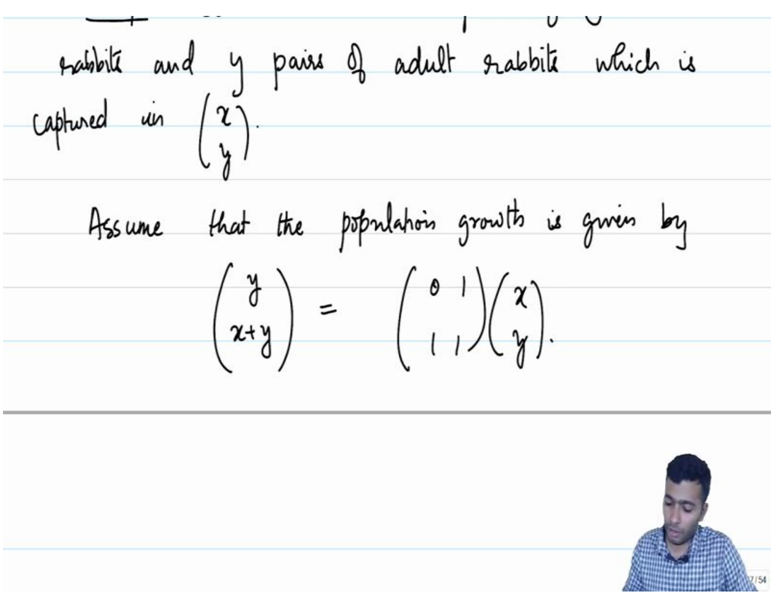

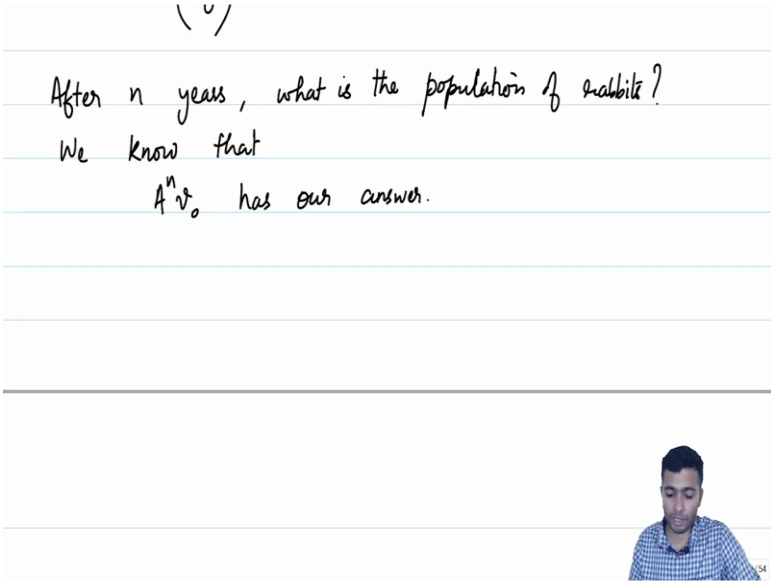

Let us apply all this to real life example. So the example here deals with what is known as Fibonacci's rabbit. This deals with the growth of the population of rabbits. Let us look at an example. So let there be x pairs of juvenile rabbits and y pairs of adult rabbits. Let there be x pairs of juvenile rabbits and y pairs of adult rabbits, which we will capture in the vector x, y. Captured in the vector x, y. Then let us assume that every year the population growth is given by left multiplication by our matrix A.

Assume that the population growth is given by y and x plus y. But what is this y and x plus y? Is nothing but multiplication of x, y by the matrix 0, 1, 1, 1 with this matrix x, y. So suppose we started off with a sample or suppose the world started off with one pair of juvenile rabbits and no adults. So v 0 v equal to 1, 0. So we would like to ask the following question. After n years, after n years what is the population of rabbits? Juvenile or and adult rabbits.

And the answer is a straightforward answer. It is A to the power n times v 0. We know that A to the power n times v 0. The first row captures the number of juveniles and the second row captures the number of the adults after n years as our answer. But then if you start trying to calculate A to the power n, in fact it is a tedious process just for this 2 cross 2 matrix to start computing A to the power n, for n very very large.

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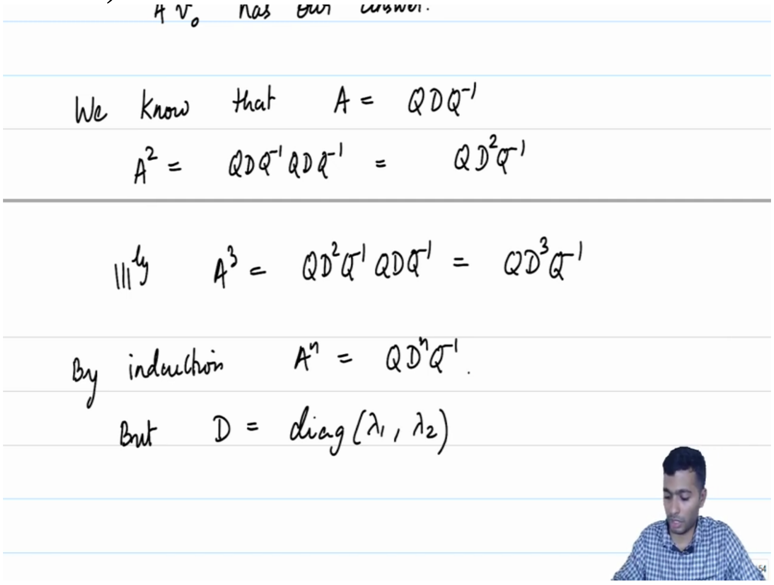

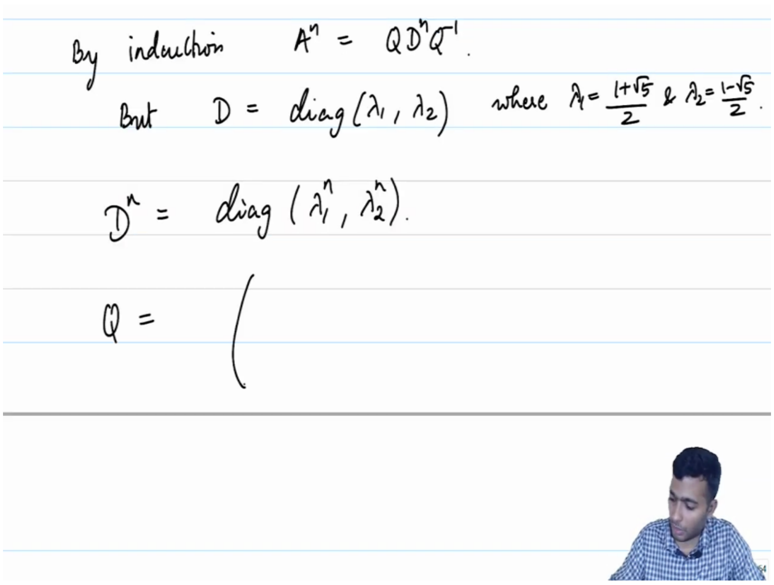

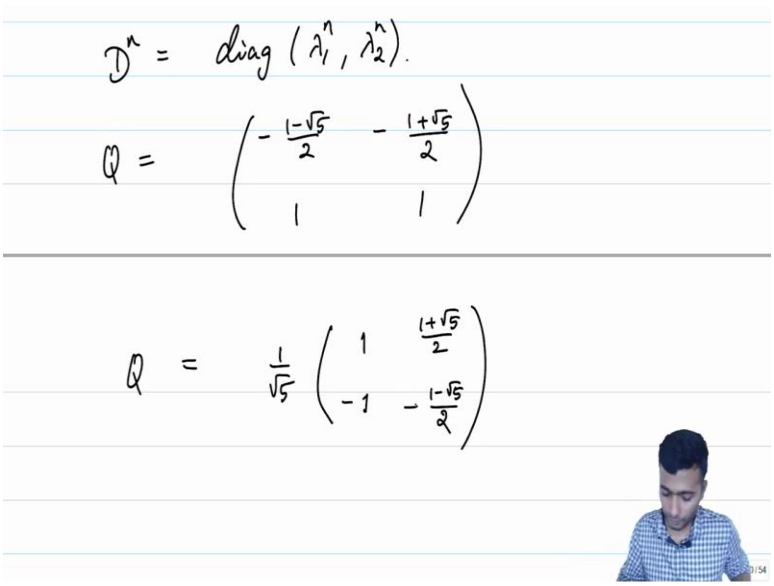

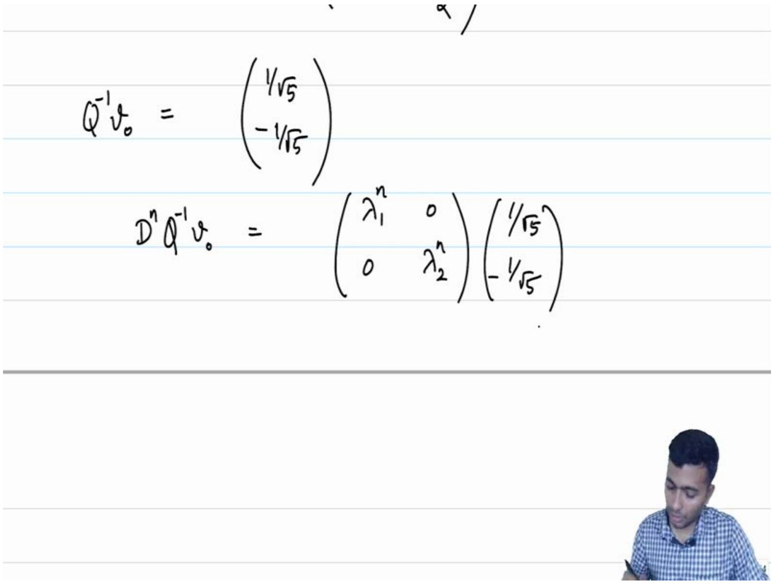

What we will do is we will use the knowledge that we have developed, we have noted that you know that A is equal to QDQ inverse. So if you recall we did mention that diagonalization was a technique which was or it is a factorization method which was beneficial for computing the powers. So you notice A square is just QDQ inverse QDQ inverse. Q inverse gives the identity and this is assumed to be QD square Q inverse.

Similarly, A cube will be just QD square Q inverse QDQ inverse, which again by the same argument gives the QD cube Q inverse. By induction you may show that A to the power n is QD to the power n Q inverse. But we know what D to the power n is, D is a very special matrix. D is a diagonal matrix consisting of the eigenvalues in its diagonal. So we know but D is equal to lambda 1. So let me not write the big expression. Let me just write diag lambda 1, lambda 2. If you recall lambda 1, lambda 2 were the eigenvalues 1 plus square root of 5 by 2.

So let me write this, where lambda 1 was 1 plus square root of 5 by 2 and lambda 2 was 1 minus square root of 5 by 2. Where lambda 1 is equal to 1 plus square root of 5 by 2 and lambda 2 is equal to 1 minus square root of 5 by 2. So what is D to the power of n? That is just going to be diagonal of lambda 1 to the power n, lambda 2 to the power n.

We have also computed what our Q is. You recall Q was just, Q is v 1, v 2. Where v 1 is minus of 1 minus square root of 5 by 2, minus of 1 minus square root of 5 by 2 and this is going to be minus of 1 plus square root of 5 by 2. And there was a 1 here. So this was our Q. So by using elementary matrices and reducing it to the epsilon form and parallely doing the same operations to the identity, one may check that I they will not do the calculations. This is just going to be 1 by root 5 times 1, minus of 1 minus square root of 5 by 2, 1 plus square root of 5 by 2 and 1 and minus 1, this is Q inverse.

Now let us calculate, so hence A to the power n v 0 is just QDQ inverse v 0. So in particular, Q inverse v 0 that is just v 0, if you recall is 1, 0. That is just going to be 1 by root 5 times. Maybe let me write it like this. This is going to be 1 by root 5, minus of 1 by root 5. And if you multiply this to A to the power n, this is going to be A to the power n, I am sorry, D to the power n, Q inverse v naught is just the lambda 1 to the power n, 0, 0, lambda 2 to the power n times 1 by root 5, and 1 minus of 1 by root 5. So let me maintain the root 5 maybe now, root 5.

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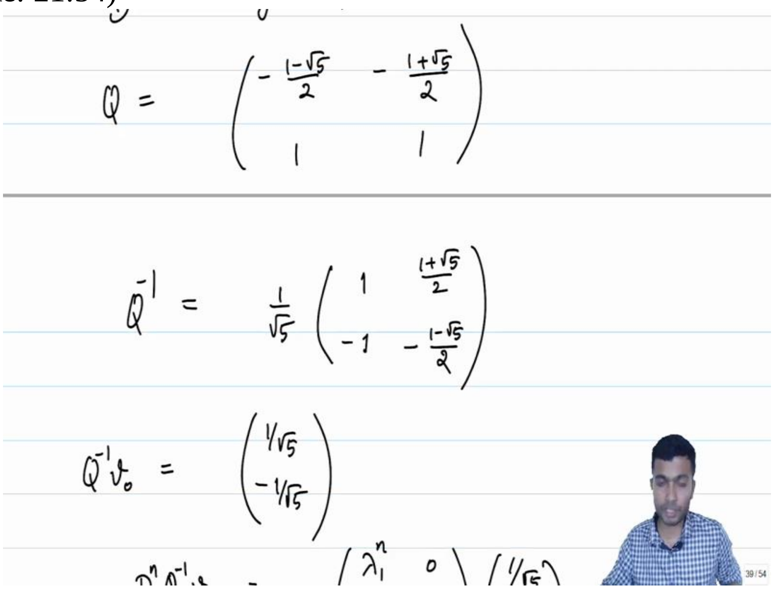

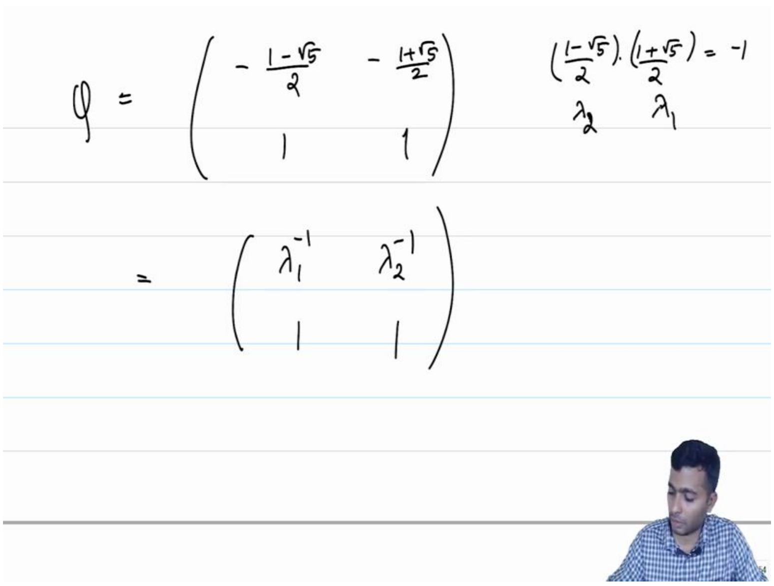

So now about Q, you notice, what was Q? Q was given by this matrix. And this we will write it down again, Q is minus of 1, Q is minus of 1 minus root 5 by 2, 1, minus of 1 plus root 5 by 2 and 1. And if you notice carefully, 1 plus root 5 by 2, 1 minus root 5 by 2 times 1 plus root 5 by 2 this is equal to minus 1. This is going to be 1 minus 4 by 4 which is minus 1. 1 minus 5 by 4 which is minus 1. And therefore, minus of 1 minus 5 by 2 is just inverse of 1 plus root 5 by 2. This is, remember this is lambda 1 and this is lambda 2. This is just going to be equal to lambda 2 inverse. So this is going to be lambda 1 inverse, lambda 2 inverse 1, 1. This is lambda 2 and this is lambda 1.

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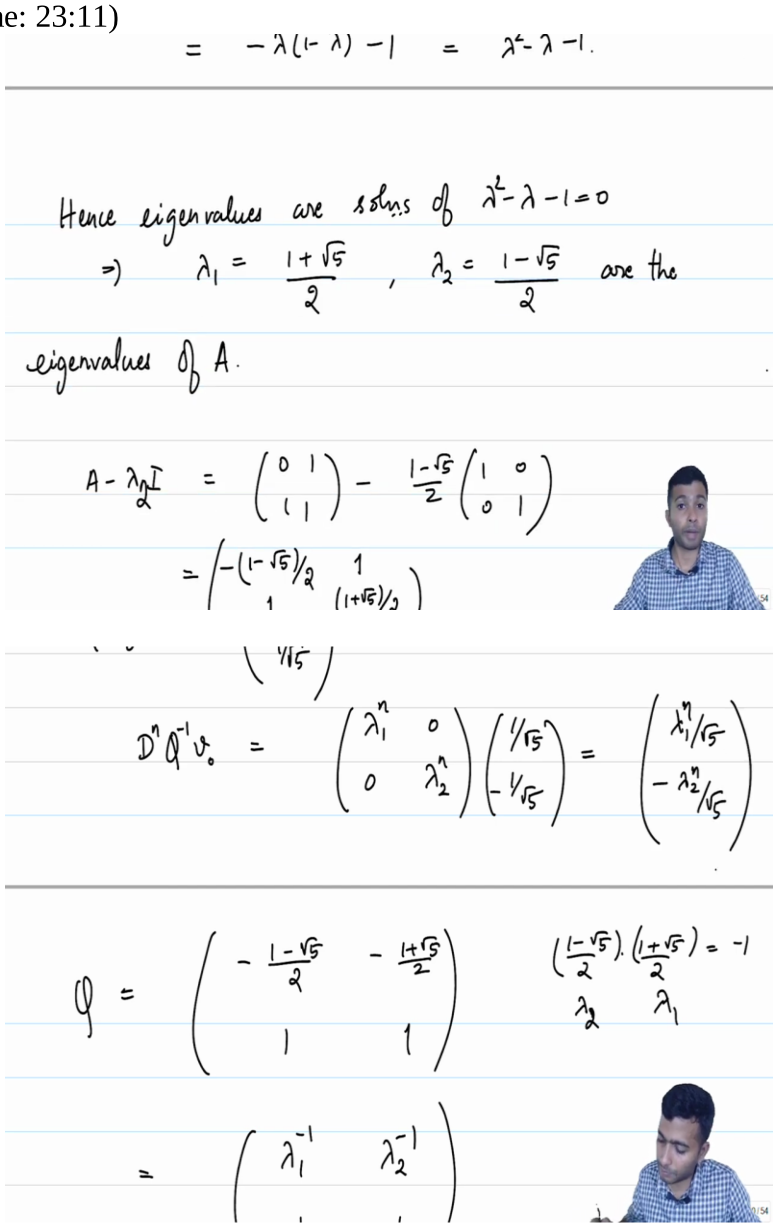

Yes, if you notice carefully, that is what our lambda 1 and lambda 2 were. Lambda 1 is 1 plus root 5 by 2 and lambda 2 is 1 minus root 5 by 2. So let us now apply this to D to the power n

Q inverse. So this, and write the explicitly, this is going to be lambda 1 to the power root 5, minus of lambda 2 to the power root, lambda 1 to the power n by root 5, lambda 2 to the power n by root 5.

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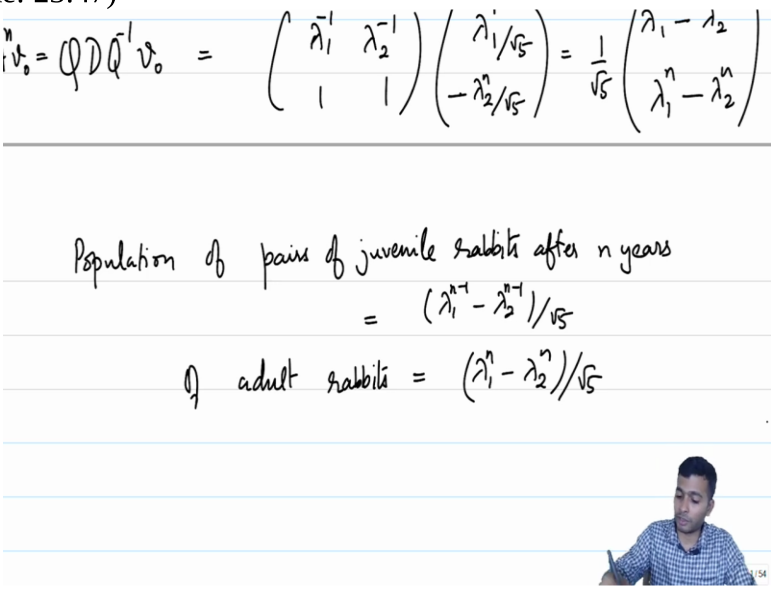

So now what is QDQ inverse v naught? This is just going to be equal to lambda 1 inverse lambda 2 inverse 1, 1. And there is a lambda 1 to the power n by root 5 and lambda 2 to the power n by root 5. This just turns out to be equal to lambda 1 to the power n minus 1 plus lambda 2 to the power n minus 1 by root 5 and lambda 1 by root 5 times lambda 1 to the power n plus lambda 2 to the power n. There is something wrong, let me just check something.

Ah, there is a minus which I have very conveniently forgotten. There will be minus here and this is going to be minus. So yes, this is going to be A to the power n applied to v naught. So the population of juvenile rabbits after n years is going to be, so the population of pairs of (juvenile) juvenile rabbits after n years, this will be equal to lambda 1 to the power n minus 1 minus lambda 2 to the power of n minus 1 by root 5. And of adult rabbits this will be just lambda 1 to the power n minus lambda 2 to the power n by root 5. If you notice, our A is matrix consisting of 0's and 1's. v 0 is also a vector consisting of 0's and 1's. So if you look at any power that will also have integer entries.

And if you multiply it to v naught, that will give you integers. So effectively this lambda 1 to the power n minus 1 minus lambda 2 to the power n minus 1 by square root of 5 are integers despite the fact that lambda 1 is an irrational number which is 1 minus square root of, 1 plus square root of 5 by 2. And lambda 2 being another irrational number which is 1 minus square root of 5 by 2. And this also gives us that lambda 1 to the power n, lambda 1 to the power n minus 1 and lambda 1 to the power n plus 1 all are related. So this gives us a very clean way of calculating all the populations after any n number of years.

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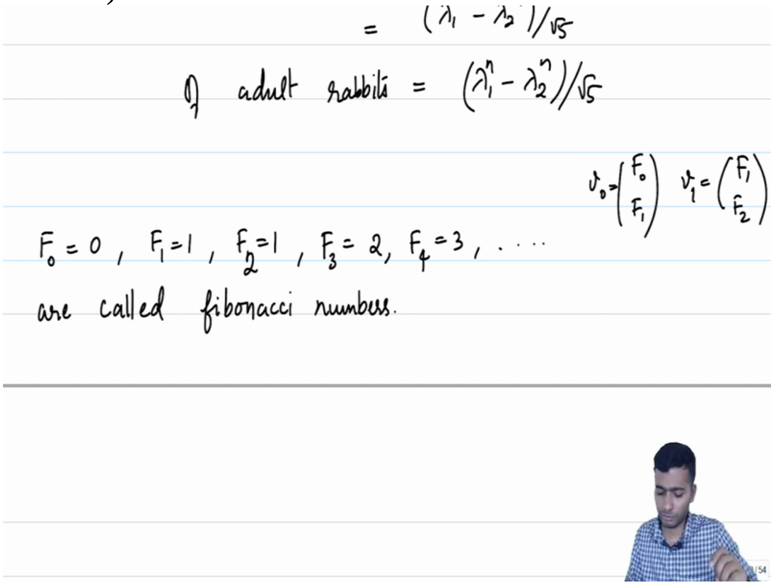

So the numbers that get popped up here are something like say F0 equal to 0, F1 is equal to 1, F2 is equal to 1, F3 is equal to 2, F4 is equal to 3 and so on, are called Fibonacci numbers. So notice that when I write F0, F1, F2 and so on; F0, F1 was our v 0, v 1 is F1, F2 and so on. This is going to be our, this will capture the population of rabbits after n years. And these numbers are called Fibonacci numbers and they are very special and they come up in many natural phenomenon, very natural in fact.

So now with examples, let us discuss what is the impact of studying vector spaces over complex spaces. So I would like to point out that working with vector spaces just over the field of scalars being real numbers is sometimes quite restrictive.

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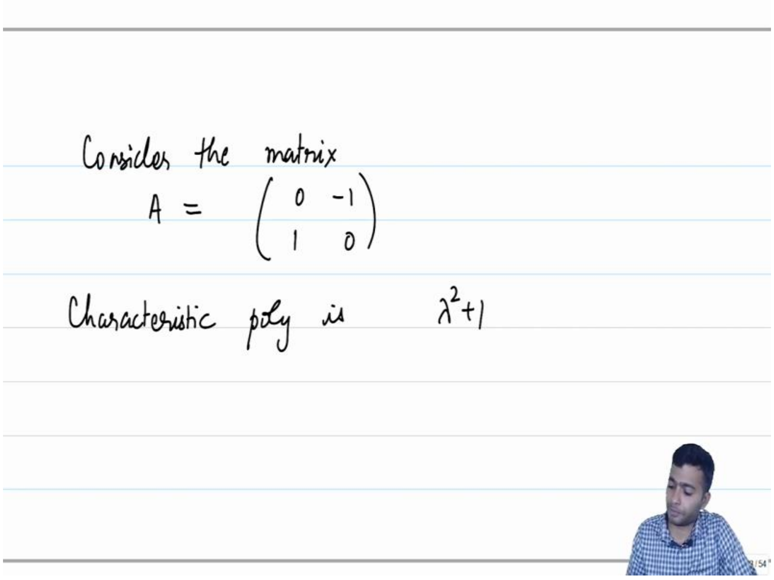

For example, let us consider this particular case, consider the matrix, the linear transformation corresponding to the matrix A which is given by say 0, minus 1, 1, 0. This is in some sense the rotation by 90 degree. So if you consider this particular matrix, then you will notice that the characteristic polynomial, this is given by lambda square plus 1. And lambda square plus 1 equal to 0 does not have any roots in real numbers. And therefore, we do not have eigenvalues over the field of scalars when it is real numbers. However, if the theory over to be developed over the field of scalars being complex numbers. Then lambda square plus 1 equal to 0 would have roots i and minus i, the complex numbers i and minus i.

And we would have been able to compute the corresponding eigenvectors in C2 instead of R2. And we would have had a much richer theory. So it is quite important to note at this juncture that it is many times far more beneficial to study vector spaces over more general fields, more general fields of scalars. In particular, it is far more powerful to study vector spaces over field of complex numbers. Which however, let us not deal with in this particular course, which I will leave you for another course in the future.